

# Preserving problems related to different means of positive operators

Patricia Szokol

joint work with Gergő Nagy

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## Definition

A mean  $M: D \times D \rightarrow D$  on an interval  $D$  is defined as a binary operation satisfying the inequalities  $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$ ,  $(x, y) \in D$ .

## Examples

- arithmetic mean;
- geometric mean;
- harmonic mean.

# Means of positive definite matrices

$\mathcal{H}$ : a complex Hilbert space;

$\mathcal{L}(\mathcal{H})$ : the  $C^*$ -algebra of all bounded linear operators on  $H$  with unit  $I$ ;

$\mathcal{L}(\mathcal{H})_{sa}$ : the vector space of the self-adjoint elements in  $\mathcal{L}(\mathcal{H})$ ;

$\mathcal{L}(\mathcal{H})_{sa}^D$ : the set of all operators in  $\mathcal{L}(\mathcal{H})_{sa}$  with spectra in  $D$  ( $D \subset \mathbb{R}$ );

An operator  $A \in \mathcal{L}(\mathcal{H})$  is positive if  $\langle Ax, x \rangle \geq 0$  is satisfied by every vector  $x \in \mathcal{H}$ ;

$\mathcal{L}(\mathcal{H})_+$  and  $\mathcal{L}(\mathcal{H})_{++}$ : stand for the set of positive and invertible positive operators in  $\mathcal{L}(\mathcal{H})$ , respectively.

## Definition

A binary operation  $\sigma : \mathcal{L}(\mathcal{H})_+ \times \mathcal{L}(\mathcal{H})_+ \rightarrow \mathcal{L}(\mathcal{H})_+$  is a Kubo-Ando mean if it has the next properties. For each elements  $A, B, C, D \in \mathcal{L}(\mathcal{H})_+$  and sequences  $(A_n), (B_n)$  in  $\mathcal{L}(\mathcal{H})_+$ :

- (i)  $I\sigma I = I$ ;
- (ii) if  $A \leq C$  and  $B \leq D$ , then  $A\sigma B \leq C\sigma D$ ;
- (iii)  $C(A\sigma B)C \leq (CAC)\sigma(CBC)$ ;
- (iv) if  $A_n \downarrow A$  and  $B_n \downarrow B$ , then  $A_n\sigma B_n \downarrow A\sigma B$ .

# Kubo-Ando mean

We see from a result of Kubo-Ando theory that for a Kubo-Ando mean  $\sigma$  and a scalar  $t > 0$  the operator  $I\sigma(tI)$  is scalar. Therefore, we can define a function  $f_\sigma: ]0, \infty[ \rightarrow [0, \infty[$ , called the generating function of  $\sigma$ , with the property

$$f_\sigma(t)I = I\sigma(tI) \quad (t > 0).$$

That result also shows that  $f_\sigma$  is operator monotone in the case  $\dim \mathcal{H} = \infty$ . Moreover,

$$(1) \quad A\sigma B = A^{1/2}f_\sigma(A^{-1/2}BA^{-1/2})A^{1/2}$$

for all  $A, B \in \mathcal{L}(\mathcal{H})_{++}$ .

$N$  symmetric norm:  $N: \mathcal{L}(\mathcal{H}) \rightarrow \mathbb{R}$  is a norm which satisfies

$$N(SAT) \leq \|S\|N(A)\|T\|$$

for all  $A, S, T \in \mathcal{L}(\mathcal{H})$ . Here  $\|\cdot\|$  denotes the usual operator norm.

## Theorem (Molnár, Szokol)

Let  $\sigma$  be a Kubo-Ando mean on  $\mathcal{L}(\mathcal{H})_+$ , such that the associated operator monotone function  $f_\sigma$  satisfies  $\lim_{t \rightarrow 0} f_\sigma(t) = 0$  and  $f_\sigma \neq \text{id} \upharpoonright_{]0, \infty[}$ . Assume that  $N: \mathcal{L}(\mathcal{H}) \rightarrow \mathbb{R}$  is a symmetric norm and  $\phi: \mathcal{L}(\mathcal{H})_+ \rightarrow \mathcal{L}(\mathcal{H})_+$  is a bijective transformation which satisfies

$$N(\phi(A)\sigma\phi(B)) = N(A\sigma B), \quad A, B \in \mathcal{L}(\mathcal{H})_+.$$

Then, there exists a unitary or antiunitary operator  $U$  on  $\mathcal{H}$ -n such that

$$\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_+).$$

## Definition

Let  $g: D \rightarrow \mathbb{R}$  be a continuous and strictly monotone function where  $D \subset \mathbb{R}$  and  $t \in (0, 1)$ . Then,  $M_g^{[t]}: D \times D \rightarrow \mathbb{R}$  defined by

$$M_g^{[t]}(x, y) = g^{-1}(tg(x) + (1 - t)g(y))$$

is called a weighted quasi-arithmetic mean. If  $t = 1/2$ , then we get a quasi-arithmetic mean, that is

$$M_g^{[\frac{1}{2}]}(x, y) := M_g(x, y) = g^{-1}\left(\frac{g(x) + g(y)}{2}\right).$$

# Generalized weighted quasi-arithmetic means

## Definition

Let  $D \subset \mathbb{R}$  be an interval. Suppose, that the functions  $f_1, f_2: D \rightarrow \mathbb{R}$  are continuous, monotone in the same sense and not simultaneously constant on any nontrivial subinterval of  $D$ . Then the function  $M_{f_1, f_2}: D \times D \rightarrow \mathbb{R}$

$$M_{f_1, f_2}(x, y) = (f_1 + f_2)^{-1}(f_1(x) + f_2(y)), \quad x, y \in D$$

defines a mean, which is called a generalized quasi-arithmetic mean in  $D$ .

If  $g: D \rightarrow \mathbb{R}$  is a strictly monotone, continuous function, and suppose that  $f_1(x) = tg(x)$  and  $f_2(x) = (1 - t)g(x)$ . Then

$$\begin{aligned} M_{f_1, f_2}(x, y) &= (f_1 + f_2)^{-1}(f_1(x) + f_2(y)) \\ &= g^{-1}(tg(x) + (1 - t)g(y)) = M_g^{[t]}(x, y). \end{aligned}$$

## Remark

*J. Matkowski: characterized the generalized arithmetic means that are quasi-arithmetic or weighted quasi-arithmetic means.*

# Generalized quasi-arithmetic means of invertible positive operators

## Definition

Let  $D \subset \mathbb{R}$  and interval and  $g: D \rightarrow \mathbb{R}$  be a strictly monotone, continuous function. The weighted quasi-arithmetic mean  $M_{g,t}$  generated by  $g$  with weight  $t \in [0, 1]$  is defined by the equality

$$M_{g,t}(A, B) = g^{-1}(tg(A) + (1 - t)g(B)), \quad A, B \in \mathcal{L}(\mathcal{H})_{sa}^D.$$

## Definition

Let  $f_1, f_2: D \rightarrow \mathbb{R}$  be continuous functions, which are monotone in the same sense and not simultaneously constant on any nontrivial interval of  $D$ . Then the general notion of (the operator theoretical version of) the (2-variable) generalized arithmetic mean generated by  $f_1$  and  $f_2$  is defined by the equality

$$M_{f_1, f_2}(A, B) = (f_1 + f_2)^{-1}(f_1(A) + f_2(B)),$$

for all operators  $A$  and  $B$  in  $\mathcal{L}(\mathcal{H})_{sa}^D$ .

## Theorem

Assume that  $\dim \mathcal{H} < \infty$  and let  $f_1, f_2: ]0, \infty[ \rightarrow \mathbb{R}$  be continuous bijections which are monotone in the same sense and  $N: \mathcal{L}(\mathcal{H}) \rightarrow \mathbb{R}$  be a symmetric norm. If  $\phi: \mathcal{L}(\mathcal{H})_{++} \rightarrow \mathcal{L}(\mathcal{H})_{++}$  is a bijective map satisfying

$$(2) \quad N(M_{f_1, f_2}(\phi(A_1), \phi(A_2))) = N(M_{f_1, f_2}(A_1, A_2))$$






for all  $A_1, A_2 \in \mathcal{L}(\mathcal{H})_{++}$ , then there is a unitary or an antiunitary operator  $U$  on  $\mathcal{H}$  such that  $\phi$  is of the form

$$\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}).$$

## Theorem

*Suppose that  $\dim \mathcal{H} < \infty$  and let  $f_1, f_2: ]0, \infty[ \rightarrow ]0, \infty[$  be continuous decreasing bijections and  $N: \mathcal{L}(\mathcal{H}) \rightarrow \mathbb{R}$  be a symmetric norm. If  $\phi: \mathcal{L}(\mathcal{H})_{++} \rightarrow \mathcal{L}(\mathcal{H})_{++}$  is a bijection satisfying (2) for all  $A_1, A_2 \in \mathcal{L}(\mathcal{H})_{++}$ , then there is a unitary or an antiunitary operator  $U$  on  $\mathcal{H}$  such that  $\phi$  is of the form*

$$\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}).$$

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