Isometries on positive definite operators with unit Fuglede-Kadison determinant

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**Case of positive definite matrices ($\mathbb{P}_n$):**

**Definition.** We say that a norm $N$ on $n \times n$ matrices ($\mathbb{M}_n$) is symmetric if

$$N(AXB) \leq \|A\|N(X)\|B\|, \quad A, B, X \in \mathbb{M}_n.$$  

- Molnár, (2013): described the structure of all surjective isometries on positive definite matrices relative to the metric $d_N$, which is defined by

$$d_N(A, B) = N(\log A^{-1/2} BA^{-1/2}), \quad A, B \in \mathbb{P}_n.$$  

- Molnár, (2013): determined all surjective isometries of the space ($\mathbb{P}_n, \delta_S$), where

$$\delta_S(A, B)^2 = \text{Tr} \log \frac{Y + I}{2\sqrt{Y}} = \left\| \log \frac{Y + I}{2\sqrt{Y}} \right\|_1$$

with $Y = A^{-1/2} BA^{-1/2}$ and $\|\|_1$ denotes the trace-norm on $\mathbb{M}_n$.  

Positive operators with unit determinant
The metrics $d_N$ and $\delta_S$ can be regarded as particular distance measures of the form

$$d_{N,f}(A, B) = N(f(A^{-1/2}BA^{-1/2})), \quad A, B \in \mathbb{P}_n,$$

where $N$ is a symmetric norm on $\mathbb{M}_n$ and $f: [0, \infty[ \rightarrow \mathbb{R}$ is an appropriate real function. We emphasize that the so-obtained function $d_{N,f}$ is a so-called **generalized distance measure**. By this concept we mean a function $d: X \times X \rightarrow [0, \infty]$ ($X$ is any set) such that for arbitrary $x, y \in X$ we have $d(x, y) = 0$ if and only if $x = y$.

- Molnár, Sz., (2015): established the complete description of 'generalized isometries' with respect to generalized distance measures of the above-formulated form on the set $\mathbb{P}_n$, where the norm $N$ is symmetric and the function $f$ satisfies some mild assumptions.
Theorem. Let \( N \) be a symmetric norm on \( \mathbb{M}_n \). Assume \( f : [0, \infty[ \to \mathbb{R} \) is a continuous function such that

1. \( f(y) = 0 \) holds if and only if \( y = 1 \);
2. there exists a number \( K > 1 \) such that

\[
|f(y^2)| \geq K |f(y)|, \quad y \in [0, \infty[.
\]

Assume that \( n \geq 3 \). If \( \phi : \mathbb{P}_n \to \mathbb{P}_n \) is a surjective map which satisfies

\[
d_{N,f}(\phi(A), \phi(B)) = d_{N,f}(A, B), \quad A, B \in \mathbb{P}_n,
\]

then there exist an invertible matrix \( T \in \mathbb{M}_n \) and a real number \( c \) such that \( \phi \) is of one of the following forms

1. \( \phi(A) = (\det A)^c TAT^*, \quad A \in \mathbb{P}_n; \)
2. \( \phi(A) = (\det A)^c TA^{-1}T^*, \quad A \in \mathbb{P}_n; \)
3. \( \phi(A) = (\det A)^c TA^{Tr}T^*, \quad A \in \mathbb{P}_n; \)
4. \( \phi(A) = (\det A)^c T(A^{Tr})^{-1}T^*, \quad A \in \mathbb{P}_n. \)
### Case of positive definite matrices ($\mathbb{P}_n$):

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Positive operators with unit determinant
Case of positive definite matrices ($\mathbb{P}_n$):

- Moakher, (2004): studied in details the manifold of $\mathbb{P}_n$ with unit determinant.

- Fletcher and Joshi (2005): investigated the same structure because of its interesting connections to the space of so-called diffusion tensors. In fact, they also studied the set $\mathbb{P}_n$ with determinant $c > 0$, which is a so-called totally geodesic submanifold of the manifold of $\mathbb{P}_n$.

- Molnár, Sz., (2015): Motivated by the previous facts, described all generalized isometries with respect to generalized distance measures on the set $\mathbb{P}_n$ with determinant 1 or with a fixed determinant $c > 0$. 
\( \mathcal{A} \): a finite von Neumann algebra acting on a complex separable Hilbert space, which is a factor, that is, its center is one dimensional;

\( \mathcal{A}^+ \): the set of positive elements in \( \mathcal{A} \);

\( \mathcal{A}^{++} \): the set of positive invertible elements in \( \mathcal{A} \).

The most natural notion of the determinant of operators on infinite dimensional Hilbert space is the Fredholm determinant.

Simon, (2005): Let \( (\lambda_n)_{n \in \Gamma} \) be a (possibly finite) sequence of the nonzero eigenvalues of the compact operator \( T - I \) counted according to their algebraic multiplicities. Then the product \( \prod_{n \in \Gamma} (1 + \lambda_n) \) exists, and \( \det T \) is given by the formula

\[
\det T = \prod_{n \in \Gamma} (1 + \lambda_n).
\]

Every finite factor admits a unique faithful tracial state \( \tau \), by which we mean a positive linear functional \( \tau : \mathcal{A} \rightarrow \mathbb{C} \) with the following properties:

(i) \( \tau(AB) = \tau(BA) \) for all \( A, B \in \mathcal{A} \);
(ii) \( \tau(A^* A) = 0 \) if and only if \( A = 0 \);
(iii) \( \tau(I) = 1 \).

Then the associated Fuglede–Kadison determinant \( \Delta_{FK} : \mathcal{A} \rightarrow \mathbb{C} \) is defined as

\[
\Delta_{FK}(A) = \exp(\tau(\log |A|)),
\]

whenever \( A \in \mathcal{A} \) is an invertible element. (Here, the operator \(|A|\) is obtained from \( A^* A \) by taking square root.)
Properties of $\Delta_{ FK}$:

- $\Delta_{ FK}(I) = 1$;
- positive homogeneous on $A^{++}$;
- multiplicative;
- $\Delta_{ FK}(A^{-1}) = \Delta_{ FK}(A)^{-1}$ for any invertible element $A \in A$;
- if $H$ is a complex Hilbert space with $\dim H = n < \infty$:
  \[
  \Delta_{ FK}(A) = \sqrt{ |\det(A)| }
  \]
  for all invertible elements of the finite von Neumann factor of $\mathbb{M}_n$.

Notation:

- $A^{++}_1$: the set of all operators $A \in A^{++}$ with $\Delta_{ FK}(A) = 1$;
- $A^{++}_c$: the collection of all operators $A \in A^{++}$ with $\Delta_{ FK}(A) = c$ for a given number $c > 0$. 

Positive operators with unit determinant
**Theorem.** Let $\mathcal{A}, \mathcal{B}$ be finite von Neumann factors, $N: \mathcal{A} \to \mathbb{R}$, $M: \mathcal{B} \to \mathbb{R}$ be complete, symmetric norms and $f, g: ]0, +\infty[ \to \mathbb{R}$ be continuous functions satisfying

(f1) $f(y) = 0$ holds if and only if $y = 1$;

(f2) there exists a number $K > 1$ such that

$$|f(y^2)| \geq K|f(y)|, \quad y \in ]0, \infty[.$$

Suppose further that $\phi: \mathcal{A}_1^{++} \to \mathcal{B}_1^{++}$ is a surjective transformation with the property

$$d_{M,f}(\phi(A), \phi(B)) = d_{N,g}(A, B), \quad A, B \in \mathcal{A}_1^{++}. \quad (1)$$

Then there exists an algebra $\ast$-isomorphism or $\ast$-antiisomorphism $\theta: \mathcal{A} \to \mathcal{B}$ and an element $T \in \mathcal{B}_1^{++}$ such that $\phi$ is of one of the following forms:

(b1) $\phi(A) = T\theta(A)T$ for all $A \in \mathcal{A}_1^{++}$;

(b2) $\phi(A) = T\theta(A^{-1})T$ for all $A \in \mathcal{A}_1^{++}$. 
Corollary.
Let $\mathcal{A}, \mathcal{B}$ be finite von Neumann factors, $c > 0$ be a scalar and $N: \mathcal{A} \to \mathbb{R}$, $M: \mathcal{B} \to \mathbb{R}$ be complete, symmetric norms. Assume that $f, g: [0, +\infty[ \to \mathbb{R}$ are continuous functions satisfying (f1)-(f2). Suppose further that $\phi: \mathcal{A}_c^{++} \to \mathcal{B}_c^{++}$ is a surjective map with the property

$$d_{M,f}(\phi(A), \phi(B)) = d_{N,g}(A, B), \quad A, B \in \mathcal{A}_c^{++}. $$

Then there exists an algebra $\ast$-isomorphism or $\ast$-antiisomorphism $\theta: \mathcal{A} \to \mathcal{B}$ and an element $T \in \mathcal{B}_1^{++}$ such that $\phi$ is of one of the following forms:

(c1) $\phi(A) = T\theta(A)T$ for all $A \in \mathcal{A}_c^{++}$;

(c2) $\phi(A) = c^2 T\theta(A^{-1})T$ for all $A \in \mathcal{A}_c^{++}$. 

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References


