

On certain classes of Schur-convex functions

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Definition

Let $t, t' \in \mathbb{R}^n$, $t = (t_1, \dots, t_n)$, $t' = (t'_1, \dots, t'_n)$. We say that t' is majorized by t and write $t' \preceq t$ if

(i) $\sum_{i=1}^n t'_i = \sum_{i=1}^n t_i$;

(ii) $\sum_{i=1}^k t'_{[i]} \leq \sum_{i=1}^k t_{[i]}$, $k = 1, \dots, n-1$

where $t_{[i]}$ denotes the i th largest coordinate of t .

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Definition

Let I be a nonempty interval. Then the function $f: I^n \rightarrow \mathbb{R}$ is Schur-convex if

$$f(t') \leq f(t), \quad \forall t' \preceq t.$$

Theorem (Hardy, Littlewood, Pólya (1929))

A necessary and sufficient condition that $t' \preceq t$ is that there exists a doubly stochastic matrix P such that $t' = Pt$.

Corollary

Let I be a nonempty interval. Then the function $f: I^n \rightarrow \mathbb{R}$ is Schur-convex if

$$f(Sx) \leq f(x)$$

for all doubly stochastic matrix S and for all $x = (x_1, \dots, x_n) \in I^n$.

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Let D be a nonempty, convex subset of a linear space X . Then, the function $f: D \times D \rightarrow \mathbb{R}$ is Schur-convex, if

$$f(tx + (1-t)y, (1-t)x + ty) \leq f(x, y)$$

for all $x, y \in D$, and $t \in [0, 1]$. And if the above inequality stands only for one fixed $t \in]0, 1[$ and f is symmetric, we say that f is t -Schur-convex.

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It is easy to check, that if $f: D \times D \rightarrow \mathbb{R}$ is a Schur-convex and

- $f(x, y) = g(x) + g(y)$, then g is a Wright-convex function;
- $f(x, y) = \max\{g(x), g(y)\}$, then g is a quasi-convex.

Let us define the function $\varphi_{x,y} : [0, 1] \rightarrow \mathbb{R}$ in the following way

$$\varphi_{x,y}(t) = f(tx + (1-t)y, (1-t)x + ty), \quad x, y \in D.$$

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Theorem

A function $f : D \times D \rightarrow \mathbb{R}$ is $\frac{1}{2}$ -Schur-convex if and only if the function $\varphi_{x,y}$ has a global minimum at $\frac{1}{2}$ for every fixed $x, y \in D$.

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Theorem

Let $f : D \times D \rightarrow \mathbb{R}$ be a symmetric function. Then f is Schur-convex if and only if for all arbitrarily fixed $x, y \in D$ the function $\varphi_{x,y}$ is monotone decreasing on $[0, \frac{1}{2}]$, monotone increasing on $[\frac{1}{2}, 1]$, and $\varphi_{x,y}$ has a global minimum at $\frac{1}{2}$.

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Remark

Olbryś: showed that the corresponding sufficiency part of the previous theorem is true in the case of Wright-convexity.

It is easy to see that the above definition can be considered in the following way: $f: D \times D \rightarrow \mathbb{R}$ is t -Schur-convex, if

$$f\left(\left(\begin{array}{cc} t & 1-t \\ 1-t & t \end{array}\right)x\right) \leq f(x).$$

for all $x \in D^2$.

Let $t = (t_1, \dots, t_n)$ such that $\sum_{i=1}^n t_i = 1$ and $t_1 \geq t_2 \geq \dots \geq t_n \geq 0$. We say that the matrix T is t -Schur matrix or shifted matrix generated by t if it is given in the following way

$$T = \begin{pmatrix} t_1 & t_2 & \dots & t_{n-1} & t_n \\ t_2 & t_3 & \dots & t_n & t_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_n & t_1 & \dots & t_{n-2} & t_{n-1} \end{pmatrix}.$$

General case

Let X be a real vector space, and $D \subset X$ be a nonempty, convex set and T be a $t = (t_1, \dots, t_n)$ -Schur matrix. A function $f : D^n \rightarrow \mathbb{R}$ is T -Schur convex, if for all $x \in D^n$,

$$f(Tx) \leq f(x).$$

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The standard n -simplex is the subset of \mathbb{R}^n given by

$$S_n = \{(t_1, \dots, t_n) \mid \sum_{i=1}^n t_i = 1, t_i \geq 0, i \in \{1, 2, \dots, n\}\}$$

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We would like to characterize Schur convexity with the help of the „monotonicity” of the function $\varphi_x : S_n \rightarrow \mathbb{R}$ ($x \in D$) defined by

$$\varphi_x(t_1, \dots, t_n) := f(Tx), \quad (t_1, \dots, t_n) \in S^n$$

where T is the $t = (t_1, \dots, t_n)$ -Schur-matrix.

Rewriting of the theorems of Judit Makó and Pál Burai:

A function $f: D \times D \rightarrow \mathbb{R}$ is $(\frac{1}{2}, \frac{1}{2})$ -Schur-convex if and only if for all arbitrarily fixed $x \in D^2$ the function $\varphi_x: S_2 \rightarrow \mathbb{R}$ defined by

$$\varphi_x(t, 1-t) = f\left(\begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix} x\right), \quad t \in \left[\frac{1}{2}, 1\right].$$

has a global minimum at $(\frac{1}{2}, \frac{1}{2})$.





A symmetric function $f: D^2 \rightarrow \mathbb{R}$ is Schur-convex if and only if for all arbitrarily fixed $x \in D^2$ the function $\varphi_x: S_2 \rightarrow \mathbb{R}$ has a global minimum at $(\frac{1}{2}, \frac{1}{2})$ and it is monotone increasing with respect to \preceq .

Theorem

Let $n \in \mathbb{N}$ and N be the $(\frac{1}{n}, \dots, \frac{1}{n})$ -Schur matrix. Then the symmetric function $f: D^n \rightarrow \mathbb{R}$ is N -Schur convex, if and only if, for all $x \in D^n$ the function $\varphi_x: S_n \rightarrow \mathbb{R}$ defined by

$$(1) \quad \varphi_x(t_1, \dots, t_n) := f(Tx), \quad (t_1, \dots, t_n) \in S^n$$

(where T is the $t = (t_1, \dots, t_n)$ -Schur matrix) has a global minimum at $(\frac{1}{n}, \dots, \frac{1}{n})$ with respect to \preceq .

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