Preserving problems related to different means of positive operators

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Introduction

Definition

A mean $M : D \times D \to D$ on an interval $D$ is defined as a binary operation satisfying the inequalities $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$, $(x, y) \in D$.

Examples

- arithmetic mean;
- geometric mean;
- harmonic mean.
Means of positive definite matrices

$\mathcal{H}$: a complex Hilbert space;
$\mathcal{L}(\mathcal{H})$: the $C^*$-algebra of all bounded linear operators on $H$ with unit $I$;
$\mathcal{L}(\mathcal{H})_{sa}$: the vector space of the self-adjoint elements in $\mathcal{L}(\mathcal{H})$;
$\mathcal{L}(\mathcal{H})^D_{sa}$: the set of all operators in $\mathcal{L}(\mathcal{H})_{sa}$ with spectra in $D$ ($D \subset \mathbb{R}$);
An operator $A \in \mathcal{L}(\mathcal{H})$ is positive if $\langle Ax, x \rangle \geq 0$ is satisfied by every vector $x \in \mathcal{H}$;
$\mathcal{L}(\mathcal{H})_+$ and $\mathcal{L}(\mathcal{H})_{++}$: stand for the set of positive and invertible positive operators in $\mathcal{L}(\mathcal{H})$, respectively.

Definition

A binary operation $\sigma : \mathcal{L}(\mathcal{H})_+ \times \mathcal{L}(\mathcal{H})_+ \rightarrow \mathcal{L}(\mathcal{H})_+$ is a Kubo-Ando mean if it has the next properties. For each elements $A, B, C, D \in \mathcal{L}(\mathcal{H})_+$ and sequences $(A_n), (B_n)$ in $\mathcal{L}(\mathcal{H})_+$:

(i) $I \sigma I = I$;
(ii) if $A \leq C$ and $B \leq D$, then $A \sigma B \leq C \sigma D$;
(iii) $C(A \sigma B)C \leq (CAC)\sigma(CBC)$;
(iv) if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \sigma B_n \downarrow A \sigma B$. 

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Preserving problems related to different means of positive operators
We see from a result of Kubo-Ando theory that for a Kubo-Ando mean \( \sigma \) and a scalar \( t > 0 \) the operator \( l \sigma (tl) \) is scalar. Therefore, we can define a function \( f_\sigma : (0, \infty) \rightarrow [0, \infty) \), called the generating function of \( \sigma \), with the property

\[
f_\sigma (t) l = l \sigma (tl) \quad (t > 0).
\]

That result also shows that \( f_\sigma \) is operator monotone in the case \( \dim \mathcal{H} = \infty \). Moreover,

\[
A \sigma B = A^{1/2} f_\sigma (A^{-1/2} BA^{-1/2}) A^{1/2}
\]

(1)

for all \( A, B \in \mathcal{L}(\mathcal{H})_{++} \).

\( N \) symmetric norm: \( N : \mathcal{L}(\mathcal{H}) \rightarrow \mathbb{R} \) is a norm which satisfies

\[
N (SAT) \leq \| S \| N (A) \| T \|
\]

for all \( A, S, T \in \mathcal{L}(\mathcal{H}) \). Here \( \| \cdot \| \) denotes the usual operator norm.
Preserver problem concerning Kubo-Ando mean

Theorem (Molnár, Szokol)

Let \( \sigma \) be a Kubo-Ando mean on \( \mathcal{L}(\mathcal{H})_+ \), such that the associated operator monotone function \( f_\sigma \) satisfies \( \lim_{t \to 0} f_\sigma(t) = 0 \) and \( f_\sigma \neq \text{id}|_{]0,\infty[} \). Assume that \( N: \mathcal{L}(\mathcal{H}) \to \mathbb{R} \) is a symmetric norm and \( \phi: \mathcal{L}(\mathcal{H})_+ \to \mathcal{L}(\mathcal{H})_+ \) is a bijective transformation which satisfies

\[
N(\phi(A)\sigma\phi(B)) = N(A\sigma B), \quad A, B \in \mathcal{L}(\mathcal{H})_+.
\]

Then, there exists a unitary or antiunitary operator \( U \) on \( \mathcal{H} \)-n such that

\[
\phi(A) = UAU^*, \quad (A \in \mathcal{L}(\mathcal{H})_+).
\]
Definition

Let $g : D \to \mathbb{R}$ be a continuous and strictly monotone function where $D \subset \mathbb{R}$ and $t \in (0, 1)$. Then, $M_g^{[t]} : D \times D \to \mathbb{R}$ defined by

$$M_g^{[t]}(x, y) = g^{-1}(tg(x) + (1 - t)g(y))$$

is called a weighted quasi-arithmetic mean. If $t = 1/2$, then we get a quasi-arithmetic mean, that is

$$M_g^{[\frac{1}{2}]}(x, y) := M_g(x, y) = g^{-1} \left( \frac{g(x) + g(y)}{2} \right).$$
Generalized weighted quasi-arithmetic means

Definition

Let $D \subset \mathbb{R}$ be an interval. Suppose, that the functions $f_1, f_2 : D \to \mathbb{R}$ are continuous, monotone in the same sense and not simultaneously constant on any nontrivial subinterval of $D$. Then the function $M_{f_1, f_2} : D \times D \to \mathbb{R}$

\[ M_{f_1, f_2}(x, y) = (f_1 + f_2)^{-1}(f_1(x) + f_2(y)), \quad x, y \in D \]

defines a mean, which is called a generalized weighted quasi-arithmetic mean in $D$.

If $g : D \to \mathbb{R}$ is a strictly monotone, continuous function, and suppose that $f_1(x) = tg(x)$ and $f_2(x) = (1 - t)g(x)$. Then

\[
M_{f_1, f_2}(x, y) = (f_1 + f_2)^{-1}(f_1(x) + f_2(y)) \\
= g^{-1}(tg(x) + (1 - t)g(y)) = M_{g}^{[t]}(x, y).
\]

Remark

J. Matkowski: characterized the generalized weighted arithmetic means that are quasi-arithmetic or weighted quasi-arithmetic means.
(Generalized) weighted quasi-arithmetic means of invertible positive operators

Definition

Let $D \subset \mathbb{R}$ and interval and $g : D \rightarrow \mathbb{R}$ be a strictly monotone, continuous function. The weighted quasi-arithmetic mean $M_{g,t}$ generated by $g$ with weight $t \in [0,1]$ is defined by the equality

$$M_{g,t}(A, B) = g^{-1}(tg(A) + (1 - t)g(B)), \quad A, B \in \mathcal{L}(\mathcal{H})_{sa}.$$

Definition

Let $f_1, f_2 : D \rightarrow \mathbb{R}$ be continuous functions, which are monotone in the same sense and not simultaneously constant on any nontrivial interval of $D$. Then the general notion of (the operator theoretical version of) the (2-variable) generalized weighted arithmetic mean generated by $f_1$ and $f_2$ is defined by the equality

$$M_{f_1,f_2}(A, B) = (f_1 + f_2)^{-1}(f_1(A) + f_2(B)),$$

for all operators $A$ and $B$ in $\mathcal{L}(\mathcal{H})_{sa}$. 
Main results

Theorem

Assume that $\dim \mathcal{H} < \infty$ and let $f_1, f_2: ]0, \infty[ \to \mathbb{R}$ be continuous bijections which are monotone in the same sense and $N: \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ be a symmetric norm. If $\phi: \mathcal{L}(\mathcal{H})_{++} \to \mathcal{L}(\mathcal{H})_{++}$ is a bijective map satisfying

\begin{equation}
N(M_{f_1,f_2}(\phi(A_1),\phi(A_2))) = N(M_{f_1,f_2}(A_1,A_2))
\end{equation}

for all $A_1, A_2 \in \mathcal{L}(\mathcal{H})_{++}$, then there is a unitary or an antiunitary operator $U$ on $\mathcal{H}$ such that $\phi$ is of the form

$$\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}).$$
Main results

Theorem

Suppose that \( \dim \mathcal{H} < \infty \) and let \( f_1, f_2 : [0, \infty[ \to [0, \infty[ \) be continuous decreasing bijections and \( N : \mathcal{L}(\mathcal{H}) \to \mathbb{R} \) be a symmetric norm. If \( \phi : \mathcal{L}(\mathcal{H})_{++} \to \mathcal{L}(\mathcal{H})_{++} \) is a bijection satisfying (2) for all \( A_1, A_2 \in \mathcal{L}(\mathcal{H})_{++} \), then there is a unitary or an antiunitary operator \( U \) on \( \mathcal{H} \) such that \( \phi \) is of the form

\[
\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}).
\]


