Preserving problems related to different means of positive operators

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Introduction

Definition

A mean $M\colon D \times D \to D$ *on an interval* D *is defined as a binary operation satisfying the inequalities* $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$, $(x, y) \in D$.

Examples

- arithmetic mean;
- geometric mean;
- harmonic mean.

Means of positive definite matrices

$\mathcal{H}$: a complex Hilbert space;
$\mathcal{L}(\mathcal{H})$: the C^*-algebra of all bounded linear operators on H with unit I;
$\mathcal{L}(\mathcal{H})_{sa}$: the vector space of the self-adjoint elements in $\mathcal{L}(\mathcal{H})$;
$\mathcal{L}(\mathcal{H})_{sa}^D$: the set of all operators in $\mathcal{L}(\mathcal{H})_{sa}$ with spectra in D ($D \subset \mathbb{R}$);
An operator $A \in \mathcal{L}(\mathcal{H})$ is positive if $\langle Ax, x \rangle \geq 0$ is satisfied by every vector $x \in \mathcal{H}$;
$\mathcal{L}(\mathcal{H})_+$ and $\mathcal{L}(\mathcal{H})_{++}$: stand for the set of positive and invertible positive operators in $\mathcal{L}(\mathcal{H})$, respectively.

Definition

A binary operation $\sigma \colon \mathcal{L}(\mathcal{H})_+ \times \mathcal{L}(\mathcal{H})_+ \to \mathcal{L}(\mathcal{H})_+$ is a Kubo-Ando mean if it has the next properties. For each elements $A, B, C, D \in \mathcal{L}(\mathcal{H})_+$ and sequences $(A_n), (B_n)$ in $\mathcal{L}(\mathcal{H})_+$:

(i) $I \sigma I = I$;

(ii) *if* $A \leq C$ *and* $B \leq D$, *then* $A \sigma B \leq C \sigma D$;

(iii) $C(A \sigma B)C \leq (CAC)\sigma(CBC)$;

(iv) *if* $A_n \downarrow A$ *and* $B_n \downarrow B$, *then* $A_n \sigma B_n \downarrow A \sigma B$.

Kubo-Ando mean

We see from a result of Kubo-Ando theory that for a Kubo-Ando mean σ and a scalar $t > 0$ the operator $I\sigma(tI)$ is scalar. Therefore, we can define a function $f_\sigma \colon]0, \infty[\to [0, \infty[$, called the generating function of σ, with the property

$$f_\sigma(t)I = I\sigma(tI) \quad (t > 0).$$

That result also shows that f_σ is operator monotone in the case $\dim \mathcal{H} = \infty$. Moreover,

$$A\sigma B = A^{1/2} f_\sigma(A^{-1/2} B A^{-1/2}) A^{1/2} \tag{1}$$

for all $A, B \in \mathcal{L}(\mathcal{H})_{++}$.
N symmetric norm: $N \colon \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ is a norm which satisfies

$$N(SAT) \leq \|S\| N(A) \|T\|$$

for all $A, S, T \in \mathcal{L}(\mathcal{H})$. Here $\|\cdot\|$ denotes the usual operator norm.

Preserver problem concerning Kubo-Ando mean

Theorem (Molnár, Szokol)

Let σ be a Kubo-Ando mean on $\mathcal{L}(\mathcal{H})_+$, such that the associated operator monotone function f_σ satisfies $\lim_{t\to 0} f_\sigma(t) = 0$ and $f_\sigma \neq \mathrm{id}|_{]0,\infty[}$. Assume that $N\colon \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ is a symmetric norm and $\phi\colon \mathcal{L}(\mathcal{H})_+ \to \mathcal{L}(\mathcal{H})_+$ is a bijective transformation which satisfies

$$N(\phi(A)\sigma\phi(B)) = N(A\sigma B), \quad A, B \in \mathcal{L}(\mathcal{H})_+.$$

Then, there exists a unitary or antiunitary operator U on $\mathcal{H}$-n such that

$$\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_+).$$

Quasi-aritmetic means

Definition

Let $g\colon D \to \mathbb{R}$ *be a continuous and strictly monotone function where* $D \subset \mathbb{R}$ *and* $t \in (0,1)$. *Then,* $M_g^{[t]}\colon D \times D \to \mathbb{R}$ *defined by*

$$M_g^{[t]}(x,y) = g^{-1}(tg(x) + (1-t)g(y))$$

is called a weighted quasi-arithmetic mean. If $t = 1/2$, *then we get a quasi-arithmetic mean, that is*

$$M_g^{\left[\frac{1}{2}\right]}(x,y) := M_g(x,y) = g^{-1}\left(\frac{g(x)+g(y)}{2}\right).$$

Generalized weighted quasi-arithmetic means

Definition

Let $D \subset \mathbb{R}$ be an interval. Suppose, that the functions $f_1, f_2 \colon D \to \mathbb{R}$ are continuous, monotone in the same sense and not simultaneously constant on any nontrivial subinterval of D. Then the function $M_{f_1,f_2} \colon D \times D \to \mathbb{R}$

$$M_{f_1,f_2}(x,y) = (f_1 + f_2)^{-1}(f_1(x) + f_2(y)), \quad x, y \in D$$

defines a mean, which is called a generalized weighted quasi-arithmetic mean in D.

If $g \colon D \to \mathbb{R}$ is a strictly monotone, continuous function, and suppose that $f_1(x) = tg(x)$ and $f_2(x) = (1-t)g(x)$. Then

$$\begin{aligned} M_{f_1,f_2}(x,y) &= (f_1 + f_2)^{-1}(f_1(x) + f_2(y)) \\ &= g^{-1}(tg(x) + (1-t)g(y)) = M_g^{[t]}(x,y). \end{aligned}$$

Remark

J. Matkowski: characterized the generalized weighted aritmetic means that are quasi-arithmetic or weighted quasi-arithmetic means.

(Generalized) weighted quasi-arithmetic means of invertible positive operators

Definition

Let $D \subset \mathbb{R}$ and interval and $g\colon D \to \mathbb{R}$ be a strictly monotone, continuous function. The weighted quasi-arithmetic mean $M_{g,t}$ generated by g with weight $t \in [0,1]$ is defined by the equality

$$M_{g,t}(A,B) = g^{-1}(tg(A) + (1-t)g(B)), \quad A, B \in \mathcal{L}(\mathcal{H})_{sa}^{D}.$$

Definition

Let $f_1, f_2\colon D \to \mathbb{R}$ be continuous functions, which are monotone in the same sense and not simultaneously constant on any nontrivial interval of D. Then the general notion of (the operator theoretical version of) the (2-variable) generalized weighted aritmetic mean generated by f_1 and f_2 is defined by the equality

$$M_{f_1,f_2}(A,B) = (f_1 + f_2)^{-1}(f_1(A) + f_2(B)),$$

for all operators A and B in $\mathcal{L}(\mathcal{H})_{sa}^{D}$.

Main results

Theorem

Assume that $\dim \mathcal{H} < \infty$ *and let* $f_1, f_2 \colon]0, \infty[\to \mathbb{R}$ *be continuous bijections which are monotone in the same sense and* $N \colon \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ *be a symmetric norm. If* $\phi \colon \mathcal{L}(\mathcal{H})_{++} \to \mathcal{L}(\mathcal{H})_{++}$ *is a bijective map satisfying*

$$N(M_{f_1,f_2}(\phi(A_1), \phi(A_2))) = N(M_{f_1,f_2}(A_1, A_2)) \tag{2}$$

for all $A_1, A_2 \in \mathcal{L}(\mathcal{H})_{++}$*, then there is a unitary or an antiunitary operator* U *on* $\mathcal{H}$ *such that* ϕ *is of the form*

$$\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}).$$

Main results

Theorem

Suppose that $\dim \mathcal{H} < \infty$ *and let* $f_1, f_2 \colon]0,\infty[\to]0,\infty[$ *be continuous decreasing bijections and* $N \colon \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ *be a symmetric norm . If* $\phi \colon \mathcal{L}(\mathcal{H})_{++} \to \mathcal{L}(\mathcal{H})_{++}$ *is a bijection satisfying* (2) *for all* $A_1, A_2 \in \mathcal{L}(\mathcal{H})_{++}$*, then there is a unitary or an antiunitary operator* U *on* $\mathcal{H}$ *such that* ϕ *is of the form*

$$\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}).$$

Refereces

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