Transformations preserving generalized quasi-arithmetic means of invertible positive operators

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joint work with Gergő Nagy

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Introduction

Definition

A mean $M\colon D \times D \to D$ *on an interval* D *is defined as a binary operation satisfying the inequalities* $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$, $(x, y) \in D$.

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Let $h: D \to \mathbb{R}$ *be a continuous and strictly monotone function where* $D \subset \mathbb{R}$ *and* $p \in (0, 1)$. *Then,* $M_h^{[p]}: D \times D \to \mathbb{R}$ *defined by*

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is called a weighted quasi-arithmetic mean. If $p=1/2$, then we get a quasi-arithmetic mean, that is

$$M_h^{[\frac{1}{2}]}(x,y):=M_h(x,y)=h^{-1}\left(\frac{h(x)+h(y)}{2}\right).$$

Definition

Let $D \subset \mathbb{R}$ be an interval. Suppose, that the functions $f_1, f_2 \colon D \to \mathbb{R}$ are continuous, monotone in the same sense and not simultaneously constant on any nontrivial subinterval of D. Then the function $M_{f_1,f_2} \colon D \times D \to \mathbb{R}$

$$M_{f_1,f_2}(x,y) = (f_1 + f_2)^{-1}(f_1(x) + f_2(y)), \quad x, y \in D$$

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If $h\colon D \to \mathbb{R}$ is a strictly monotone, continuous function, and suppose that $f_1(x) = ph(x)$ and $f_2(x) = (1-p)h(x)$. Then

$$\begin{aligned} M_{f_1,f_2}(x,y) &= (f_1 + f_2)^{-1}(f_1(x) + f_2(y)) \\ &= h^{-1}(ph(x) + (1-p)h(y)) = M_h^{[p]}(x,y). \end{aligned}$$

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Remark

J. Matkowski: characterized the generalized weighted aritmetic means that are quasi-arithmetic or weighted quasi-arithmetic means.

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Definition

Let $D \subset \mathbb{R}$ be an interval, $f_1, f_2 \colon D \to \mathbb{R}$ continuous functions, which are monotone in the same sense and not simultaneously constant on any nontrivial interval of D. Then the general notion of (the operator theoretical version of) the (2-variable) Matkowski mean generated by f_1 and f_2 is defined by the equality

$$M_{f_1,f_2}(A, B) = (f_1 + f_2)^{-1}(f_1(A) + f_2(B)),$$

for all operators A and B in $\mathcal{L}(\mathcal{H})_{sa}^D$.

Main result

N: unitary invariant norm, i.e. $N\colon \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ is a norm which satisfies $N(UAV) = N(A)$ for all $A \in \mathcal{L}(\mathcal{H})$ and unitary operators U, V.

Theorem

Let D be any of the sets $\mathbb{R}$, $]0,\infty[$ and $f, g\colon]0,\infty[\to D$ be continuous bijections which are monotone in the same sense and satisfy that $|\lim_{x\to 0}(f(x)+g(x))| = \infty$. Moreover, fix a unitary invariant norm $N\colon \mathcal{L}(\mathcal{H}) \to \mathbb{R}$. If $\phi\colon \mathcal{L}(\mathcal{H})_{++} \to \mathcal{L}(\mathcal{H})_{++}$ is a bijection satisfying

$$N(M_{f_1,f_2}(\phi(A),\phi(B))) = N(M_{f_1,f_2}(A,B)) \tag{1}$$

for all $A, B \in \mathcal{L}(\mathcal{H})_{++}$, then there is a unitary or an antiunitary operator U on $\mathcal{H}$ such that ϕ is of the form

$$\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}).$$

Sketch of the proof 1.

We consider the case when $D =]0, \infty[$.
By the assumptions:

$$N(M_{f_1,f_2}(\phi(A), \phi(B))) = N(M_{f_1,f_2}(A, B)), \quad A, B \in \mathcal{L}(\mathcal{H})_{++},$$

which means that for all $A, B \in \mathcal{L}(\mathcal{H})_{++}$

$$N((f_1 + f_2)^{-1}(f_1(\phi(A)) + f_2(\phi(B))) = N((f_1 + f_2)^{-1}(f_1(A) + f_2(B)).$$

Consider the maps $\psi_1, \psi_2 \colon \mathcal{L}(\mathcal{H})_{++} \to \mathcal{L}(\mathcal{H})_{++}$ given by

$$\psi_i(A) = f_i(\phi(f_i^{-1}(A))) \quad (A \in \mathcal{L}(\mathcal{H})_{++},\ i = 1, 2).$$

Then, ψ_1 and ψ_2 are bijective and posses the following property

$$(2) \qquad N((f_1 + f_2)^{-1}(\psi_1(A) + \psi_2(B))) = N((f_1 + f_2)^{-1}(A + B))$$

for all $A, B \in \mathcal{L}(\mathcal{H})_{++}$.

Sketch of the proof 2.

Moreover, by the conditions of Theorem, the function $g = (f_1 + f_2)^{-1} \colon]0, \infty[\to]0, \infty[$ is continuous, strictly monotone decreasing for which $\lim_{\alpha \to \infty} g(\alpha) = 0$.

By a Lemma of M. Gaál and G. Nagy

Let D be any of the sets $]0, \infty[$, $\mathbb{R}$ and $g \colon D \to]0, \infty[$ be a continuous strictly decreasing function such that $\lim_{\alpha \to \infty} g(\alpha) = 0$. If $A, B \in \mathcal{L}(\mathcal{H})_{sa}^D$, then $A \leq B$ exactly when $N(g(A + X)) \geq N(g(B + X))$ for all $X \in \mathcal{L}(\mathcal{H})_{sa}^D$.

Applying this Lemma one has that

$$\begin{aligned}
A \leq B &\iff N(g(A + X)) \geq N(g(B + X)) \ \ \forall X \in \mathcal{L}(\mathcal{H})_{++} \\
&\iff N(g(\psi_1(A) + \psi_2(X))) \geq N(g(\psi_1(B) + \psi_2(X))) \ \ \forall X \in \mathcal{L}(\mathcal{H})_{++} \\
&\iff N(g(\psi_1(A) + Y)) \geq N(g(\psi_1(B) + Y)) \ \ \forall Y \in \mathcal{L}(\mathcal{H})_{++} \\
&\iff \psi_1(A) \leq \psi_1(B).
\end{aligned}$$

Sketch of the proof 3.

By the previous observation we infer that ψ_1 and ψ_2 are order automorphisms of $\mathcal{L}(\mathcal{H})_{++}$. Such transformations of $\mathcal{L}(\mathcal{H})_{++}$ are described in a paper of Lajos Molnár. Applying those results to ψ_1, ψ_2, we deduce that there are invertible linear or conjugate-linear operators T_1, T_2 on $\mathcal{H}$ such that

$$\psi_1(A) = T_1 A T_1^*,\ \psi_2(A) = T_2 A T_2^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}). \tag{3}$$

Then by (2) and the latter conclusion

$$N(g(T_1 A_1 T_1^* + T_2 A_2 T_2^*)) = N(g(A_1 + A_2)) \quad (A_1, A_2 \in \mathcal{L}(\mathcal{H})_{++}).$$

Plugging $A_2 = (1/k)I$ $(k \in \mathbb{N})$ in this equality and tending to ∞ with k, we see that

$$N(g(T_1 A T_1^*)) = N(g(A)).$$

Using polar decomposition of T_1 it can be shown that T_1 is unitary.

Generalized n variable weighted quasi-arithmetic means

Definition

Let $n \in \mathbb{N} \setminus \{1\}$, $D \subset \mathbb{R}$ an interval and $f_i : D \to \mathbb{R}$ $(i = 1, \ldots, n)$ continuous functions which are monotone in the same sense and not simultaneously constant on any nontrivial subinterval of D. Then the n-variable Matkowski mean generated by $f_1, \ldots, f_n$ is defined by the equality

$$M_{f_1,\ldots,f_n}(A_1, \ldots, A_n) = (f_1 + \ldots + f_n)^{-1}(f_1(A_1) + \ldots + f_n(A_n)).$$

for all operators $A_i \in \mathcal{L}(\mathcal{H})^D_{sa}$ $(i = 1, \ldots, n)$.

Theorem

Let D be any of the sets $\mathbb{R}$, $]0,\infty[$ and $f_1,\ldots,f_n\colon]0,\infty[\to D$ be continuous bijections which are monotone in the same sense and satisfy that $|\lim_{x\to 0}(f_1(x)+\ldots+f_n(x))|=\infty$. Moreover, fix a unitary invariant norm $N\colon \mathcal{L}(\mathcal{H})\to\mathbb{R}$. If $\phi\colon \mathcal{L}(\mathcal{H})_{++}\to\mathcal{L}(\mathcal{H})_{++}$ is a bijection satisfying

$$N(M_{f_1,\ldots,f_n}(\phi(A_1),\ldots,\phi(A_n)))=N(M_{f_1,\ldots,f_n}(A_1,\ldots,A_n)) \tag{4}$$

for all $A_1,\ldots,A_n\in\mathcal{L}(\mathcal{H})_{++}$, then there is a unitary or an antiunitary operator U on $\mathcal{H}$ such that ϕ is of the form

$$\phi(A)=UAU^* \quad (A\in\mathcal{L}(\mathcal{H})_{++}).$$

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