On certain classes of Schur-convex functions

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Definition

Let I be a nonempty interval. Then the function $f : I^n \to \mathbb{R}$ is Schur-convex if

$$f(Sx) \leq f(x)$$

for all doubly stohastic matrix S and for all $x = (x_1, \cdots, x_n) \in I^n$.

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Let D be a nonempty, convex subset of a linear space X. Then, the function $f : D \times D \to \mathbb{R}$ is Schur-convex, if

$$f(tx + (1-t)y, (1-t)x + ty) \leq f(x, y)$$

for all $x, y \in D$, and $t \in [0, 1]$. And if the above inequality stands only for one fixed $t \in]0, 1[$ and f is symmetric, we say that f is t-Schur-convex.

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It is easy to check, that if $f : D \times D \to \mathbb{R}$ is a Schur-convex and

- $f(x, y) = g(x) + g(y)$, then g is a Wright-convex function;
- $f(x, y) = \max\{g(x), g(y)\}$, then g is a quasi-convex.

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Let us define the function $\varphi_{x,y} : [0,1] \to \mathbb{R}$ in the following way

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Theorem

A function $f : D \times D \to \mathbb{R}$ is $\frac{1}{2}$-Schur-convex if and only if the function $\varphi_{x,y}$ has a global minimum at $\frac{1}{2}$ for every fixed $x, y \in D$.

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Let $f : D \times D \to \mathbb{R}$ be a symmetric function. Then f is Schur-convex if and only if for all arbitrarily fixed $x, y \in D$ the function $\varphi_{x,y}$ is monotone decreasing on $[0, \frac{1}{2}]$, monotone increasing on $[\frac{1}{2}, 1]$, and $\varphi_{x,y}$ has a global minimum at $\frac{1}{2}$.

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Remark

Olbryś: showed that the corresponding sufficiency part of the previous theorem is true in the case of Wright-convexity.

Motivation

It is easy to see that the above definition can be considered in the following way: $f : D \times D \to \mathbb{R}$ is t-Schur-convex, if

$$f\left(\begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix} x\right) \leq f(x).$$

for all $x \in D^2$.

Let $t = (t_1, \dots, t_n)$ such that $\sum_{i=1}^n t_i = 1$ and $t_1 \geq t_2 \geq \dots \geq t_n \geq 0$. We say that the matrix T is *t-Schur* matrix or shifted matrix generated by t if it is given in the following way

$$T = \begin{pmatrix} t_1 & t_2 & \dots & t_{n-1} & t_n \\ t_2 & t_3 & \dots & t_n & t_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_n & t_1 & \dots & t_{n-2} & t_{n-1} \end{pmatrix}.$$

General case

Let X be a real vector space, and $D \subset X$ be a nonempty, convex set and T be a $t = (t_1, \cdots, t_n)$-Schur matrix. A function $f : D^n \to \mathbb{R}$ is *T-Schur convex*, if for all $x \in D^n$,

$$f(Tx) \leq f(x).$$

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The standard n-simplex is the subset of $\mathbb{R}^n$ given by

$$S_n = \{(t_1, \ldots, t_n) \mid \sum_{i=1}^{n} t_i = 1, t_i \geq 0, i \in \{1, 2, \ldots, n\}\}$$

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We would like to characterize Schur convexity with the help of the „monotonicity" of the function $\varphi_x \colon S_n \to \mathbb{R}$ $(x \in D)$ defined by

$$\varphi_x(t_1, \dots, t_n) := f(Tx), \qquad (t_1, \dots, t_n) \in S^n$$

where T is the $t = (t_1, \cdots, t_n)$-Schur-matrix.

General case

Let $t, t' \in \mathbb{R}^n$, $t = (t_1, \ldots, t_n)$, $t' = (t'_1, \ldots, t'_n)$. We say that t' is *majorized* by t and write $t' \preceq t$ if

(i) $\sum_{i=1}^n t'_i = \sum_{i=1}^n t_i$;

(ii) $\sum_{i=1}^k t'_{[i]} \leq \sum_{i=1}^k t_{[i]}$, $k = 1, \ldots, n-1$

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Definition

Let $\lambda \in [0,1]$ and Q be a permutation matrix that just interchanges two coordinates. Then, a linear transformation is called a $T-$ transform if the matrix of the transformation has of the form

$$T = \lambda I + (1-\lambda)Q.$$

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Lemma (Muirhead (1903), Hardy, Littlewood, Pólya (1934))

If $x \preceq y$, then x can be derived from y by successive applications of a finite number of T-transform.

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Theorem (Hardy, Littlewood, Pólya (1929))

A necessary and sufficient condition that $x \preceq y$ is that there exist a doubly stochastic matrix P such that $x = yP$.

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Theorem (Hardy, Littlewood, Pólya (1929))

A necessary and sufficient condition that $x \preceq y$ is that there exist a doubly stochastic matrix P such that $x = yP$.

Remark

In general, the matrix P is not unique.

Corollary (Rado, 1952)

The set $\{x : x \preceq y\}$ is the convex hull of points obtained by permuting the components of y.

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$$\varphi_x(t, 1-t) = f\left(\begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix} x\right), \quad t \in \left[\frac{1}{2}, 1\right].$$

has a global minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

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A symmetric function $f\colon D^2 \to \mathbb{R}$ is Schur-convex if and only if for all arbitrarily fixed $x \in D^2$ the function $\varphi_x\colon S_2 \to \mathbb{R}$ has a global minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and it is monotone increasing with respect to $\preceq$.

Theorem

Let $n \in \mathbb{N}$ and N be the $\left(\frac{1}{n}, \cdots, \frac{1}{n}\right)$-Schur matrix. Then the symmetric function $f : D^n \to \mathbb{R}$ is N-Schur convex, if and only if, for all $x \in D^n$ the function $\varphi_x : S_n \to \mathbb{R}$ defined by

$$(1) \qquad \varphi_x(t_1, \ldots, t_n) := f(Tx), \qquad (t_1, \cdots, t_n) \in S^n$$

(where T is the $t = (t_1, \cdots, t_n)$-Schur matrix) has a global mininum at $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ with respect to $\preceq$.

Sketch of the proof

- It is easy to show, that if $t = (t_1, \dots, t_n) \in S_n$ and assume that $t_1 \geq \dots \geq t_n$, then $(t_1, \dots, t_n) \succeq (\frac{1}{n}, \dots \frac{1}{n})$.

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- Let $t = (t_1, t_2, \cdots, t_n) \in S_n$ be arbitrary. If we sort its coordinates in descending order, (i.e. $t_1 \geq \dots \geq t_n$), then have that $(t_1, \dots, t_n) \succeq (\frac{1}{n}, \dots \frac{1}{n})$.

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- Now, assume that f is N-Schur convex, i.e. for all $x \in D^n$

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- Then, we easily have that

$$\varphi_x\left(\frac{1}{n}, \cdots, \frac{1}{n}\right) = f(N(Tx)) \leq f(Tx) = \varphi_x(t_1, \cdots, t_n).$$

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- On the other hand assume that for all $x \in D^n$, φ_x has a global minimum at $(\frac{1}{n}, \cdots, \frac{1}{n})$. Then,

$$f(Nx) = \varphi_x(\frac{1}{n}, \cdots, \frac{1}{n}) \leq \varphi_x(1, 0, \cdots, 0) = f(Ix) = f(x),$$

which proves that f is N-Schur convex.

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