Stability of a functional equation on Banach lattices

Patricia Szokol

joint work with Nutefe Kwami Agbeko

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Introduction

Definition

A vector space $V$ endowed with a partial order is called a Riesz space if for any $x, y, z \in V$ the following properties satisfy:

1. Translation invariance: $x \leq y$ implies $x + z \leq y + z$;
2. Positive homogeneity: For any nonnegative scalar $\alpha$, $x \leq y$ implies $\alpha x \leq \alpha y$;
3. For any pair of vectors $x, y \in V$ there exists a least upper bound (denoted by $x \lor y$) in $V$ with respect to the partial order ($\leq$).

Example 1: $(\mathbb{R}^n, \leq_1)$, where $x \leq_1 y \iff x_i \leq y_i$, for all $i = 1, \ldots, n$.

Example 2: $(\mathbb{R}^2, \leq_2)$, where $x \leq_2 y \iff (x_1 < y_1)$ or $(x_1 = y_1$ and $x_2 \leq y_2)$.

Example 3: $(V, \leq_3)$, where $V$ is the set of all real functions and $f \leq_3 g \iff f(x) \leq g(x)$, $(x \in \mathbb{R})$. 

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Definition

Let $V$ be a Riesz space and $f \in V$. Then,

1. the positive part of $f$: $f^+ = 0 \vee f$;
2. the negative part of $f$: $f^- = (-f) \vee 0$;
3. the modulus of $f$: $|f| = f^+ \vee f^-$. 

Definition

Let $V$ be a Riesz space and $\|\cdot\|$ be a norm. We say, that $(V, \|\cdot\|)$ is a normed Riesz space if $|f| \leq |g|$ implies that $\|f\| \leq \|g\|$.

If a normed Riesz space is complete, then it is called Banach lattice.

Example

Let $X$ be a topological space, $Y$ a Banach lattice and $C(X,Y)$ the space of bounded, continuous functions from $X$ to $Y$ with the supremum norm.

Then, $C(X,Y)$ becomes a Banach lattice with the pointwise order.
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Let $\varepsilon, p, q \in (0, \infty)$ be three numbers, $G_1$ and $G_2$ be two Riesz spaces, $d(\cdot, \cdot)$ be a metric defined on $G_2$, $\Delta^*_G, \Delta^{**}_G \in \{\wedge G, \vee G\}$ and $\Delta^*_G, \Delta^{**}_G \in \{\wedge G, \vee G\}$ be four lattice operations.
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$$d((F((\tau^q|x|)\Delta^*_G(\eta^q|y|))) \Delta^*_G(F((\tau^q|x|)\Delta^{**}_G(\eta^q|y|))), (\tau^pF(|x|)) \Delta^{**}_G(\eta^pF(|y|))) \leq \delta$$

for all $x, y \in G_1$ and all $\tau, \eta \in [0, \infty)$,
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$$d\left((F((\tau^q|x|)\Delta^*_G_1(\eta^q|y|)) \Delta^*_G_2(F((\tau^q|x|)\Delta^{**}_G_1(\eta^q|y|)))\right),$$

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for all $x, y \in G_1$ and all $\tau, \eta \in [0, \infty)$, then there exists an operation-preserving functional $T : G_1 \rightarrow G_2$,
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\]
for all $x, y \in G_1$ and all $\tau, \eta \in [0, \infty)$, then there exists an operation-preserving functional $T : G_1 \rightarrow G_2$, i.e. a functional $T$ such that
\[
(T ((\tau^q|x|) \Delta^*_{G_1}(\eta^q|y|))) \Delta^*_G (T ((\tau^q|x|) \Delta^{**}_{G_1}(\eta^q|y|))) = \\
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$$d\left( (F((\tau^q|x|)\Delta^*_{G_1}(\eta^q|y|))) \Delta^*_G \left( F((\tau^q|x|)\Delta^{**}_{G_1}(\eta^q|y|)) \right) , \right.$$

$$\left. (\tau^p F(|x|)) \Delta^{**}_G \left( \eta^p F(|y|) \right) \right) \leq \delta$$

for all $x, y \in G_1$ and all $\tau, \eta \in [0, \infty)$, then there exists an operation-preserving functional $T : G_1 \to G_2$, i.e. a functional $T$ such that

$$\left( T ((\tau^q |x|) \Delta^*_{G_1} (\eta^q |y|)) \right) \Delta^*_G \left( T ((\tau^q |x|) \Delta^{**}_{G_1} (\eta^q |y|)) \right) =$$

$$\left( \tau^p T (|x|) \right) \Delta^{**}_G \left( \eta^p T (|y|) \right),$$

with the property that

$$d\left( T(x), F(x) \right) \leq \varepsilon$$

for all $x \in G_1$ and all $\tau, \eta \in [0, \infty)$?
Remark

If $T : G_1 \to G_2$ is an operation-preserving functional, then for all $x \in G_1$ and all $\tau \in [0, \infty)$

$$T(\tau^q |x|) = \tau^p T(|x|).$$
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If \( \tau = \eta = 1 \), then the above problem reduces to the problem posed and treated by Nutefe Agbeko Kwami.
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All along $(\mathcal{X}, \wedge, \vee)$ will stand for a normed Riesz space and $(\mathcal{Y}, \wedge, \vee)$ for a Banach lattice with $\mathcal{X}^+$ and $\mathcal{Y}^+$ their respective positive cones.
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Definition

We say, that a functional $H : \mathcal{X} \to \mathcal{Y}$ is cone-related if $H(\mathcal{X}^+) \subset \mathcal{Y}^+$, where $H(\mathcal{X}^+) = \{H(|x|) : x \in \mathcal{X}\}$. 
Theorem

Let \((p, q) \in (0, \infty) \times (0, \infty)\) be a pair of real numbers, and \(F : \mathcal{X} \rightarrow \mathcal{Y}\) a cone-related functional for which there are numbers \(\vartheta > 0\) and \(\alpha\) with \(q\alpha \in (p, \infty)\) such that for all \(x, y \in \mathcal{X}\) and \(\tau, \eta \in [0, \infty)\)

\[
\|\tau^p F((\tau q|x|) \Delta^\ast \mathcal{X}(\eta q|y|)) - (\tau^p F(|x|)) \Delta^\ast \mathcal{Y}(\eta|y|)\| \leq 2(p-1)\vartheta(\|x\|^{\alpha} + \|y\|^{\alpha})
\]

Then the sequence \((2^{np} F(2^{-nq}|x|))_{n \in \mathbb{N}}\) is a Cauchy sequence for every \(x \in \mathcal{X}\).

Let \(T : \mathcal{X}^+ \rightarrow \mathcal{Y}^+\) be a functional, defined by

\[
T(|x|) = \lim_{n \to \infty} 2^{np} F(2^{-nq}|x|), \quad x \in \mathcal{X}.
\]

Then \(T\) both is an operation-preserving functional and satisfies inequality

\[
\|T(|x|) - F(|x|)\| \leq 2p\vartheta 2q^{\alpha} - 2p\|x\|^{\alpha}, \quad x \in \mathcal{X}.
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Theorem

Let \((p, q) \in (0, \infty) \times (0, \infty)\) be a pair of real numbers, and \(F : \mathcal{X} \to \mathcal{Y}\) a cone-related functional for which there are numbers \(\vartheta > 0\) and \(\alpha\) with \(q\alpha \in (p, \infty)\) such that for all \(x, y \in \mathcal{X}\) and \(\tau, \eta \in [0, \infty)\)

\[
\| (F (|\tau^q x|) \Delta^*_\mathcal{X} (|\eta^q y|)) \Delta^*_\mathcal{Y} (F (|\tau^q x|) \Delta^*_\mathcal{X} (|\eta^q y|)) - (\tau^p F (|x|)) \Delta^*_\mathcal{Y} (\eta^p F (|y|)) \| \leq 2^{(p-1)\vartheta} (\|x\|^\alpha + \|y\|^\alpha),
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Then the sequence \(2^{np} F (2^{nq} |x|)\) \(n \in \mathbb{N}\) is a Cauchy sequence for every \(x \in \mathcal{X}\).

Let \(T : \mathcal{X} \Rightarrow \mathcal{Y}\) be a functional, defined by

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\|((F (|\tau^q\, x|)) \Delta^*_x (\eta^q \, |y|)) \Delta^*_y (F (|\tau^q\, x|) \Delta^*_x (\eta^q \, |y|))) - (\tau^p \, F (|x|)) \Delta^*_y (\eta^p \, F (|y|))\| \\
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\[
\| T (|x|) - F (|x|) \| \leq \frac{2^p \vartheta}{2q\alpha - 2p} \|x\|^\alpha, \quad x \in \mathcal{X}.
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Let \((p, q) \in (0, \infty) \times (0, \infty)\), a pair of real numbers and \(F : \mathcal{X} \to \mathcal{Y}\) a cone-related functional for which there are numbers \(\beta \in [0, \infty), \vartheta > 0\) and \(\alpha\) with \(q\alpha \in (0, p)\) such that for all \(x, y \in \mathcal{X}\) and all \(\tau, \eta \in [0, \infty)\)

\[
\|F((\tau q|x|) \ast X(\eta q|y|)) \ast Y(F((\tau q|x|) \ast X(\eta q|y|))) - (\tau p F(|x|)) \ast Y(\eta p F(|y|))\| \\
\leq \beta + \vartheta 2^{-p - 1} + \beta q 2^{q - 2} q^2 \alpha - 2 q^2 \alpha.
\]

Then the sequence \((2 - np F(2 n q|x|))\) is a Cauchy sequence for every fixed \(x \in \mathcal{X}\).

Let \(T : \mathcal{X}^+ \to \mathcal{Y}^+\) be a functional, defined by \(T(|x|) = \lim_{n \to \infty} 2 - np F(2 n q|x|), x \in \mathcal{X}\).

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\[
\left\| (F ((\tau^q |x|) \Delta_{\mathcal{X}}^* (\eta^q |y|))) - (\tau^p F (|x|)) \Delta_{\mathcal{Y}}^* (\eta^p F (|y|)) \right\| \\
\leq \beta + \vartheta 2^{-(p+1)} (\|x\|^\alpha + \|y\|^\alpha).
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\| (F ((\tau^q |x|) \Delta_{\mathcal{X}}^* (\eta^q |y|))) \| \Delta_{\mathcal{Y}}^* (F ((\tau^q |x|) \Delta_{\mathcal{X}}^* (\eta^q |y|)))) - (\tau^p F (|x|)) \Delta_{\mathcal{Y}}^* (\eta^p F (|y|)) \| \\
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Then the sequence \((2^{-np} F (2^{nq} |x|))_{n \in \mathbb{N}}\) is a Cauchy sequence for every fixed \(x \in \mathcal{X}\).
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\[
\begin{align*}
\|(F((\tau^q|x|) \Delta^*_\mathcal{X} (\eta^q |y|))) - (\tau^p F(|x|)) \Delta^*_\mathcal{Y} (\eta^p F(|y|))\| & \leq \beta + \vartheta 2^{-(p+1)} (\|x\|^\alpha + \|y\|^\alpha).
\end{align*}
\]

Then the sequence \((2^{-np} F (2^{nq} |x|))_{n \in \mathbb{N}}\) is a Cauchy sequence for every fixed \(x \in \mathcal{X}\). Let \(T : \mathcal{X}^+ \to \mathcal{Y}^+\) be a functional, defined by

\[
T(|x|) = \lim_{n \to \infty} 2^{-np} F (2^{nq} |x|), \quad x \in \mathcal{X}.
\]
Main results

Theorem

Let \((p, q) \in (0, \infty) \times (0, \infty)\), a pair of real numbers and \(F : \mathcal{X} \to \mathcal{Y}\) a cone-related functional for which there are numbers \(\beta \in [0, \infty), \vartheta > 0\) and \(\alpha\) with \(q\alpha \in (0, p)\) such that for all \(x, y \in \mathcal{X}\) and all \(\tau, \eta \in [0, \infty)\)

\[
\left\| (F (|x|) \Delta^*_\mathcal{X} (\eta^q |y|)) - (\tau^p F (|x|)) \Delta^*_\mathcal{Y} (\eta^p F (|y|)) \right\| \leq \beta + \vartheta 2^{-(p+1)} (\|x\|^\alpha + \|y\|^\alpha).
\]

Then the sequence \((2^{-np} F (2^{nq} |x|))_{n \in \mathbb{N}}\) is a Cauchy sequence for every fixed \(x \in \mathcal{X}\). Let \(T : \mathcal{X}^+ \to \mathcal{Y}^+\) be a functional, defined by

\[
T (|x|) = \lim_{n \to \infty} 2^{-np} F (2^{nq} |x|), \quad x \in \mathcal{X}.
\]

Then \(T\) both is an operation-preserving functional and satisfies inequality

\[
\| T (|x|) - F (|x|) \| \leq \frac{\beta 2^p}{2^p - 1} + \frac{\vartheta \|x\|^\alpha 2^{q\alpha}}{2^p - 2^{q\alpha}}, \quad x \in \mathcal{X}.
\]
Theorem

Let \((X, d)\) be a complete metric space, \(S\) an appropriate set and \(f : S \to X\), \(G : S \to S\), \(H : X \to X\) and \(\delta : S \to [0, \infty)\) functions that satisfy the inequality

\[d(H(f(G(x)))), f(x)) \leq \delta(x), \quad x \in S.\]
Theorem

Let \((X, d)\) be a complete metric space, \(S\) an appropriate set and \(f : S \rightarrow X, G : S \rightarrow S, H : X \rightarrow X\) and \(\delta : S \rightarrow [0, \infty)\) functions that satisfy the inequality

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**Theorem**

Let \((X, d)\) be a complete metric space, \(S\) an appropriate set and \(f : S \rightarrow X\), \(G : S \rightarrow S\), \(H : X \rightarrow X\) and \(\delta : S \rightarrow [0, \infty)\) functions that satisfy the inequality

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\]

If \(H\) is continuous and satisfies the inequality

\[
d(H(u), H(v)) \leq \varphi(d(u, v)), \quad u, v \in X,
\]

for a certain non-decreasing subadditive function \(\varphi : [0, \infty) \rightarrow [0, \infty)\) and for every \(x \in S\) the following series is convergent

\[
\sum_{j=0}^{\infty} \varphi^j(\delta(G^j(x))),
\]
then there exists a unique function $F : S \rightarrow X$ solution of the functional equation

$$H(F(G(x))) = F(x), \quad x \in S,$$

and satisfying the following inequality:

$$d(F(x), f(x)) \leq \sum_{j=0}^{\infty} \phi_j(\delta(G_j(x))).$$

Furthermore, the function $F$ is given by

$$F(x) = \lim_{n \rightarrow \infty} H^n(f(G^n(x))).$$
Theorem

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Fix arbitrarily $\tau, \eta \in (0, 2)$ and consider the function

$$F : [0, \infty) \rightarrow [0, \infty), \quad F(x) = x^{\alpha+1}, \quad \alpha = \frac{p}{q}.$$
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Since $F$ is increasing the first equality in the chain below is valid, entailing the subsequent relations:

$$|F ((\tau^q x) \lor (\eta^q y)) - (\tau^p F(x)) \land (\eta^p F(y))| =$$

$$\left| (\tau^q x)^{\alpha+1} \lor (\eta^q y)^{\alpha+1} - (\tau^p x^{\alpha+1}) \land (\eta^p y^{\alpha+1}) \right|$$

$$\leq (\tau^q x)^{\alpha+1} \lor (\eta^q y)^{\alpha+1} + (\tau^p x^{\alpha+1}) \land (\eta^p y^{\alpha+1})$$

$$\leq (2^q x)^{\alpha+1} \lor (2^q y)^{\alpha+1} + (2^p x^{\alpha+1}) \land (2^p y^{\alpha+1})$$

$$\leq 2^{p+q}(x^{\alpha+1} \lor y^{\alpha+1}) + 2^{p+q}(x^{\alpha+1} \land y^{\alpha+1}) = 2^{p+q} (x^{\alpha+1} + y^{\alpha+1})$$

for all $x, y \in [0, \infty)$. 
We have mentioned (in general) that if $T$ is an operation-preserving functional, i.e.

$$T ((\tau^q x) \lor (\eta^q y)) = (\tau^p T (x)) \land (\eta^p T (y)),$$

then $T (\tau^q x) = \tau^p T (x)$ for all $x \in [0, \infty)$ and all $\tau \in [0, \infty)$. Let $T : [0, \infty) \to [0, \infty)$ be such a function.
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$$\sup_{x \in (0, \infty)} \frac{|F(x) - T(x)|}{2^p + q x^{\alpha+1}} = \sup_{x \in (0, \infty)} \frac{|x^{\alpha+1} - T((x^{1/q})^q)|}{2^p + q x^{\alpha+1}} = \sup_{x \in (0, \infty)} \frac{|x^{\alpha+1} - x^\alpha T(1)|}{2^p + q x^{\alpha+1}} = \frac{1}{2^p + q} \sup_{x \in (0, \infty)} \left| 1 - \frac{T(1)}{x} \right| = \infty.$$


