ISOMETRIES ON POSITIVE DEFINITE OPERATORS WITH UNIT FUGLEDE-KADISON DETERMINANT

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Abstract. In this paper, we explore the structure of certain surjective generalized isometries (which are transformations that leave any given member of a large class of generalized distance measures invariant) of the set of positive invertible elements in a finite von Neumann factor with unit Fuglede-Kadison determinant. We conclude that any such map originates from either an algebra *-isomorphism or an algebra *-antiisomorphism of the underlying operator algebra.

1. Introduction and preliminaries

Throughout the paper, $\mathcal{A}$ denotes a finite von Neumann algebra acting on a complex separable Hilbert space. We shall assume that $\mathcal{A}$ is a *factor*, that is, its center is one dimensional. The symbol $\mathcal{A}_{sa}$ stands for the self-adjoint part of $\mathcal{A}$. The set of positive and positive invertible elements in $\mathcal{A}$ will be denoted by $\mathcal{A}^+$ and $\mathcal{A}^{++}$, respectively. The symbol I will stand for the unit of $\mathcal{A}$. Furthermore, we recall that if S is a set and $d : S \times S \to [0, +\infty[$ is a function satisfying $d(x, y) = 0$ if and only if $x = y$ $(x, y \in S)$, then d is termed a generalized distance measure. Clearly, a generalized distance measure may not be a metric in the usual sense, because we require neither the symmetry nor the triangle inequality.

In [7], Moakher studied in details the manifold of n by n positive definite (PD) matrices with unit determinant. Moreover, in the paper [3], the authors investigated the same structure because of its interesting connections to the space of so-called diffusion tensors. In fact, they also studied the set of all PD matrices with determinant $c > 0$, which is a so-called totally geodesic submanifold of the manifold of PD matrices. Motivated by these facts, in [9], among others, the problem of establishing the complete description of 'generalized isometries' with respect to generalized distance measures of the form

$$d_{N,g}(A, B) = N(g(A^{-1/2}BA^{-1/2})) \tag{1.1}$$

was solved on the set of PD matrices with determinant 1, where the norm N and the function g satisfy some mild assumptions (see the details below).

As a matter of fact, there is an infinite dimensional theory of the determinant. There are several notions of the determinant of operators on infinite dimensional Hilbert spaces, the most natural one being the Fredholm determinant. This is

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an extension of the usual determinant to trace class operators perturbed by the identity. There are several equivalent definitions for the Fredholm determinant $\det T$ of such an operator T. For example, on [11, p. 33], it is mentioned that one of them can be given as follows. Let $(\lambda_n)_{n\in N}$ be a sequence of the nonzero eigenvalues of the compact operator $T - I$ counted according to their algebraic multiplicities. Then the product $\prod_{n\in N}(1+\lambda_n)$ exists, and $\det T$ is given by the formula

$$\det T = \prod_{n\in N}(1+\lambda_n).$$

The Fredholm determinant has the disadvantage that it is defined only for quite small subsets of operator algebras. Another definition of the determinant — which will be given below — does not have this disadvantage.

The theory of the determinant in finite von Neumann factors, which has been developed by Fuglede and Kadison [4], is a bit more involved. Every finite factor admits a unique faithful tracial state τ, by which we mean a positive linear functional $\tau : \mathcal{A} \to \mathbb{C}$ with the following properties: (i) $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{A}$; (ii) $\tau(A^*A) = 0$ if and only if $A = 0$; (iii) $\tau(I) = 1$. Then the associated Fuglede-Kadison determinant $\Delta_{FK} \colon \mathcal{A} \to \mathbb{C}$ is defined as

$$\Delta_{FK}(A) = \exp(\tau(\log |A|)),$$

whenever $A \in \mathcal{A}$ is an invertible element. (Here, the operator $|A|$ is obtained from A^*A by taking square root.) Notable properties of Δ_{FK} are that $\Delta_{FK}(I) = 1$ and Δ_{FK} is positive homogeneous on $\mathcal{A}^{++}$. It is also positive and continuous on the general linear group of $\mathcal{A}$, moreover, by [4, p. 529], it is multiplicative, as well. It follows that for any invertible element $A \in \mathcal{A}$, we have $\Delta_{FK}(A^{-1}) = \Delta_{FK}(A)^{-1}$.

The quantity Δ_{FK} extends naturally the n-th root of the absolute value of the usual determinant, where n is the dimension of the underlying space. Indeed, it is very easy to check that for a complex Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = n < \infty$, one has $\Delta_{FK}(A) = \sqrt[n]{|\det(A)|}$ for all invertible elements A of the finite von Neumann factor of n-by-n complex matrices, because the unique faithful tracial state on this structure is just the canonical (unnormalized) trace divided by n. Consequently, a completely analogous von Neumann algebraic counterpart of the problem in [9] (mentioned at the end of the second paragraph) can be formulated in finite von Neumann factors, and one may ask how the corresponding results in [9] survive when the setting becomes much more general? In this paper, we answer the latter question and, at the same time, we also handle the case $n = 2$, which is completely missing from [9].

In what follows, denote by $\mathcal{A}_1^{++}$ the set of positive invertible elements with unit Fuglede-Kadison determinant. We consider those generalized distance measures on the set $\mathcal{A}_1^{++}$ which admit the form (1.1) and describe the structure of all transformations (called generalized isometries) of $\mathcal{A}_1^{++}$ which preserve one of the above type generalized distance measures. In this paper, the functions N, g satisfy the following properties. The equality $N(|A|) = N(A)$ $(A \in \mathcal{A})$ holds and $N \colon \mathcal{A} \to \mathbb{R}$ is symmetric meaning that

$$N(AXB) \leq \|A\| N(X) \|B\| \quad (A, B, X \in \mathcal{A}).$$

We remark that each symmetric norm is easily seen to be unitarily invariant, furthermore these properties are known to be equivalent in the case where $\mathcal{A}$ is a full matrix algebra (see, e.g. [2, Proposition IV.2.4.]). As for the map $g :]0, +\infty[\to \mathbb{R}$, it is continuous and satisfies

- (g1) $g(y) = 0$ if and only if $y = 1$ ($y \in]0, +\infty[$);
- (g2) there exists a real number $K > 1$ such that $|g(y^2)| \geq K|g(y)|$ holds for every $y \in]0, +\infty[$.

It is obvious that if a continuous function $g :]0, +\infty[\to \mathbb{R}$ has the property (g1), then $d_{N,g}$ is a generalized distance measure on $\mathcal{A}_1^{++}$.

We note that in the case where $\mathcal{A}$ is a full matrix algebra, the set of the above conditions for N, g is equivalent to the collection of those that are required in [9, Theorem 5]. Therefore, it can be seen very easily that our structural result, Theorem 2.1 provides a substantial generalization of the one just cited. Just as in the paper [9], beside the set of positive invertible elements with unit Fuglede-Kadison determinant, we will also consider, for a given number $c > 0$, the collection $\mathcal{A}_c^{++}$ of all operators $A \in \mathcal{A}^{++}$ with $\Delta_{FK}(A) = c$. And we will present the corresponding structural theorem of "generalized isometries".

We remark that the conditions (g1)-(g2) concerning the numerical function g basically come from the requirement that we want to cover some particularly important and widely used distance measures (on the cone of n-by-n positive definite matrices), as demonstrated by the forthcoming examples.

Example 1.1. If $g(y) = \log y$ ($y > 0$) and $N(.) = \|.\|$ is the operator norm (or, in other words, spectral norm), then

$$d_{N,g}(A, B) = \| \log \left(A^{-1/2} B A^{-1/2} \right) \|,$$

which is called the Thompson part metric.

Example 1.2. By the choice $g(y) = \log((y + 1)/(2\sqrt{y}))$ ($y > 0$) and letting $N(.) = \|.\|_1$ to be the trace-norm, we arrive at

$$d_{N,g}(A, B) = \left\| \log \left((1/2)((A^{-1/2} B A^{-1/2})^{1/2} + (A^{-1/2} B A^{-1/2})^{-1/2}) \right) \right\|_1,$$

which is the symmetric Stein-divergence.

For further examples the reader can consult with the paper [9].

2. Main results

Our first result concerning generalized distance measures discussed in the previous section reads as follows.

Theorem 2.1. *Let $\mathcal{A}, \mathcal{B}$ be finite von Neumann factors. Suppose that $N \colon \mathcal{A} \to \mathbb{R}$ and $M \colon \mathcal{B} \to \mathbb{R}$ are complete, symmetric norms such that*

$$N(|A|) = N(A), M(|B|) = M(B), \qquad A \in \mathcal{A},\ B \in \mathcal{B}.$$

Assume that $f, g \colon]0, +\infty[\to \mathbb{R}$ are continuous functions both satisfying (g1)-(g2). Suppose further that $\phi \colon \mathcal{A}_1^{++} \to \mathcal{B}_1^{++}$ is a surjective transformation with the property

$$d_{M,f}(\phi(A), \phi(B)) = d_{N,g}(A, B), \qquad A, B \in \mathcal{A}_1^{++}. \tag{2.1}$$

Then there exists an algebra *-isomorphism or *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$ and an element $T \in \mathcal{B}_1^{++}$ such that ϕ is of one of the following forms:

(a1) $\phi(A) = T\theta(A)T$ *for all* $A \in \mathcal{A}_1^{++}$;
(b1) $\phi(A) = T\theta(A^{-1})T$ *for all* $A \in \mathcal{A}_1^{++}$.

Using the previous theorem, we easily obtain the structure of "generalized isometries" between the spaces $\mathcal{A}_c^{++}$ and $\mathcal{B}_c^{++}$.

Corollary 2.2. *Let $\mathcal{A}, \mathcal{B}$ be finite von Neumann factors and $c > 0$ be a scalar. Suppose that $N\colon \mathcal{A} \to \mathbb{R}$ and $M\colon \mathcal{B} \to \mathbb{R}$ are complete, symmetric norms such that*

$$N(|A|) = N(A), M(|B|) = M(B), \qquad A \in \mathcal{A},\ B \in \mathcal{B}.$$

Assume that $f, g\colon]0, +\infty[\to \mathbb{R}$ are continuous functions both satisfying (g1)-(g2). Suppose further that $\phi\colon \mathcal{A}_c^{++} \to \mathcal{B}_c^{++}$ is a surjective map with the property

$$d_{M,f}(\phi(A), \phi(B)) = d_{N,g}(A, B), \qquad A, B \in \mathcal{A}_c^{++}.$$

Then there exists an algebra *-isomorphism or *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$ and an element $T \in \mathcal{B}_1^{++}$ such that ϕ is of one of the following forms:

(a2) $\phi(A) = T\theta(A)T$ *for all* $A \in \mathcal{A}_c^{++}$;
(b2) $\phi(A) = c^2 T\theta(A^{-1})T$ *for all* $A \in \mathcal{A}_c^{++}$.

3. **Proofs**

The main steps of the arguments below basically follow the ones that appeared in [9], however we must adjust them to the setting of von Neumann algebras. We rely heavily on a general Mazur-Ulam type result which appeared in [8]. To present this result, we need the definition of the so-called point-reflection geometries, which were defined by Manara and Marchi [6].

Definition 3.1. Let $\mathcal{X}$ be a set equipped with a binary operation $\diamond$ which satisfies the following conditions:

(p1) $a \diamond a = a$ for every $a \in \mathcal{X}$;
(p2) $a \diamond (a \diamond b) = b$ for every $a, b \in \mathcal{X}$;
(p3) the equation $x \diamond a = b$ has a unique solution $x \in \mathcal{X}$ for any given $a, b \in \mathcal{X}$.

In this case, the pair $(\mathcal{X}, \diamond)$ (or $\mathcal{X}$ itself) is called a point-reflection geometry.

Now we are in a position to present the mentioned general Mazur-Ulam type result.

Proposition 3.2. [8, Theorem 3] *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are sets equipped with binary operations $\diamond$ and $\star$, respectively, with which they form point-reflection geometries. Let d and ρ be generalized distance measures on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Select $a, b \in \mathcal{X}$, set*

$$L_{a,b} := \{x \in X \ : \ d(a, x) = d(x, b \diamond a) = d(a, b)\},$$

and assume the following:

(b1) $d(b \diamond x, b \diamond x') = d(x', x)$ *holds for every* $x, x' \in \mathcal{X}$;
(b2) $\sup\{d(x, b) \ : \ x \in L_{a,b}\} < \infty$;

(b3) *there is a constant* $K > 1$ *such that*

$$d(x, b \diamond x) \geq Kd(x, b), \qquad x \in L_{a,b};$$

Suppose that $\phi\colon \mathcal{X} \to \mathcal{Y}$ *is a surjective map with the properties that*

$$\rho(\phi(x), \phi(x')) = d(x, x'), \qquad x, x' \in \mathcal{X}$$

and also that

(b4) *the element* $c \in \mathcal{Y}$ *with* $c \star \phi(a) = \phi(b \diamond a)$ *satisfies* $\rho(c \star y, c \star y') = \rho(y', y)$ *for every* $y, y' \in \mathcal{Y}$.

Then necessarily

$$\phi(b \diamond a) = \phi(b) \star \phi(a).$$

Proof of Theorem 2.1. First observe that by the property (2.1) concerning the generalized distance measures $d_{N,g}, d_{M,f}$, the map ϕ is injective. We next prove that ϕ is continuous with respect to the metric induced by the operator norm. Let (X_n) be a sequence in $\mathcal{A}_1^{++}$ such that $X_n \to X \in \mathcal{A}_1^{++}$ $(n \to \infty)$ in this metric. Then $X^{-1/2}X_nX^{-1/2} \to I$, and hence

$$d_{N,g}(X, X_n) = N(g(X^{-1/2}X_nX^{-1/2})) \to N(g(I)) = 0$$

yielding that

$$d_{M,f}(\phi(X), \phi(X_n)) = M(f(\phi(X)^{-1/2}\phi(X_n)\phi(X)^{-1/2})) \to 0.$$

It follows that $f(\phi(X)^{-1/2}\phi(X_n)\phi(X)^{-1/2}) \to 0$ in the operator norm. By the continuity of f and properties (g1)-(g2), it is easy to verify that we necessarily have

$$\phi(X)^{-1/2}\phi(X_n)\phi(X)^{-1/2} \to I,$$

i.e., $\phi(X_n) \to \phi(X)$ in the operator norm and then we obtain the continuity of ϕ with respect to the metric in question.

Next, we are going to apply Proposition 3.2. In order to do so, we must define point-reflection geometries on $\mathcal{A}_1^{++}, \mathcal{B}_1^{++}$ in the following way: $A \diamond B := AB^{-1}A$ for all $A, B \in \mathcal{A}_1^{++}$; and $C \star D := CD^{-1}C$ for every $C, D \in \mathcal{B}_1^{++}$. Indeed, it is easy to check that the properties $(p1) - (p2)$ are satisfied. Concerning $(p3)$, we mention that the well-known Ricatti equation, $XA^{-1}X = B$ has a unique positive invertible solution X for all such elements A, B of a C^*-algebra, which is just the geometric mean of A and B, i.e.

$$X = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

This is called Anderson-Trapp theorem, the original source containing it is [1]. Moreover, X has unit Fuglede-Kadison determinant, if A, B are positive invertible operators in a finite von Neumann factor with $\Delta_{FK}(A) = \Delta_{FK}(B) = 1$. Indeed, X is positive; moreover, taking the Fuglede-Kadison determinant of both sides of the Ricatti equation and using the multiplicativity of that determinant, we get that $(\Delta_{FK}(X))^2 = 1$. Consequently, $\Delta_{FK}(X) = 1$.

Now we are in a position to check that conditions (b1)-(b4) are all satisfied with the choice $\mathcal{X} = \mathcal{A}_1^{++}$, $\mathcal{Y} = \mathcal{B}_1^{++}$ and $d = d_{N,g}$, $\rho = d_{M,f}$. Concerning (b1), we assert that the equalities

$$d_{N,g}(A^{-1}, B^{-1}) = d_{N,g}(B, A), \quad d_{N,g}(TAT^*, TBT^*) = d_{N,g}(A, B) \tag{3.1}$$

hold for all operators $A, B \in \mathcal{A}_1^{++}$ and invertible element $T \in \mathcal{A}$. Indeed, let $A, B \in \mathcal{A}_1^{++}$ and consider the polar decomposition $B^{-1/2}A^{1/2} = U|B^{-1/2}A^{1/2}|$. We see that $|A^{1/2}B^{-1/2}|^2 = U|B^{-1/2}A^{1/2}|^2U^*$ and then compute

$$\begin{aligned} d_{N,g}(A^{-1}, B^{-1}) &= N(g(A^{1/2}B^{-1}A^{1/2})) = N(g(|B^{-1/2}A^{1/2}|^2)) \\ &= N(g(U^*|A^{1/2}B^{-1/2}|^2U)) = N(U^*g(|A^{1/2}B^{-1/2}|^2)U) \\ &= N(g(B^{-1/2}AB^{-1/2})) = d_{N,g}(B, A). \end{aligned}$$

Now, for an arbitrary invertible element $T \in \mathcal{A}$, we deduce

$$\begin{aligned} &((TAT^*)^{-1/2}TBT^*(TAT^*)^{-1/2})^2 \\ &= (TAT^*)^{-1/2}TBT^*(TAT^*)^{-1}TBT^*(TAT^*)^{-1/2}. \end{aligned}$$

For $X = A^{-1/2}BT^*(TAT^*)^{-1/2}$, one has

$$XX^* = A^{-1/2}BA^{-1}BA^{-1/2} = (A^{-1/2}BA^{-1/2})^2.$$

Hence, using the polar decomposition $X = V|X|$, we compute

$$\begin{aligned} &(TAT^*)^{-1/2}TBT^*(TAT^*)^{-1/2} \\ &= ((TAT^*)^{-1/2}TBT^*(TAT^*)^{-1}TBT^*(TAT^*)^{-1/2})^{1/2} \\ &= ((TAT^*)^{-1/2}TBA^{-1}BT^*(TAT^*)^{-1/2})^{1/2} \\ &= (X^*X)^{1/2} = |X| = V^*|X^*|V = V^*(A^{-1/2}BA^{-1/2})V. \end{aligned}$$

It readily follows that $d_{N,g}(TAT^*, TBT^*) = d_{N,g}(A, B)$ holds for any $A, B \in \mathcal{A}_1^{++}$, completing the proof of (3.1). Let us now select two arbitrary elements A, B of $\mathcal{A}_1^{++}$. By (3.1), the condition (b1) is satisfied for the pair A, B.

As for the condition (b2), let us consider the set $\mathcal{H}$ of those elements $X \in \mathcal{A}_1^{++}$ for which we have

$$\begin{aligned} d_{N,g}(A, X) &= N(g(A^{-1/2}XA^{-1/2})) \\ &= N(g(A^{-1/2}BA^{-1/2})) = d_{N,g}(A, B). \end{aligned}$$

(Clearly, $L_{A,B} \subset \mathcal{H}$.) We show that the corresponding set of numbers

$$\begin{aligned} &N(g(X^{-1/2}BX^{-1/2})) = d_{N,g}(X, B) \\ &= d_{N,g}(B^{-1}, X^{-1}) = N(g(B^{1/2}X^{-1}B^{1/2})) \end{aligned}$$

is bounded. Indeed, since $N(g(A^{-1/2}XA^{-1/2}))$ is constant on $\mathcal{H}$ and N is symmetric and hence equivalent to the operator norm $||.||$, the set

$$\{\|g(A^{-1/2}XA^{-1/2})\| : X \in \mathcal{H}\}$$

is bounded. It is easy to see that (g1), (g2) imply

$$\lim_{y\to 0} g(y), \lim_{y\to\infty} g(y) \in \{-\infty, \infty\}.$$

Then it follows easily that there are positive numbers m, M such that $mI \leq A^{-1/2}XA^{-1/2} \leq MI$ holds for all $X \in \mathcal{H}$. Clearly, we then have another pair $m', M' > 0$ of scalars such that $m'I \leq X \leq M'I$ and finally a third one m'', M''

such that $m''I \leq B^{1/2}X^{-1}B^{1/2} \leq M''I$ is satisfied by each $X \in \mathcal{H}$. Since g is continuous, it is bounded on the interval $[m'', M'']$ and this implies that the set

$$\{N(g(B^{1/2}X^{-1}B^{1/2}))\colon X \in \mathcal{H}\}$$

is so. We conclude that the condition (b2) is fulfilled.

To verify that the requirement (b3) holds, we recall that g satisfies (g2), the symmetric norm N is monotone increasing on $\mathcal{A}^+$ by [8, Lemma 12] and $N(|A|) = N(A)$ for every $A \in \mathcal{A}$. Consequently,

$$N(g(C^2)) = N(|g(C^2)|) \geq KN(|g(C)|) = KN(g(C))$$

is satisfied by any $C \in \mathcal{A}_1^{++}$. Now, pick an element $X \in \mathcal{A}_1^{++}$ and let $Y = X^{-1/2}BX^{-1/2}$. Then we easily have that

$$\begin{aligned} d_{N,g}(X, B \diamond X) &= N(g(X^{-1/2}BX^{-1}BX^{-1/2})) = N(g(Y^2)) \\ &\geq KN(g(Y)) = KN(g(X^{-1/2}BX^{-1/2})) = Kd_{N,g}(X,B). \end{aligned}$$

Moreover, using the argument which we have employed to verify that (b1) holds, we obtain that the condition (b4) is also satisfied.

Applying Proposition 3.2, one has that ϕ satisfies

$$\phi(AB^{-1}A) = \phi(A)\phi(B)^{-1}\phi(A), \qquad A, B \in \mathcal{A}_1^{++}.$$

Now consider the transformation $\psi\colon \mathcal{A}_1^{++} \to \mathcal{A}_1^{++}$ defined by

$$\psi(A) = \phi(I)^{-1/2}\phi(A)\phi(I)^{-1/2}, \qquad A \in \mathcal{A}_1^{++}.$$

It is easy to see that ψ is a continuous bijection of $\mathcal{A}_1^{++}$ which satisfies

$$\psi(AB^{-1}A) = \psi(A)\psi(B)^{-1}\psi(A), \qquad A, B \in \mathcal{A}_1^{++}$$

and has the additional property that $\psi(I) = I$. By substituting $A = I$ in the last displayed equality, we obtain that $\psi(B^{-1}) = \psi(B)^{-1}$, which implies that ψ is a Jordan triple isomorphism of $\mathcal{A}_1^{++}$, i.e., a bijective map with the property that

$$\psi(ABA) = \psi(A)\psi(B)\psi(A), \qquad A, B \in \mathcal{A}_1^{++}.$$

Consider the transformation $\tilde{\psi} : \mathcal{A}^{++} \to \mathcal{B}^{++}$ defined by

$$\tilde{\psi}(A) = \Delta_{FK}(A) \cdot \psi\left(\frac{A}{\Delta_{FK}(A)}\right), \qquad A \in \mathcal{A}^{++}.$$

One can check easily that $\tilde{\psi}$ is also a Jordan triple isomorphism which extends ψ. Moreover, applying the continuity property of the Fuglede-Kadison determinant mentioned in the introduction, we deduce that $\tilde{\psi}$ is continuous, as well.

Assume now that the factor $\mathcal{A}$ is not of type I_2. Then we apply [8, Theorem 5] which tells us that there is an algebra *-isomorphism or *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$ and a continuous tracial linear functional $l : \mathcal{A} \to \mathbb{C}$ such that we have either

$$\tilde{\psi}(A) = e^{l(\log A)}\theta(A), \qquad A \in \mathcal{A}^{++}$$

or

$$\tilde{\psi}(A) = e^{l(\log A)}\theta(A^{-1}), \qquad A \in \mathcal{A}^{++}.$$

Furthermore, considering the Jordan decomposition $l = l_1 - l_2$, it is not difficult to check that the positive linear functionals l_1 and l_2 of the finite factor $\mathcal{A}$ are both tracial, too. Therefore, using [5, 8.2.8. Theorem], we infer that l is necessarily a scalar multiple of the unique tracial state τ on $\mathcal{A}$. The conclusion of Theorem 2.1 now follows easily in this case.

Finally, assume that $\mathcal{A}$ is of type I_2. Then $\mathcal{A}$ is isomorphic to the algebra of all 2 by 2 complex matrices. It follows from the statement of [8, Lemma 16] that $\tilde{\psi}(A) = \exp(F(\log A))$ $(A \in \mathcal{A}^{++})$ with some linear commutativity preserving map $F : \mathcal{A}_{sa} \to \mathcal{B}_{sa}$. This yields that $\mathcal{B}$ is also a factor of type I_2 and the result [10, Theorem 2] applies. Now the proof can be completed very easily.

□

Using Theorem 2.1, it is now simple to verify the second main result in the paper.

Sketch of the proof of Corollary 2.2. Consider the transformation $\Phi\colon \mathcal{A}_1^{++} \to \mathcal{B}_1^{++}$ defined by

$$\Phi(A) = \frac{1}{c}\phi(cA), \quad A \in \mathcal{A}_1^{++}.$$

It is easy to check that Φ satisfies (2.1) and it is surjective. It follows that Theorem 2.1 can be applied, which implies that there exists an algebra *-isomorphism or *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$ and an element $T \in \mathcal{B}_1^{++}$ such that Φ is of one of the forms (a1),(b1). Hence, using the definition of Φ, we get the desired forms of ϕ. □

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