CONVEXITY GENERATED BY SPECIAL CIRCULANT MATRICES

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ABSTRACT. In this paper we investigate symmetric T-Schur convex functions, where T is a circulant doubly stochastic matrix generated by a vector. In a particular case we give a complete characterization of that kind of functions.[1]

1. INTRODUCTION

The notion of Schur-convex functions is a widely studied area of convex analysis, which was introduced by Issai Schur, in 1923 [16]. That class of functions is interesting not only from theoretical point of view but the obtained results on convexity are very useful in the area of optimization and application as well ([1, 7, 11]). Moreover, we emphasized, that Schur-convex functions appear not only in the area of mathematics, but they play also an important role in information and communication theory (see e.g. [4, 6] and the references cited there).

To introduce the standard definition of Schur-convex functions we recall that an $n \times n$ matrix is doubly stochastic, if it consists of nonnegative elements and its each rows and columns sums to 1. Then, we say that $f\colon I^n \to \mathbb{R}$ ($I \subset \mathbb{R}$ is an interval) is Schur-convex if

$$f(Sx) \leq f(x),$$

for all doubly stochastic matrix S and $x \in I^n$.

In [2], the first and second authors have considered the two-variables Schur-convex functions in linear spaces. In that case, the definition reads as follows: if D is a nonempty, convex subset of a linear space X, then a function $f\colon D \times D \to \mathbb{R}$ is Schur-convex, if

$$f(tx + (1-t)y, (1-t)x + ty) \leq f(x, y)$$

for all $x, y \in D$, and $t \in [0, 1]$.

If the above inequality stands only for one fixed $t \in]0, 1[$ and f is symmetric, we say that f is t-Schur-convex.

It is easy to check, that the definition of Schur-convexity leads to the ones of Wright-, and quasi-convexity. Indeed, if $f\colon D \times D \to \mathbb{R}$ is a Schur-convex function and

- $f(x, y) = g(x) + g(y)$, then g is a Wright-convex function [17];
- $f(x, y) = \max\{g(x), g(y)\}$, then g is a quasi-convex.

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We emphasize, that Wright-, and quasi-convex functions create also very important and widely studied classes of functions (see e.g [8, 10, 13, 15]).

In [2], the authors introduced the function $\varphi_{x,y} : [0,1] \to \mathbb{R}$, which is defined by

$$\varphi_{x,y}(t) = f(tx + (1-t)y, (1-t)x + ty), \qquad x, y \in D,$$

and they characterized the $\frac{1}{2}$-Schur-convex, and Schur-convex functions with the help of global minima and monotonicity of φ, respectively. The theorems read as follows.

Theorem 1 (Burai, Makó [2])**.** *A function $f\colon D \times D \to \mathbb{R}$ is $\frac{1}{2}$-Schur-convex if and only if the function $\varphi_{x,y}$ has a global minimum at $\frac{1}{2}$ for every fixed $x, y \in D$.*

Theorem 2 (Burai, Makó [2])**.** *Let $f\colon D \times D \to \mathbb{R}$ be a symmetric function. Then f is Schur-convex if and only if for all arbitrarily fixed $x, y \in D$ the function $\varphi_{x,y}$ is monotone decreasing on $[0, \frac{1}{2}]$, monotone increasing on $[\frac{1}{2}, 1]$, and $\varphi_{x,y}$ has a global minimum at $\frac{1}{2}$.*

We remark, that Olbryś showed that the corresponding sufficiency part of the previous theorem is true in the case of Wright-convexity [12].

Our aim is to generalize the two previous theorems. To do this, we need to introduce the definition of T-Schur-convex functions and a notion of order between vectors. We emphasize that until now we have only partial result of the mentioned problems. Namely, we proved a result, which is analogous to Theorem 1. Firstly, we recall the following definition, which was introduced in [5].

Definition 1. Let $t, t' \in \mathbb{R}^n,\ t = (t_1, \dots, t_n),\ t' = (t'_1, \dots, t'_n)$. We say that t' is *majorized* by t and write $t' \preceq t$ if

(i) $\sum_{i=1}^n t_{[i]} = \sum_{i=1}^n t'_{[i]}$,
(ii) $t_1 \geq t_2 \geq \cdots \geq t_n$ and $t'_1 \geq t'_2 \geq \cdots \geq t'_n$
(iii) $\sum_{i=1}^k t_{[i]} \leq \sum_{i=1}^k t'_{[i]},\ k = 1, \dots, n-1$;

In fact, the order $\preceq$ was studied by many mathematicians and there are many interesting and nice results were obtained concerning this notion (see e.g. [5, 9]). We recall only one of them, which was verified by Hardy, Littlewood and Pólya in 1929. It reads as follows. If $x, y \in \mathbb{R}^n$, then $x \preceq y$ if and only if there exists an $n \times n$ doubly stochastic matrix S such that $x = Sy$. Using this result, one can obtain that $f\colon I^n \to \mathbb{R}$ is Schur-convex if

$$f(x) \leq f(y) \Longleftrightarrow x \preceq y.$$

It is easy to see that Theorems appearing in [2] can be formulated applying the introduced order $\preceq$. For the simplicity let S_n denote the standard n-simplex, i.e.

$$S_n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid \sum_{i=1}^n t_i = 1, t_i \geq 0, i \in \{1, 2, \dots, n\}\} \tag{1}$$

Corollary 1. A function $f\colon D \times D \to \mathbb{R}$ is $\left(\frac{1}{2}, \frac{1}{2}\right)$-Schur-convex if and only if for all arbitrarily fixed $x \in D^2$ the function $\varphi_x\colon S_2 \to \mathbb{R}$ defined by

$$\varphi_x(t, 1-t) = f\left(\begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix} x\right), \quad t \in \left[\frac{1}{2}, 1\right].$$

has a global minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Corollary 2. A symmetric function $f\colon D^2 \to \mathbb{R}$ is Schur-convex if and only if for all arbitrarily fixed $x \in D^2$ the function $\varphi_x\colon S_2 \to \mathbb{R}$ has a global minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and it is monotone increasing with respect to $\preceq$.

Now, we turn to the case of n-variables case, where $n > 2$. Firstly, we introduce a special class of doubly stochastic matrices.

Definition 2. We say that a matrix T is *t-Schur* if it is generated by the vector $t = (t_1, \dots, t_n)$ with $\sum_{i=1}^n t_{[i]} = 1$ in the following way,

$$T = \begin{pmatrix} t_1 & t_2 & \dots & t_{n-1} & t_n \\ t_n & t_1 & \dots & t_{n-2} & t_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_2 & t_3 & \dots & t_n & t_1 \end{pmatrix}. \tag{2}$$

We emphasize that t-Schur matrices are a special class of circulant matrices, as well, which are investigated by many mathematicians (see e.g. [3, 14]).

Now, we define Schur-convex functions with respect to a Schur-matrix.

Definition 3. Let X be a real vector space, and $D \subset X$ be a nonempty, convex set. Let T be a fixed Schur matrix. A function $f : D^n \to \mathbb{R}$ is *T-Schur convex,* if for all $x \in D^n$,

$$f(Tx) \leq f(x) \tag{3}$$

In two-dimensional case every doubly stochastic matrix is a circulant one, as well. This easy observation gives the motivation to consider the previous definition of T-Schur-convex functions.

2. Main results

Now, we are in a position to present the generalization of Theorem 1.

Theorem 3. *Let $n \in \mathbb{N}$ and N be the following matrix:*

$$N = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \dots \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots \frac{1}{n} \\ \vdots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \dots \frac{1}{n} \end{pmatrix}.$$

Then the symmetric function $f : D^n \to \mathbb{R}$ is N-Schur convex, if and only if, for all $x \in D^n$ the function $\varphi_x : S_n \to \mathbb{R}$ defined by

$$\varphi_x(t_1, \dots, t_n) := f(Tx), \qquad (t_1, \dots, t_n) \in S^n \tag{4}$$

has a global mininum at $(\frac{1}{n}, \dots, \frac{1}{n})$ with respect to $\preceq$.

Proof: Let $t = (t_1, \dots, t_n) \in S_n$ and assume that $t_1 \geq \dots \geq t_n$. First we prove that $(t_1, \dots, t_n) \succeq (\frac{1}{n}, \dots \frac{1}{n})$. Since $\sum_{i=1}^n t_{[i]} = \sum_{i=1}^n \frac{1}{n} = 1$, we can get that (i) holds. Now, we are going to prove that

$$\sum_{i=1}^k t_i \geq \frac{k}{n}, \quad k \in \{1, \dots, n\}.$$

Let us assume that there exists a $k \in \{1, \dots, n\}$ such that $\sum_{i=1}^{k} t_i < \frac{k}{n}$. Since $t_1 \geq \cdots \geq t_n$, we get that

$$k \cdot t_k \leq \sum_{i=1}^{k} t_i < \frac{k}{n},$$

which implies that $t_k < 1/n$. Hence the inequality $t_i < 1/n$ holds for all $i = k+1, \dots, n$, as well. Consequently,

$$\sum_{i=k+1}^{n} t_i < \frac{n-k}{n}.$$

It yields, that

$$\sum_{i=1}^{n} t_i = \sum_{i=1}^{k} t_i + \sum_{i=k+1}^{n} t_i < \frac{k}{n} + \frac{n-k}{n} = 1,$$

which is a contradiction.

Now assume that f is N-Schur convex, i.e., for all $x \in D^n$,

$$f(Nx) \leq f(x).$$

Let $t = (t_1, \dots, t_n) \in S_n$ be arbitrary. Ordering it the way $t_1 \geq t_2 \geq \cdots \geq t_n$, we have that $(t_1, \dots, t_n) \succeq (\frac{1}{n}, \cdots \frac{1}{n})$ and consider the Schur-matrix T which is indicated the vector t. Substituting in the previous inequality x by Tx, and using the property that $NT = N$ for all Schur-matrix T, we easily get that:

$$\varphi_x\left(\frac{1}{n}, \dots, \frac{1}{n}\right) = f(Nx) = f(N(Tx)) \leq f(Tx) = \varphi_x(t_1, \dots, t_n),$$

which proves our statement.

On the other hand, assume that for all $x \in D^n$, φ_x has a global minimum at $\left(\frac{1}{n}, \cdots, \frac{1}{n}\right)$. Then,

$$f(Nx) = \varphi_x\left(\frac{1}{n}, \cdots, \frac{1}{n}\right) \leq \varphi_x(1, 0, \cdots, 0) = f(Ix) = f(x),$$

which proves that f is N-Schur convex. This completes the proof.

Finally, we would like to mention that our aim is to generalize Theorem 2, which would give a characterization of n-variables ($n > 2$), symmetric Schur-convex functions with the help of the monotonicity of function inroduced in (4).

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