THE MOMENTS OF b-ADDITIVE FUNCTIONS IN CANONICAL NUMBER SYSTEMS

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Abstract. The aim of the present paper is the estimation of the dth moment of additive functions in canonical number systems. These number systems are generalizations of the decimal number system to arbitrary polynomials having integer coefficient. We call a function additive (with respect to a number system) if it only acts on the digits of an expansion. The sum-of-digits function, as a special additive function, has been analyzed in the case of q-adic number systems by Delange and number systems in number fields by Gittenberger and Thuswaldner. The present paper is a generalization of these results to arbitrary additive functions in canonical number systems.

1. Introduction

Let $q \geq 2$ be a positive integer, then we define the sum-of-digits function s_q as follows

$$s_q(z) = \sum_{h=0}^{\ell} a_h \quad \text{for} \quad z = \sum_{h=0}^{\ell} a_h q^h.$$

This function has been studied from different aspects and the first result is due to Delange [7]. This result deals with the arithmetic mean of $s_q(z)$. In particular, Delange was able to show that

$$\frac{1}{N} \sum_{z<N} s_q(z) = \frac{q-1}{2} \log_q N + \Phi(\log_q N),$$

where Φ is a continuous, nowhere differentiable, 1-periodic function. The variance of $s_q(z)$ was computed independently by Kennedy and Cooper [17] and Kirschenhofer [18]. They proved that

$$V_N = \frac{1}{N} \sum_{z<N} s_q^2(z) - \frac{1}{N^2} \left(\sum_{z<N} s_q(z) \right)^2 = \left(\frac{q-1}{2} \right)^2 \log_q N + \Phi(\log_q N).$$

Finally a formula for the d-th moment was established by Grabner, Kirschenhofer, Prodinger and Tichy [12].

Later all these results have been generalized to the case of canonical number systems. Now we briefly summarize the most important results in this direction. For this we need to introduce some further notation. Let $\mathcal{K}$ be a number field of degree n and $\mathbb{Z}_{\mathcal{K}}$ be its ring of integers. Denote by $D_{\mathcal{K}}$ the discriminant of $\mathcal{K}$. Let $b \in \mathbb{Z}_{\mathcal{K}}$ and $\mathcal{N} := \{0, 1, \ldots, |\mathrm{N}(b)| - 1\}$, where $\mathrm{N}(b)$ denotes the norm of b over $\mathbb{Q}$. Then the pair $(b, \mathcal{N})$ is called a canonical number system in $\mathbb{Z}_{\mathcal{K}}$ if each $0 \neq z \in \mathbb{Z}_{\mathcal{K}}$ admits a finite and unique representation of the form

$$z = \sum_{h=0}^{\ell} a_h b^h$$

with $a_h \in \mathcal{N}$ for $0 \leq h \leq \ell$ and $a_\ell \neq 0$ if $\ell \neq 0$. Furthermore we call b the base and $\mathcal{N}$ the set of digits.

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Similarly to the definition above, we define the sum-of-digits function s_b by

$$s_b(z) = \sum_{h=0}^{\ell} a_h \quad \text{for} \quad z = \sum_{h=0}^{\ell} a_h b^h.$$

A characterization for all possible bases together with an algorithm for determining bases was given by Kovács and Pethő [19]. Unfortunately this characterization depends on the integral basis of the field. This algorithm was improved and simplified in some cases by Akiyama and Pethő in [3]. A completely new algorithm for the solutions of this problem was given by Brunotte [5]. Explicit characterizations for some classes of number fields were given by Gilbert [9] and in a series of papers by Kátai, Kovács and Szabó [14–16].

For the Gaussian integers Kátai and Szabó [16] showed that the possible bases b are of the form $b = -u \pm i$ with $u \in \mathbb{N}$. Grabner, Kirschenhofer and Prodinger [11] generalized Delange's result to the Gaussian integers. In particular, they showed that

$$\frac{1}{N} \sum_{|z|^2 < N} s_b(z) = \frac{\pi u^2}{2} \log_{u^2+1} N + \Phi(\log_{u^2+1} N) + \mathcal{O}\left(N^{-\frac{1}{2}} \log N\right),$$

where the sum is extended over Gaussian integers z. In order to generalize this result to arbitrary canonical number systems we have to define the proper area of summation. Therefore we define the Minkowski-embedding $\phi(z)$ of $\mathcal{K}$ into $\mathbb{R}^n$ by

$$\phi(z) := (z^{(1)}, \dots, z^{(s)}, \Re z^{(s+1)}, \Im z^{(s+1)}, \dots, \Re z^{(s+t)}, \Im z^{(s+t)}), \tag{1.1}$$

where $z^{(1)}, \dots, z^{(s)}$ are the real and $z^{(s+1)}, \dots, z^{(s+t)}$ are the complex conjugates of $z \in \mathcal{K}$. We define the set $C(X_1, \dots, X_s, X_{s+1}, \dots, X_{s+t}) \subset \mathbb{R}^n$ as generalization of the area of summation from above. That is, let $C(X_1, \dots, X_s, X_{s+1}, \dots, X_{s+t})$ consist of all vectors

$$(x_1, \dots, x_s, x_{s+1}, y_{s+1}, \dots, x_{s+t}, y_{s+t}) \in \mathbb{R}^n,$$

whose coordinates satisfy

$$\begin{aligned} |x_j| &\le X_j && (1 \le j \le s), \\ x_{s+j}^2 + y_{s+j}^2 &\le X_{s+j} && (1 \le j \le t). \end{aligned}$$

With the help of this set we define

$$M\left(X_1, \dots, X_s, X_{s+1}, \dots, X_{s+t}\right) = \{z \in \mathbb{Z}_{\mathcal{K}} : \phi(z) \in C(X_1, \dots, X_s, X_{s+1}, \dots, X_{s+t})\}.$$

We need a special version of the last set. In particular, we want to restrict the volume of our set by N and put together all those points having a similar (up to a constant) maximum length ℓ of expansion and to have a parameter x at hand to smoothly increase this length between two integer values. Since the complex conjugates arise in pairs having the same norm we have to distinguish two cases according to whether we have a completely real extension ($s = 0$) or not ($s \neq 0$). If $s \neq 0$ then we choose an x with $1 < x < \left|b^{(1)}\right|$ and set $x_1(x) = x$ and

$$\begin{aligned} x_i(x) &= a_i x + c_i; & a_i &= \frac{\left|b^{(i)}\right| - 1}{\left|b^{(1)}\right| - 1}, & c_i &= \frac{\left|b^{(1)}\right| - \left|b^{(i)}\right|}{\left|b^{(1)}\right| - 1} & & (i = 2, \dots, s); \\ x_i(x) &= a_i x + c_i; & a_i &= \frac{\left|b^{(i)}\right|^2 - 1}{\left|b^{(1)}\right| - 1}, & c_i &= \frac{\left|b^{(1)}\right| - \left|b^{(i)}\right|^2}{\left|b^{(1)}\right| - 1} & & (i = s+1, \dots, s+t). \end{aligned}$$

On the other hand, if $s = 0$ then we take an x such that $1 < x < \left|b^{(1)}\right|^2$ and set $x_1(x) = x$ and

$$x_i(x) = a_i x + c_i; \qquad a_i = \frac{\left|b^{(i)}\right|^2 - 1}{\left|b^{(1)}\right|^2 - 1}, \qquad c_i = \frac{\left|b^{(1)}\right|^2 - \left|b^{(i)}\right|^2}{\left|b^{(1)}\right|^2 - 1} \qquad (i = 2, \dots, t).$$

Finally we write for short

$$M(b, \ell, x) := M\left(\left|b^{(1)}\right|^{\ell} x_1(x), \dots, \left|b^{(s+t)}\right|^{\ell} x_{s+t}(x)\right), \tag{1.2}$$

where ℓ is a positive integer.

As we will see below we will relate the parameters N, ℓ and x defining the set $M(b,\ell,x)$ such that its volume increases with N, the elements have similar (up to a constant) maximum length ℓ and x is the parameter responsible for interpolation between two integers.

Thuswaldner [24] generalized the result of Delange [7] on the arithmetic mean to arbitrary canonical number systems. He showed that

$$\frac{1}{N}\sum_{z\in M(b,K,x)} s_b(z) = \frac{2^s\pi^t}{\sqrt{D_{\mathcal{K}}}}\frac{\mathrm{N}(b)-1}{2}\log_{\mathrm{N}(b)} N + \Phi\left(\log_{\mathrm{N}(b)} N\right) + \mathcal{O}\left(N^{-\frac{1}{n}}\log_{\mathrm{N}(b)} N\right).$$

The d-th moment of the sum-of-digits function was considered by Gittenberger and Thuswaldner [10], who could show that

$$\frac{1}{N}\sum_{z\in M(b,K,x)} (s_b(z))^d$$
$$= \frac{2^s\pi^t}{\sqrt{D_{\mathcal{K}}}}\left(\frac{\mathrm{N}(b)-1}{2}\right)^d \log^d_{\mathrm{N}(b)} N + \sum_{j=0}^{d-1}\log^j_{\mathrm{N}(b)} N\Phi_j\left(\log_{\mathrm{N}(b)} N\right) + \mathcal{O}\left(N^{-\frac{1}{n}}\log^d_{\mathrm{N}(b)} N\right).$$

2. Definitions and Results

The objective of this paper is the generalization of the last result by Gittenberger and Thuswaldner [10] in two directions. First we want to consider number systems in a quotient ring of the ring of polynomials over the integers. The second direction is to replace the sum-of-digits function by an arbitrary additive function with respect to a given number system.

To state our results we first have to define the relevant number systems in quotient rings of the ring of polynomials over the integers.

Definition 2.1. Let $p\in\mathbb{Z}[X]$ be monic of degree n and let $\mathcal{N}$ be a subset of $\mathbb{Z}$. The pair $(p,\mathcal{N})$ is called a number system if for every $g\in\mathbb{Z}[X]\setminus\{0\}$ there exist unique $\ell\in\mathbb{N}$ and $a_h(g)\in\mathcal{N}, h=0,\dots,\ell; a_\ell(g)\neq 0$ such that

$$g\equiv\sum_{h=0}^{\ell} a_h(g)X^h \pmod{p}. \tag{2.1}$$

In this case the integers $a_h(g)$ are called the digits and $\ell=\ell(a)$ the length of the representation.

This concept was introduced in [22] and was studied among others in [1,2,19,20]. It was proved in [2], that $\mathcal{N}$ must be a complete residue system modulo $p(0)$ including 0 and the zeroes of p are lying outside or on the unit circle. However, following the argument of the proof of Theorem 6.1 of [22], which deals with the case p square free, one can prove that non of the zeroes of p are lying on the unit circle (*cf.* [23]).

If p is irreducible then we may replace X by one of the roots β of p. Then we are in the case of $\mathbb{Z}[X]/(p)\cong\mathbb{Z}[\beta]$ being an integral domain in an algebraic number field (*cf.* Section 1). Then we may also denote the number system by the pair$(\beta,\mathcal{N})$ instead of $(p,\mathcal{N})$. For example, let $q\geq 2$ be a positive integer. Then $(p,\mathcal{N})$ with $p=X-q$ gives a number system in $\mathbb{Z}$, which corresponds to the number system $(q,\mathcal{N})$. Furthermore for u a positive integer and $p=X^2+2u\,X+(u^2+1)$ we get number systems in $\mathbb{Z}[i]$.

Now we would like to consider these more general number systems and consider additive functions within them. These functions were introduced by Gel′fond [8] and studied among others by Delange [6] and Kátai [13].

Definition 2.2. Let $(p,\mathcal{N})$ be a number system and g as in (2.1). A function $f:\mathbb{Z}[X]\to\mathbb{R}$ is called *additive* if $f(0)=0$ and

$$f(g)=\sum_{h=0}^{\ell} f(a_h(g)X^h).$$

Furthermore we call a function f *strictly additive* if $f(0)=0$ and the function value is independent of the positions of the digits, *i.e*,

$$f(g)=\sum_{h=0}^{\ell} f(a_h(g)).$$

Clearly the sum-of-digits function s_p is a special case of a strictly additive function with

$$s_p(g)=\sum_{h=0}^{\ell} a_h(g),$$

where again g has a representation as in (2.1).

After defining the analogues of number systems and additive functions in these number systems, we need a generalization of the set $M(X_1,\dots,X_{s+t})$ from above. Therefore we take a closer look at the structure of $\mathbb{Z}[X]/(p)$ and start by factoring p by

$$p:=\prod_{i=1}^{r} p_i^{m_i}$$

with $p_i\in\mathbb{Z}[X]$ irreducible and $\deg p_i=n_i$. Then we define by

$$\mathcal{R}:=\mathbb{Z}[X]/(p)=\bigoplus_{i=1}^{r}\mathcal{R}_i \quad\text{with}\quad \mathcal{R}_i=\mathbb{Z}[X]/\left(p_i^{m_i}\right)$$

for $i=1,\dots,r$ the $\mathbb{Z}$-module under consideration and in the same manner by

$$\mathcal{K}:=\mathbb{Q}[X]/(p)=\bigoplus_{i=1}^{r}\mathcal{K}_i \quad\text{with}\quad \mathcal{K}_i=\mathbb{Q}[X]/\left(p_i^{m_i}\right)$$

for $i=1,\dots,r$ the corresponding vector space. Finally we denote by $\overline{\mathcal{K}}$ the completion of $\mathcal{K}$ according to the usual Euclidean distance.

In order to properly state our results we need a bounded area similar to $M(b,k,x)$ in (1.2), which is also compatible with the length of expansion. To this end we denote by β_{ik} the roots of p_i for $i=1,\dots,r$ and $k=1,\dots,n_i$. We may assume that these roots are ordered such that for (s_i,t_i) being the index of p_i (*i.e.*, s_i being the number of real roots and t_i being the number of pairs of complex roots, respectively) we have that $\beta_{i1},\dots,\beta_{is_i}$ are the real roots and $(\beta_{i,s_i+1},\beta_{i,s_i+t_i+1}),\dots,(\beta_{i,s_i+t_i},\beta_{i,s_i+2t_i})$ are the pairs of complex roots of p_i.

Now we define the parameters which help us bounding the length of the expansion of an element $g\in\mathcal{R}$. For this purpose let $g\in\mathbb{Z}[X]$ be a polynomial, and put

$$B_{ijk}(g):=\left.\frac{\mathrm{d}^{j-1}g}{\mathrm{d}X^{j-1}}\right|_{X=\beta_{ik}} \qquad (i=1,\dots,r;j=1,\dots,m_i;k=1,\dots,s_i+t_i).$$

Then the following proposition connects the length of the expansion of $g\in\mathbb{Z}[X]$ with the values $B_{ijk}(g)$ defined above.

Proposition 2.3 ([21])**.** *Assume that $(p,\mathcal{N})$ is a number system. Let $N=\max\{|a|:a\in\mathcal{N}\}$ and set*

$$M(g):=\max\left\{\frac{\log|B_{ijk}(g)|}{\log|\beta_{ik}|}:i=1,\dots,r;j=1,\dots,m_i;k=1,\dots,n_i\right\}.$$

If $g\in\mathbb{Z}[X]$ is of degree at most $n-1$, then there exists a constant $c>0$ and for any $\varepsilon>0$ there exists $L=L(\varepsilon)$ such that if $\ell(g)>L$ then

$$|\ell(g)-M(g)|\leq c. \tag{2.2}$$

After providing a bound for the length of expansion we want to generalize the above mentioned definition of $\mathcal{M}(b,\ell,x)$ and therefore the set C to this new situation. Therefore we need a generalization of the Minkowski embedding. Since it is more convenient to start at the bottom level we define ϕ_{ij} for $i=1,\dots,r$ and $j=1,\dots,m_i$ by

$$\phi_{ij}(z)=\left(B_{i,j,1}(z),\dots,B_{i,j,s_i}(z),\Re B_{i,j,s_i+1}(z),\Im B_{i,j,s_i+1}(z),\dots,\Re B_{i,j,s_i+t_i}(z),\Im B_{i,j,s_i+t_i}(z)\right).$$

Then we combine them to get ϕ_i for $i = 1, \dots, r$ and ϕ, *i.e.*

$$\phi_i(z) = (\phi_{i1}(z), \dots, \phi_{im_i}(z))$$

and

$$\phi(z) = (\phi_1(z), \dots, \phi_t(z)). \tag{2.3}$$

We note that for the case of $\mathcal{K}$ being a separable algebraic number field, *i.e.* $m_i = 1$ for $i = 1, \dots, r$, the definition of ϕ coincides with the one in (1.1).

A central step in the proof will be the switch from a sum over elements in the lattice $\mathcal{R}$ to an integral over a bounded area $\mathcal{C}(X_1, \dots, X_n) \subset \mathbb{R}^n$. In the same manner as in Section 1 we split vectors in $\mathbb{R}^n$ up into its components according to the embeddings ϕ_i and ϕ_{ij}. In particular, for fixed $\mathbf{x} \in \mathbb{R}^n$ we write

$$\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_r) = (\mathbf{x}_{11}, \dots, \mathbf{x}_{rm_r})$$

and

$$\mathbf{x}_{ij} = (x_{ij1}, \dots, x_{i,j,s_i}, x_{i,j,s_i+1}, y_{i,j,s_i+1}, \dots, x_{i,j,s_i+t_i}, y_{i,j,s_i+t_i})$$

where $\mathbf{x}_i \in \mathbb{R}^{m_i n_i}$, $\mathbf{x}_{ij} \in \mathbb{R}^{n_i}$, and $x_{ijk}, y_{ijk} \in \mathbb{R}$ respectively.

We shall use lattice theory in $\mathbb{R}^n$. Therefore we define the bounded area $\mathcal{C} \subset \mathbb{R}^n$ and use our projections and embeddings to gain the "bounded area" in $\mathcal{R}$. Thus for X_{ijk} with $i = 1, \dots, r$, $j = 1, \dots, m_i$, $k = 1, \dots, n_i$ let $\mathcal{C}(X_{111}, \dots, X_{r,m_r,n_r})$ be the set of points $\mathbf{x} \in \mathbb{R}^n$ such that

$$\begin{aligned} |x_{ijk}| &\le X_{ijk} \quad (k = 1, \dots, s_i), \\ x_{ijk}^2 + y_{ijk}^2 &\le X_{ijk}^2 \quad (k = s_i + 1, \dots, s_i + t_i). \end{aligned}$$

Then $\mathcal{M}(X_{111}, \dots, X_{r,m_r,n_r})$ is defined by

$$\mathcal{M}(X_{111}, \dots, X_{r,m_r,n_r}) := \{z \in \mathcal{R} : \phi(z) \in \mathcal{C}(X_{111}, \dots, X_{r,m_r,n_r})\}.$$

Now we have to guarantee that the area $\mathcal{C}$ grows smoothly with respect to the length of expansions of the corresponding elements in $\mathcal{M}$. Therefore we set for $0 < x < 1$

$$\begin{aligned} x_{ik}(x) &= (|\beta_{ik}| - 1)x + 1 \quad (i = 1, \dots, r, k = 1, \dots, s_i), \\ x_{ik}(x) &= (|\beta_{ik}|^2 - 1)x + 1 \quad (i = 1, \dots, r, k = s_i + 1, \dots, s_i + t_i). \end{aligned}$$

Note that since $|\beta_{ik}| > 1$ by [2], we have that $x_{ik}(x) \ge 0$ for $x \ge 0$. Finally we fix a positive integer ℓ and set

$$X_{ijk} = |\beta_{ik}|^\ell x_{ik}(x)$$

for $i = 1, \dots, r$, $j = 1, \dots, m_i$ and $k = 1, \dots, n_i$ and write for short

$$\mathcal{M}(p, \ell, x) := \mathcal{M}(X_{111}, \dots, X_{r,m_r,n_r}). \tag{2.4}$$

Our main result is the following

Theorem 2.4. *Let $(p, \mathcal{N})$ be a number system and $M = |p(0)|$. Furthermore let f be a strictly additive function in $(p, \mathcal{N})$ and μ_f be the mean of the values of f, i.e.,*

$$\mu_f := \frac{1}{|\mathcal{N}|} \sum_{a \in \mathcal{N}} f(a).$$

If we set

$$N = M^\ell \prod_{i=1}^{r} \prod_{k=1}^{s_i+t_i} (x_{ik}(x))^{m_i},$$

then we have

$$\frac{1}{N} \sum_{z \in \mathcal{M}(p,\ell,x)} (f(z))^d = c(p)\mu_f^d \log_M^d(N) + \sum_{j=0}^{d-1} \log_M^j(N) \Phi_j(\log_M N) + \mathcal{O}\left(N^{-\frac{1}{n}} \log_M^d N\right),$$

where $c(p)$ is a constant depending only on the ring $\mathcal{R}$ and thus on p and $\Phi_0, \dots, \Phi_{d-1}$ are continuous periodic functions of period 1.

We note that this theorem reduces to the results of Thuswaldner [24] and Gittenberger and Thuswaldner [10] by setting p accordingly. Therefore it can be seen as a direct generalization of these results.

Remark 2.5. We could generalize this result further, to additive function. However, the statement would be rather technical, so we do not give it here.

3. Proof of Theorem 2.4

In the present proof we want to apply Delange's method (*cf.* [7]) and thus follow the ideas in [11] and [24].We start with the definition of the fundamental domain as

$$\mathcal{F} := \left\{ z \in \mathcal{K} \middle| z = \sum_{h \geq 1} a_h X^{-h}, a_h \in \mathcal{N} \right\}.$$

It can be easily seen (*cf.* [4]) that $\mathcal{F}$ is compact.

As it is shown in Proposition 2.3 the length of expansion is uniformly bounded. Thus let $X_{ijk} = |\beta_{ik}|$ for $i = 1, \ldots, r$, $j = 1, \ldots, m_i$ and $k = 1, \ldots, n_i$. Then let $L := \max_z\{\ell(z)\}$, where the maximum is taken over all $z \in \mathcal{R}$ with $(z + \mathcal{F}) \cap \mathcal{M}(X_{111}, \ldots, X_{r,m_r,s_r+t_r}) \neq \emptyset$. Furthermore let $\mathcal{F}_\ell$ be the set of elements having at most ℓ digits in their fractional part, *i.e.*,

$$\mathcal{F}_\ell := \left\{ z \in \mathcal{K} \middle| z = \sum_{h=-L}^{\ell} a_h(z) X^{-h}, a_h \in \mathcal{N} \right\}.$$

Clearly by the definition of L we get that all elements of $\mathcal{M}(X_{111}, \ldots, X_{r,m_r,s_r+t_r})$ having at most ℓ digits in their fractional part are contained in $\mathcal{F}_\ell$. Let us define by $S_d(N)$ the sum we want to estimate, *i.e.*,

$$S_d(N) = S_d(p, \ell, x) = \sum_{z \in \mathcal{M}(p,\ell,x)} (f(z))^d. \tag{3.1}$$

Using our definition of $\mathcal{F}_K$ we can shift the "decimal dot" and rewrite the sum $S_d(p, \ell, x)$ as

$$S_d(N) = \sum_{h_1,\ldots,h_d=-L}^{\ell} \sum_{z \in \mathcal{M}(p,0,x) \cap \mathcal{F}_\ell} f(a_{h_1}(z)) \cdots f(a_{h_d}(z)).$$

Now we use Delange's method together with the definition of our embedding ϕ defined in (2.3) to rewrite the sum into an integral. Thus

$$S_d(N) = c(p) M^\ell \sum_{h_1,\ldots,h_d=-L}^{\ell} \int_{\mathcal{M}(p,0,x)} f(a_{h_1}(\phi^{-1}(z))) \cdots f(a_{h_d}(\phi^{-1}(z))) \mathrm{d}\lambda_d(z) + \mathcal{O}\left(\ell^d M^{\ell \frac{n-1}{n}}\right),$$

where λ_d denotes the d-dimensional Lebesgue measure. Noting that the functions $a_h(\phi^{-1}(z))$ are constant for every piece of the tiling of $\mathbb{R}^n$ induced by the translates of $\phi(X^{-\ell}\mathcal{F})$ we get that the only difference of the sum and the integral are caused by the elements intersecting the boundary of $\mathcal{M}(p, 0, x)$, whose number is $\mathcal{O}\left(M^{\ell \frac{n-1}{n}}\right)$. Since the product in the integrand is bounded and we have $\mathcal{O}(\ell^d)$ summands, the order of magnitude for the error term is $\mathcal{O}\left(\ell^d M^{\ell \frac{n-1}{n}}\right)$. The factor $c(p)M^\ell$ is due to the volume of the fundamental domain of the lattice induced by the elements of $\mathcal{F}_\ell$.

In the next step we want replace $f(a_h(\phi^{-1}(z)))$ by its mean μ_f. This centralization will help us separating the terms belonging to the periodic fluctuation from those not belonging to it. In particular, we set $L_h(z) = f(a_h(\phi^{-1}(z))) - \mu_f$ and get

$$\begin{aligned} S_d(N) &= c(p)M^\ell \sum_{h_1,\dots,h_d=-L}^{\ell} \int_{\mathcal{M}(p,0,x)} \prod_{i=1}^{d} (L_{h_i}(z) + \mu_f)\,\mathrm{d}\lambda_d(z) + \mathcal{O}\left(\ell^d M^{\ell\frac{n-1}{n}}\right) \\ &= c(p)M^\ell \sum_{h_1,\dots,h_d=-L}^{\ell} \int_{\mathcal{M}(p,0,x)} \sum_{i=0}^{d} \mu_f^{d-i} \tau_i(L_{h_1}(z),\dots,L_{h_d}(z))\mathrm{d}\lambda_d(z) + \mathcal{O}\left(\ell^d M^{\ell\frac{n-1}{n}}\right), \end{aligned}$$

where τ_i denotes the ith elementary symmetric function.

Now we interchange summation and integration and separate the summand corresponding to $i = 0$. This will become our main term, whereas we will transform the rest to get the fluctuating part. Thus we get

(3.2)
$$\begin{aligned} S_d(N) &= c(p)M^\ell \sum_{h_1,\dots,h_d=-L}^{\ell} \int_{\mathcal{M}(p,0,x)} \mu_f^d \mathrm{d}\lambda_d(z) \\ &\quad + c(p)M^\ell \sum_{i=1}^{d} \mu_f^{d-i} \sum_{h_1,\dots,h_d=-L}^{\ell} \int_{\mathcal{M}(p,0,x)} \tau_i\left(L_{h_1}(z),\dots,L_{h_d}(z)\right)\mathrm{d}\lambda_d(z) \\ &\quad + \mathcal{O}\left(\ell^d M^{\ell\frac{n-1}{n}}\right) \\ &= c(p)M^\ell \mu_f^d (L+\ell+1)^d \lambda_d(\mathcal{M}(p,0,x)) \\ &\quad + c(p)M^\ell \sum_{i=1}^{d} \mu_f^{d-i} \binom{d}{i} (L+\ell+1)^{d-i} \sum_{h_1,\dots,h_i=-L}^{\ell} \int_{\mathcal{M}(p,0,x)} L_{h_1}(z)\cdots L_{h_i}(z)\mathrm{d}\lambda_d(z) \\ &\quad + \mathcal{O}\left(\ell^d M^{\ell\frac{n-1}{n}}\right). \end{aligned}$$

We focus on the integral. Since f is completely additive we note that the integrals over $L_{h_1}\dots L_{h_i}$ only depend on the number of factors and how many of the numbers $h_1,\dots,h_d$ are pairwise equal. Thus the integrand has the shape $L_{h_1}^{w_1}(z)\cdots L_{h_j}^{w_j}(z)$ for some $w_1+\cdots+w_j = i$. Then the inner sum can be rewritten as

$$\sum_{j=1}^{i} \sum_{w_1+\cdots+w_j=i} \sum_{h_1,\dots,h_j=-L}^{\ell} \int_{\mathcal{M}(p,0,x)} L_{h_1}^{w_1}(z)\cdots L_{h_j}^{w_j}(z)\mathrm{d}\lambda_d(z), \tag{3.3}$$

where the innermost sum runs over all j-tuples of pairwise non-equal numbers $h_1,\dots,h_j$.

In the next step we want to replace L_h^w by its expectation $Q(w)$. We note that $Q(w)$ does not depend on h, since f is completely additive, and is zero for $w \equiv 1 \pmod 2$. Then for $\eta = \max h_j - 1$ and all $\xi \in \mathcal{F}_\eta$ the integral

$$\int_{\overline{\phi(\xi+X^{-\eta}\mathcal{F})}} \left(L_{h_1}^{w_1}(z) - Q(w_1)\right)\cdots\left(L_{h_j}^{w_j}(z) - Q(w_j)\right)\mathrm{d}\lambda_d(z) = 0, \tag{3.4}$$

since the mean of the term with index $\eta+1$ is zero while all other factors are constant. Hence,

$$\int_{\mathcal{M}(p,0,x)} \left(L_{h_1}^{w_1}(z) - Q(w_1)\right)\cdots\left(L_{h_j}^{w_j}(z) - Q(w_j)\right)\mathrm{d}\lambda_d(z) = \mathcal{O}\left(M^{-\frac{\max_r h_r}{n}}\right), \tag{3.5}$$

since by (3.4) the only part, which contributes to the integral comes from those fundamental domains intersecting the boundary of $\mathcal{M}(p,0,x)$.

Now the idea is to split the integral in (3.3) to get representations in terms of the form (3.5). One of these splitting steps is

$$\begin{aligned}\sum_{h_1,\ldots,h_j=-L}^{\ell}&\int_{\mathcal{M}(p,0,x)} L_{h_1}^{w_1}(z),\ldots,L_{h_j}^{w_j}(z)\mathrm{d}\lambda_d(z)\\ &=\sum_{h_1,\ldots,h_j=-L}^{\ell}\int_{\mathcal{M}(p,0,x)}\left(L_{h_1}^{w_1}(z)-Q(w_1)\right)L_{h_2}^{w_2}(z)\cdots L_{h_j}^{w_j}(z)\mathrm{d}\lambda_d(z)\\ &\quad+\sum_{h_1,\ldots,h_j=-L}^{\ell}Q(w_1)\int_{\mathcal{M}(p,0,x)}L_{h_2}^{w_2}(z)\cdots L_{h_j}^{w_j}(z)\mathrm{d}\lambda_d(z).\end{aligned}$$

Now if we continue this step we get expressions of the form

$$\sum_{h_1,\ldots,h_j=-L}^{\ell}Q(w_1)\cdots Q(w_a)\int_{\mathcal{M}(p,0,x)}\left(L_{h_{a+1}}^{w_{a+1}}-Q(w_{a+1})\right)\cdots\left(L_{h_j}^{w_j}-Q(w_j)\right)\mathrm{d}\lambda_d(z),$$

where $1\leq a<j\leq i$. Since $Q(w)$ is zero for $w\equiv 1 \pmod 2$ it suffices to consider those cases, where $w_1,\ldots,w_a$ are all even and not less than 2. Furthermore $w_1+\cdots+w_a\leq i$ implies that $a\leq\frac{i}{2}$. Since the summands only depend on $h_{a+1},\ldots,h_j$ we substitute it into (3.2) and obtain

$$\begin{aligned}&\tilde{c}(p)M^{\ell}(L+\ell+1)^{a+d-i}Q(w_1)\cdots Q(w_a)\\ &\qquad\times\sum_{h_{a+1},\ldots,h_j=-L}^{\ell}\int_{\mathcal{M}(p,0,x)}\left(L_{h_{a+1}}^{w_{a+1}}-Q(w_{a+1})\right)\cdots\left(L_{h_j}^{w_j}-Q(w_j)\right)\mathrm{d}\lambda_d(z).\end{aligned}\tag{3.6}$$

The summation over the integral will provide us with the fluctuating function. Since the integral is bounded by (3.5), we may let ℓ tend to infinity in order to get a more general periodic function. Thus replacing the corresponding part in (3.6) by

$$\Psi(x)=\sum_{h_{a+1},\ldots,h_j=-L}^{\infty}\int_{\mathcal{M}(p,0,x)}\left(L_{h_{a+1}}^{w_{a+1}}-Q(w_{a+1})\right)\cdots\left(L_{h_j}^{w_j}-Q(w_j)\right)\mathrm{d}\lambda_d(z)$$

causes an error of order $\mathcal{O}\left(\ell^d M^{\ell/n}\right)$, which disappears in the error term of (3.2).

We recall that N should be related to the volume, ℓ to the length and x to the interpolation between ℓ and $\ell+1$. Thus we need a map that associates every N a pair ℓ and x with $0<x<1$. Since

$$1\leq\prod_{i=1}^{r}\prod_{k=1}^{s_i+t_i}(x_{ik}(x))^{m_i}\leq\prod_{i=1}^{r}\prod_{k=1}^{s_i+t_i}|\beta_{ik}|^{m_i}=|p(0)|=M$$

we denote y as in [24] by

$$y=\prod_{i=1}^{r}\prod_{k=1}^{s_i+t_i}(x_{ik}(x))^{m_i}=M^{\{\log_M N\}},$$

where $\{x\}$ denotes the fractional part of $x\in\mathbb{R}$. Then $y=P(x)$ is a polynomial consisting of positive and strictly monotone factors. Hence $P(x)$ is positive and strictly monotone in $[0,1]$ and thus invertible. By our definition of y we have

$$P^{-1}(M^{\{\log_M N\}})=x.$$

We define a new function δ as

$$\delta(x)=M^{-\{x\}}\Psi(P^{-1}(M^{\{x\}})),$$

which is obviously continuous and periodic with period 1. Since $\log_M N = [\log_M N] + \{\log_M N\}$ and $\ell = [\log_M N]$ we get

$$\begin{aligned}(L+\ell+1)^{a+d-i} &= (\log_M N - \{\log_M N\} + L + 1)^{a+d-i}\\ &= \sum_{j=0}^{a+d-i}\binom{a+d-i}{j}\log_M^j N(L+1-\{\log_M N\})^{a+d-i-j}.\end{aligned}\tag{3.7}$$

Plugging this in (3.6) yields

$$\begin{aligned}&c(p)Q(w_1)\cdots Q(w_a)N\sum_{j=0}^{a+d-i}\binom{a+d-i}{j}\log_M^j N(L+1-\{\log_M N\})^{a+d-i-j}\delta(\log_M N)\\ &\quad+\mathcal{O}(N^{\frac{n-1}{n}}\log_M^d N)\\ &= N\sum_{j=0}^{a+d-i}\log_M^j N\delta_j(\log_M N)+\mathcal{O}(N^{\frac{n-1}{n}}\log_M^d N),\end{aligned}$$

where we set $\delta_j(x) = c(p)Q(w_1)\cdots Q(w_a)\binom{a+d-i}{j}(L+1-\{x\})^{a+d-i-j}\delta(x)$ for $j = 0,\dots,a+d-i$. Noting that there are only finitely many summands of this kind we conclude that the contribution to $S_d(p,\ell,x)$ coming from the terms in the second line of (3.2) has the form

$$N\sum_{j=0}^{d-1}\log_M^j N\tilde{\Phi}_j(\log_M N),$$

where the $\tilde{\Phi}_j$ are finite sums of periodic functions of period 1 and hence periodic functions of period 1, too. Thus it remains to investigate the term corresponding to $i = 0$ in (3.2). Applying (3.7) again we get

$$c(p)\mu_f^d M^\ell\prod_{i=1}^{r}\prod_{k=1}^{s_i+t_i}(x_{ik}(x))^{m_i}(L+\ell+1)^d = c(p)\mu_f^d N\log_M^d N + N\sum_{j=0}^{d-1}\log_M^j N\overline{\Phi}_j(\log_M N),$$

where $\overline{\Phi}_j$ are periodic functions of period 1. Setting $\Phi_j(x) = \overline{\Phi}_j + \tilde{\Phi}_j$ for $j = 0,\dots,d-1$ we derive

$$\begin{aligned}S(N) &= \sum_{z\in\mathcal{S}(p,\ell,x)}(f(z))^d\\ &= c(p)\mu_f^d N\log_M^d N + \sum_{j=0}^{d-1}N\log_M^j N\Phi_j(\log_M N)+\mathcal{O}\left(N^{\frac{n-1}{n}}\log_M^d N\right)\end{aligned}$$

and the theorem is proved.

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