UPPER BOUND ON THE SOLUTION TO $F_n^{(2\kappa)} = \pm F_m^{(2\kappa)}$ WITH NEGATIVE SUBSCRIPTS

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ABSTRACT. In this paper, we provide an explicit upper bound on the absolute value of the solutions $n < m < 0$ to the Diophantine equation $F_n^{(k)} = \pm F_m^{(k)}$, assuming k is even. Here $\{F_n^{(k)}\}_{n\in\mathbb{Z}}$ denotes the k-generalized Fibonacci sequence. The upper bound depends only on the term of k.

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1. Introduction

Let $k \geq 2$ be a positive integer. The k-generalized Fibonacci sequence $\{F_n^{(k)}\}_{n\in\mathbb{Z}}$ is defined by

$$F_{-k+2}^{(k)} = \cdots = F_0^{(k)} = 0,\ F_1^{(k)} = 1, \tag{1}$$

and by

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \qquad \text{for all } n \in \mathbb{Z}.$$

The case $k = 2$ gives the Fibonacci sequence. There exist several results in the literature connected to the sequence $\{F_n^{(k)}\}$, but less deal with problems when negative subscripts are considered. To construct the sequence in reverse direction using, for example (1) as initial values, one can apply the recurrence relation

$$F_{-t}^{(k)} = -F_{-t+1}^{(k)} - \cdots - F_{-t+k-1}^{(k)} + F_{-t-k}^{(k)},$$

where the index $-t$ emphasizes its negativity ($t \in \mathbb{Z}^+$, $t \geq k-1$). The main problem with negative subscripts is that while the characteristic polynomial $T_k(x) = x^k - x^{k-1} - \cdots - x - 1$ (of $\{F_n^{(k)}\}_{n=n_0}^{\infty}$) has a (positive real) dominating zero, the characteristic polynomial $\overline{T}_k(x) = x^k + x^{k-1} + \cdots + x - 1$ of the reverse sequence has no one if k is odd. Other difficulty in computations is that, although $\overline{T}_k(x)$ possesses a dominating zero (a negative real number) if k is even but the dominance is not strong. This paper supposes that k is even, and according to the previous note, we face this difficulty.

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Bravo et al. [2], extending the results of [1], determined the total multiplicity of Tribonacci sequence $\{T_n\}_{n\in\mathbb{Z}} = \{F_n^{(3)}\}_{n\in\mathbb{Z}}$. In particular, they solved the Diophantine equation $T_n = T_m$ for negative subscripts.

Pethő [5] proved that the Diophantine equation

$$F_n^{(k)} = F_m^{(\ell)}$$

possesses only finitely many solutions $(n, m) \in \mathbb{Z}^2$ for fixed $k \geq \ell \geq 2$. This result is ineffective, the proof is based on the theory of S-unit equations. An effective finiteness results of [5] states that if k and ℓ are given positive even integers, and the integers n and m satisfy

$$F_n^{(k)} = \pm F_m^{(\ell)},$$

then $|m|, |n| < C$, where the constant C depends only on k, ℓ, and on the zeros of $T_k(x)$ and $T_\ell(x)$.

This paper is devoted to investigate the equation

$$F_n^{(k)} = \pm F_m^{(k)} \tag{2}$$

for negative subscripts n and m if $k = 2\kappa$ is even. We explicitly give an upper bound B_k such that the solutions satisfy $|n|, |m| < B_k$. This bound is huge, and cannot be applied to eliminate the solutions to (2). But, in fact, it bounds explicitly the size of the solutions only in the term of k. In the proof, we do not use Baker method. The precise result is given here.

Theorem 1. *If k is even, and $n < m < 0$ satisfy (2), then*

$$|m| < 6^{k^4+1}\log(5k^2(1 + 3^{d_k/k})) + 1, \quad |n| < |m| + d_k, \tag{3}$$

where $d_k = 6^{k^4+1}\log(9k)$.

Note that this theorem is true even for the Fibonacci sequence. Moreover Pethő [5] solved (2) if $k = 4$, and (3) obviously fulfils in this case, too. Hence it is sufficient to justify the theorem for $k \geq 6$. The method follows the proof of Theorem 5, Case ii of [5]. In order to have this paper self-contained, we will refer the necessary details from that proof.

Now we introduce some more notations in order to fix a few subject materials. It is known that the polynomial $T_k(x)$ has simple zeros, the largest one in absolute value is a positive real number denoted by α_1, which is greater than 1. $T_k(x)$ is a Pisot polynomial, i.e. all zeros but α_1 lie inside the unit circle. The other zeros are complex non-real numbers, except if k is even. In this case there exist a second real zero, say α_k, which is negative, and has the unique smallest absolute value among all the zeros. If two zeros have common absolute value then they form a complex conjugate pair (see [5]). Hence the zeros of the characteristic polynomial $T_k(x)$ can be ordered by

$$|\alpha_k| < |\alpha_{k-1}| = |\alpha_{k-2}| < \cdots < |\alpha_3| = |\alpha_2| < \alpha_1$$

assuming k is even.

For any $k \geq 2$, Dresden and Du [3] gave the simplified explicit formula

$$F_n^{(k)} = \sum_{j=1}^{k} g_k(\alpha_j)\alpha_j^{n-1} \qquad \text{for } n \geq 0, \tag{4}$$

where

$$g_k(x) = \frac{x-1}{2+(k+1)(x-2)}. \tag{5}$$

Note that (4) is also true for any $n \in \mathbb{Z}$.

Keeping that $k = 2\kappa$ is even, we consider equation (2) for negative subscripts $n < m < 0$. Clearly, $F_{-k+1}^{(k)} = 1$, $F_{-k}^{(k)} = -1$, and $F_{-k-1}^{(k)} = 0$ follow from (1) and (1). Hence without loss of generality we may suppose $m \leq -k+2$.

Here we list up a few estimates will be used later. The first three lemmata do not depend on the parity of $k \geq 2$.

Lemma 1. *For $k \geq 2$ the following inequalities hold.*

$$2 - \frac{1}{2^{k-1}} < \ \alpha_1 \ < 2 - \frac{1}{2^k}.$$

Proof. This is Lemma 3.6, and a consequence of Theorem 3.9 in [6]. □

Lemma 2. *If $j \neq 1$, then $3^{-1/k} < |\alpha_j|$.*

Proof. See Lemma 2.1 in [4]. □

Lemma 3. *If $|\alpha_j| > |\alpha_i|$, then*

$$\frac{|\alpha_j|}{|\alpha_i|} > c_k := 1 + 6^{-k^4}.$$

Proof. A simple computation shows that the statement is true for $k = 2, 3, \ldots, 9$. If $k \geq 10$, then we follow the proof of Lemma 2.2 in [4]. Note that $c_k = 1 + 8^{-k^4}$ was given there. The improvement is based on the fact that $1 + 4 \cdot 3^{2/k} < 6$ if $k \geq 10$ (see (2.5) and afterwards in [4]). □

Lemma 4. *Assume that $k \geq 4$ is even. Then*

$$|\alpha_k| < \frac{2k-1}{2k+1}.$$

Proof. Consider the graph of the polynomial function

$$f_k(x) = (x-1)T_k(x) = x^{k+1} - 2x^k + 1.$$

For negative x the function is increasing and reaches its relative maximum at $x_0 = 0$. Put $a_k = -(2k-1)/(2k+1)$. It is sufficient to show that $f_k(a_k) = a_k^k(a_k - 2) + 1 < 0$. Equivalently we prove

$$\frac{2k+1}{6k+1} < \left(\frac{2k-1}{2k+1}\right)^k. \tag{6}$$

The left-hand side is a decreasing sequence which tends to $1/3$ (as $k \to \infty$). The right-hand side is an increasing sequence, and it tends to $1/e$. Since (6) holds if $k = 4$, then it holds for all even $k > 4$, too. $\square$

2. Proof of Theorem 1

2.1. **Preparation.** First we carry out a few preliminary computations. We arrange these results in lemmata as follows. In the sequel, assume that k is even.

Lemma 5. *Suppose that $k \geq 2$ is even. Then $|g_k(\alpha_k)| > \frac{2(1+3^{-1/k})}{6k+3}$ holds.*

Proof. Apply (5), which together with the fact $-1 < \alpha_k < 0$ and Lemmata 2, 4 provides

$$\begin{aligned} |g_k(\alpha_k)| &= \frac{|\alpha_k - 1|}{|2 + (k+1)(\alpha_k - 2)|} = \frac{1-\alpha_k}{-2 + (k+1)(2-\alpha_k)} \\ &> \frac{1 + 3^{-1/k}}{-2 + (k+1)\left(3 - \frac{2}{2k+1}\right)} = \frac{(2k+1)(1+3^{-1/k})}{6k^2 + 3k - 1} \\ &> \frac{2(1+3^{-1/k})}{6k+3}. \end{aligned}$$

$\square$

Lemma 6. *For $2 \leq j \leq k$ (k is even) we have $|g_k(\alpha_j)| < \frac{2}{k-1}$.*

Proof. Use again (5). Then we have

$$\begin{aligned} |g_k(\alpha_j)| &= \frac{|\alpha_j - 1|}{|2 + (k+1)(\alpha_j - 2)|} \leq \frac{|\alpha_j| + 1}{|-2k + (k+1)\alpha_j|} \\ &< \frac{2}{|2k - (k+1)|\alpha_j||} < \frac{2}{2k - (k+1)} = \frac{2}{k-1}. \end{aligned}$$

$\square$

Lemma 7. *If $k \geq 2$ is even, then $|g_k(\alpha_1)| < \frac{2^k-1}{2(2^k-k-1)}$.*

Proof. Observe that Lemma 1 implies $-(k+1)/2^{k-1} < (k+1)(\alpha_1-2) < -(k+1)/2^k$. Thus $(k+1)(\alpha_1-2)$ is always negative, but its absolute value is less than or equal to $3/2$ if $k \geq 2$. Combining this argument with function (5) and Lemma 1, it leads to

$$\begin{aligned} |g_k(\alpha_1)| &= \frac{|\alpha_1 - 1|}{|2+(k+1)(\alpha_1-2)|} < \frac{1-2^{-k}}{2+(k+1)(\alpha_1-2)} \\ &< \frac{1-2^{-k}}{2-(k+1)2^{1-k}} = \frac{2^k-1}{2(2^k-k-1)}. \end{aligned}$$

□

Lemma 8. *If $k \geq 6$ is even and $2 \leq j \leq k-1$, then $\frac{|g_k(\alpha_j)|}{|g_k(\alpha_k)|} < 4.26$.*

Proof. Lemmata 5 and 6 show

$$\frac{|g_k(\alpha_j)|}{|g_k(\alpha_k)|} < \frac{(6k+3)3^{1/k}}{(k-1)(3^{1/k}+1)}.$$

The last expression is monotone decreasing and tends to 3. Hence its value at $k=6$ gives the upper bound indicated in the lemma. □

Lemma 9. *If $k \geq 6$ is even, then $\frac{|g_k(\alpha_1)|}{|g_k(\alpha_k)|} < 0.453(2k+1)$.*

Proof. Lemmata 5 and 7 imply

$$\frac{|g_k(\alpha_1)|}{|g_k(\alpha_k)|} < \frac{3}{2}(2k+1) \cdot \frac{2^k-1}{2(2^k-k-1)} \cdot \frac{1}{1+3^{-1/k}}.$$

The last two fractions are monotone decreasing, and each tends to $1/2$ (as $k \to \infty$). Hence ruling out $2k+1$, the remaining part at $k=6$ confirms the statement of the lemma. □

Lemma 10. *Assume that $k \geq 2$ is even. Then $\frac{|\alpha_1|}{|\alpha_k|} > 2$ follows.*

Proof. Comparing the bounds of the numerator and denominator (see Lemmata 1 and 4), we see

$$\frac{|\alpha_1|}{|\alpha_k|} > \frac{2k+1}{2k-1} \cdot \frac{2^k-1}{2^{k-1}},$$

and this sequence decreasingly tends to 2. □

2.2. **The proof.** Now turn our attention to the principal part of the proof. Recall $n < m \leq -k+2$, and $k \geq 6$. Hence $m \leq -4$. We combine equation (2) and the explicit formula (4) with (5), which admit

$$g_k(\alpha_k)\alpha_k^{m-1} + \sum_{j=1}^{k-1} g_k(\alpha_j)\alpha_j^{m-1} = \pm\left(g_k(\alpha_k)\alpha_k^{n-1} + \sum_{j=1}^{k-1} g_k(\alpha_j)\alpha_j^{n-1}\right),$$

or equivalently

$$g_k(\alpha_k)\alpha_k^{m-1}\left(1 \mp \alpha_k^{n-m}\right) = \sum_{j=1}^{k-1} g_k(\alpha_j)\alpha_j^{m-1}\left(-1 \pm \alpha_j^{n-m}\right).$$

It leads immediately to

$$1 \mp \alpha_k^{n-m} = \sum_{j=1}^{k-1} \frac{g_k(\alpha_j)}{g_k(\alpha_k)}\left(\frac{\alpha_j}{\alpha_k}\right)^{m-1}\left(-1 \pm \alpha_j^{n-m}\right). \tag{7}$$

In the first phase, we will bound $|n-m|$ (recall that $n - m < 0$). Therefore take the absolute value of the sides of (7) to conclude

$$\begin{aligned} \left|1 \mp \alpha_k^{n-m}\right| &\leq \sum_{j=2}^{k-1}\left|\frac{g_k(\alpha_j)}{g_k(\alpha_k)}\right|\left|\frac{\alpha_j}{\alpha_k}\right|^{m-1}\left|-1 \pm \alpha_j^{n-m}\right| \\ &\quad + \left|\frac{g_k(\alpha_1)}{g_k(\alpha_k)}\right|\left|\frac{\alpha_1}{\alpha_k}\right|^{m-1}\left|-1 \pm \alpha_1^{n-m}\right|. \end{aligned} \tag{8}$$

On the right-hand side of the above formula we have separated the term corresponding to α_1 since this odd one out addend requires different treatment.

Clearly, for the left-hand side $0 < |\alpha_k|^{n-m} - 1 \leq \left|1 \mp \alpha_k^{n-m}\right|$ holds. For the right-hand side (in short, RHS) we apply Lemmata 5-10. Besides we also need an additional argument presented by

$$\left|-1 \pm \alpha_1^{n-m}\right| \leq \alpha_1^{-1} + 1 < \frac{1}{2 - \frac{1}{2^{k-1}}} + 1 = \frac{2^{k-1}}{2^k - 1} + 1.$$

The last sequence is decreasing, and $k \geq 6$ implies that it does not exceed $1 + 32/63 < 1.51$. Thus

$$\begin{aligned} RHS &\leq (k-2)\cdot\left(4.26c_k^{m-1}\left(1 + c_k^{n-m}|\alpha_k|^{n-m}\right)\right) \\ &\qquad + 0.453(2k+1)\cdot 2^{m-1}\cdot 1.51 \\ &< 4.304k - 8.498 + 4.26(k-2)c_k^{n-m}|\alpha_k|^{n-m}, \end{aligned} \tag{9}$$

where in the second inequality we used the fact that $c_k^{m-1} < 1$ (the definition of c_k is given at Lemma 3), and $m - 1 \leq -5$. Consequently

$$|\alpha_k|^{n-m} - 1 < 4.304k - 8.498 + 4.26(k-2)c_k^{n-m}|\alpha_k|^{n-m}.$$

Add $+1$ to both sides, and divide the inequality by $c_k^{n-m}|\alpha_k|^{n-m}$, which together with the fact $1 < c_k^{n-m}|\alpha_k|^{n-m}$ (since $1 > |\alpha_j| > c_k|\alpha_k|$ for $2 \le j \le k-1$) yields

$$c_k^{m-n} < 4.304k - 7.498 + 4.26(k-2) < 9k - 16.$$

Finally we find

$$(10)\quad m-n = |n-m| < \frac{\log(9k-16)}{\log c_k} < \frac{\log(9k-16)}{\frac{1}{6}\cdot 6^{-k^4}} = 6^{k^4+1}\log(9k).$$

Put $d_k = 6^{k^4+1}\log(9k)$.

In the second phase of the proof, we return to (7), and knowing the upper bound (10) on $|n-m|$ we target to bound m and n. Clearly,

$$(11)\qquad \left|1 \mp \alpha_k^{n-m}\right| \cdot |\alpha_k|^{m-1} = \left|\sum_{j=1}^{k-1} \frac{g_k(\alpha_j)}{g_k(\alpha_k)}\, \alpha_j{}^{m-1}\left(-1 \pm \alpha_j^{n-m}\right)\right|.$$

First observe that

$$\left|1 \mp \alpha_k^{n-m}\right| \ge |\alpha_k|^{-1} - 1 > \frac{2k+1}{2k-1} - 1 = \frac{2}{2k-1}.$$

Similarly, as we handled (8), and obtained (9), we treat the right-hand side of (11) which we denote by RHS_1. So

$$\begin{aligned}
RHS_1 \quad &\le \quad (k-2)\cdot 4.26 \cdot |\alpha_{k-1}|^{m-1}(1+|\alpha_k|^{n-m}) \\
&\qquad\qquad +0.453(2k+1)|\alpha_1|^{m-1}(1+\alpha_1^{n-m}) \\
&\le \quad 4.26(k-2)|\alpha_{k-1}|^{m-1}(1+|\alpha_k|^{-d_k}) \\
&\qquad\qquad +0.453(2k+1)\left(\frac{2^{k-1}}{2^k-1}\right)^5\left(1+\left(\frac{2^{k-1}}{2^k-1}\right)\right) \\
&\le \quad 4.26(k-2)c_k^{m-1}|\alpha_k|^{m-1}(1+|\alpha_k|^{-d_k}) + 0.024(2k+1).
\end{aligned}$$

Combining (11) and the two previous arguments, together with Lemma 4 it yields

$$\frac{2}{4k^2-1} < 4.26\frac{k-2}{2k+1}c_k^{m-1}(1+|\alpha_k|^{-d_k}) + 0.024\left(\frac{2k+1}{2k-1}\right)^{m-1},$$

and then

$$\frac{1}{4k^2} < \left(1.065(1+|\alpha_k|^{-d_k}) + 0.012\right)c_k^{m-1}.$$

Indeed, $c_k = 1 + 8^{-k^4} < (2k+1)/(2k-1)$. Now

$$\frac{1}{k^2} < \left(4.26(1+|\alpha_k|^{-d_k}) + 0.048\right)c_k^{m-1} < 4.27(1+|\alpha_k|^{-d_k})c_k^{m-1},$$

which, via $|\alpha_k| > 3^{-1/k}$ leads to
$$c_k^{|m-1|} < 4.27k^2(1+3^{d_k/k}).$$
Hence
$$|m-1| < \frac{\log(5k^2(1+3^{d_k/k}))}{\log(1+6^{-k^4})} < 6^{k^4+1}\log(5k^2(1+3^{d_k/k})).$$
Then the proof is complete.

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