

On Composite Rational Functions

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Abstract. In this paper we characterize all composite lacunary rational functions having at most 3 distinct zeroes and poles and we also provide some examples in case of 4, 5, 6 and 7 singularities.

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1 Introduction

In this article we deal with a problem related to decompositions of polynomials and rational functions. A classical result by Ritt [32] states that if there is a polynomial $f \in \mathbb{C}[X]$ satisfying certain tameness properties and

$$f = g_1 \circ g_2 \circ \cdots \circ g_r = h_1 \circ h_2 \circ \cdots \circ h_s,$$

then $r = s$ and $\{\deg g_1, \dots, \deg g_r\} = \{\deg h_1, \dots, \deg h_r\}$. Ritt's fundamental result has been investigated, extended and applied in various wide-ranging contexts (see e.g. [5, 12, 15, 16, 19, 20, 23, 24, 26, 27, 36, 37]). The above statement is not true for rational functions. It is not true that all complete decompositions of a rational function have the same length. Gutierrez and Sevilla [23] provided an example with rational coefficients as follows

$$\begin{aligned} f &= \frac{x^3(x+6)^3(x^2-6x+36)^3}{(x-3)^3(x^2+3x+9)^3}, \\ f &= g_1 \circ g_2 \circ g_3 = x^3 \circ \frac{x(x-12)}{x-3} \circ \frac{x(x+6)}{x-3}, \\ f &= h_1 \circ h_2 = \frac{x^3(x+24)}{x-3} \circ \frac{x(x^2-6x+36)}{x^2+3x+9}. \end{aligned}$$

We would like to emphasize that combinations of Siegel's [34] and Faltings' [18] finiteness theorems, related to integral and rational points on algebraic curves, and

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Ritt's result have yielded many nice results in Diophantine number theory (see e.g. [2, 7, 8, 9, 11, 14, 20, 25, 28, 29, 35]).

In his book [30, 31] Rédei introduced lacunary polynomials over finite fields. He characterized certain fully reducible lacunary polynomials over finite fields and he used this theory to obtain applications to areas of algebra and number theory.

In case of lacunary polynomials, that is when the number of terms of the polynomial is considered to be fixed while the degrees and coefficients may vary, Erdős [17] and independently Rényi posed the following conjecture. If $h(x)^2$ has boundedly many terms, then the same is true for $h(x) \in \mathbb{C}[X]$. Schinzel [33] gave a proof in a more general case, namely when $h(x)^d$ has boundedly many terms. Schinzel made the conjecture that if $g(h(x))$ has boundedly many terms, then it holds also for $h(x)$. This latter conjecture has been proved by Zannier [38]. Fuchs and Zannier [22] extended the problem, they considered lacunary rational functions which are decomposable. An other possibility to think about lacunarity is that one considers the number of zeros and poles of a rational function in reduced form to be bounded. In this case Fuchs and Pethő [21] obtained results related to the structure of such decomposable rational functions. We note that their proof was algorithmic.

In this paper we provide some computational experiments that we obtained by using a MAGMA [10] implementation of the algorithm of Fuchs and Pethő. We not only compute the appropriate varieties, but we also provide parametrizations of the possible solutions. We remark that algorithms have been developed earlier to find decompositions of a given rational function (see e.g. [1, 3, 4]). In [3], Ayad and Fleischmann implemented a MAGMA code to find decompositions, as an example they considered the rational function

$$f = \frac{x^4 - 8x}{x^3 + 1}$$

and they obtained that $f(x) = g(h(x))$, where

$$g = \frac{x^2 + 4x}{x + 1} \quad \text{and} \quad h = \frac{x^2 - 2x}{x + 1}.$$

At the end of the paper we show that this concrete decomposition corresponds to a point on certain variety.

Our paper is organized as follows. In Section 2 we present the result of Fuchs and Pethő [21] and we introduce some notation. In Section 3 we describe our algorithm and provide some information about the computation we have done. In Section 4 we prove Theorem 1 that we state below. First we define equivalence of rational functions. Two rational functions $f_1(x) = \prod_{i=1}^n (x - \alpha_i^{(1)})$ and $f_2(x) = \prod_{i=1}^n (x - \alpha_i^{(2)})$ are equivalent if there exist $a_{i,j} \in \mathbb{Q}$, $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, n+1\}$ such that

$$\alpha_i^{(1)} = a_{i,1}\alpha_1^{(2)} + a_{i,2}\alpha_2^{(2)} + \dots + a_{i,n}\alpha_n^{(2)} + a_{i,n+1},$$

for all $i \in \{1, 2, \dots, n\}$.

Theorem 1. Let k be an algebraically closed field of characteristic zero. If $f, g, h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \geq 2$, g not of the shape $(\lambda(x))^m$, $m \in \mathbb{N}$, $\lambda \in PGL_2(k)$, and f has 3 zeros and poles altogether, then f is equivalent to one of the following rational functions

- (a) $\frac{(x-\alpha_1)^{k_1}(x+1/4-\alpha_1)^{2k_2}}{(x-1/4-\alpha_1)^{2k_1+2k_2}}$ for some $\alpha_1 \in k$ and $k_1, k_2 \in \mathbb{Z}, k_1 + k_2 \neq 0$,
- (b) $\frac{(x-\alpha_1)^{2k_1}(x+\alpha_1-2\alpha_2)^{2k_2}}{(x-\alpha_2)^{2k_1+2k_2}}$ for some $\alpha_1, \alpha_2 \in k$ and $k_1, k_2 \in \mathbb{Z}, k_1 + k_2 \neq 0$.

In Section 5 we deal with systems related to the case $n = 4$. Finally, in Section 6 we provide some examples with $n \in \{5, 6, 7\}$.

2 Auxiliary results

Fuchs and Pethő [21] proved the following theorem.

Theorem A. Let k be an algebraically closed field of characteristic zero. Let n be a positive integer. Then there exists a positive integer J and, for every $i \in \{1, \dots, J\}$, an affine algebraic variety V_i defined over \mathbb{Q} and with $V_i \subset \mathbb{A}^{n+t_i}$ for some $2 \leq t_i \leq n$, such that:

(i) If $f, g, h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \geq 2$, g not of the shape $(\lambda(x))^m$, $m \in \mathbb{N}$, $\lambda \in PGL_2(k)$, and f has at most n zeros and poles altogether, then there exists for some $i \in \{1, \dots, J\}$ a point $P = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_{t_i}) \in V_i(k)$, a vector $(k_1, \dots, k_{t_i}) \in \mathbb{Z}^{t_i}$ with $k_1 + k_2 + \dots + k_{t_i} = 0$ or not depending on V_i , a partition of $\{1, \dots, n\}$ in $t_i + 1$ disjoint sets $S_\infty, S_{\beta_1}, \dots, S_{\beta_{t_i}}$ with $S_\infty = \emptyset$ if $k_1 + k_2 + \dots + k_{t_i} = 0$, and a vector $(l_1, \dots, l_n) \in \{0, 1, \dots, n-1\}^n$, also both depending only on V_i , such that

$$f(x) = \prod_{j=1}^{t_i} (\omega_j / \omega_\infty)^{k_j}, \quad g(x) = \prod_{j=1}^{t_i} (x - \beta_j)^{k_j}$$

and

$$h(x) = \begin{cases} \beta_j + \frac{\omega_j}{\omega_\infty} & (j = 1, \dots, t_i), & \text{if } k_1 + k_2 + \dots + k_{t_i} \neq 0 \\ \frac{\beta_{j_1}\omega_{j_2} - \beta_{j_2}\omega_{j_1}}{\omega_{j_2} - \omega_{j_1}} & (1 \leq j_1 < j_2 \leq t_i), & \text{otherwise,} \end{cases}$$

where

$$\omega_j = \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m}, \quad j = 1, \dots, t_i$$

and

$$\omega_\infty = \prod_{m \in S_\infty} (x - \alpha_m)^{l_m}.$$

Moreover, we have $\deg h \leq (n-1)/\max\{t_i-2, 1\} \leq n-1$.

(ii) Conversely for given data $P \in V_i(k), (k_1, \dots, k_{t_i}), S_\infty, S_{\beta_1}, \dots, S_{\beta_{t_i}}, (l_1, \dots, l_n)$ as described in (i) one defines by the same equations rational functions f, g, h with f having at most n zeros and poles altogether for which $f(x) = g(h(x))$ holds.

(iii) The integer J and equations defining the varieties V_i are effectively computable only in terms of n .

The method of proof of the above Theorem is effective. It provides an algorithm to obtain all possible decompositions of rational functions with at most n zeros and poles altogether.

We introduce some notation. Without loss of generality we may assume that f and g are monic. Let

$$f(x) = \prod_{i=1}^n (x - \alpha_i)^{f_i}$$

with pairwise distinct $\alpha_i \in k$ and $f_i \in \mathbb{Z}$ for $i = 1, \dots, n$. Similarly, let

$$g(x) = \prod_{j=1}^t (x - \beta_j)^{k_j}$$

with pairwise distinct $\beta_j \in k$ and $k_j \in \mathbb{Z}$ for $j = 1, \dots, t$ and $t \in \mathbb{N}$. Therefore we have

$$\prod_{i=1}^n (x - \alpha_i)^{f_i} = f(x) = g(h(x)) = \prod_{j=1}^t (h(x) - \beta_j)^{k_j}.$$

We shall write $h(x) = p(x)/q(x)$ with $p, q \in k[x], p, q$ coprime. Fuchs and Pethő showed that if $S_\infty \neq \emptyset$ then

$$q(x) = \prod_{m \in S_\infty} (x - \alpha_m)^{l_m}$$

and there is a partition of the set $\{1, \dots, n\} \setminus S_\infty$ in t disjoint non empty subsets $S_{\beta_1}, \dots, S_{\beta_t}$ such that

$$h(x) = \beta_j + \frac{1}{q(x)} \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m}, \quad (2.1)$$

where $l_m \in \mathbb{N}$ satisfies $l_m k_j = f_m$ for $m \in S_{\beta_j}$, and this holds true for every $j = 1, \dots, t$. We obtain at least two different representations of h , since we assumed that g is not of the shape $(\lambda(x))^m$. Hence we get at least one equation of the form

$$\beta_i + \frac{1}{q(x)} \prod_{r \in S_{\beta_i}} (x - \alpha_r)^{l_r} = \beta_j + \frac{1}{q(x)} \prod_{s \in S_{\beta_j}} (x - \alpha_s)^{l_s}. \quad (2.2)$$

If $S_\infty = \emptyset$ then we have

$$(p(x) - \beta_j q(x))^{k_j} = \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{f_m}.$$

Now we have that $t \geq 3$, otherwise g is in the special form we excluded. Siegel's identity provides the equations in this case. That is if $1 \leq j_1 < j_2 < j_3 \leq t$, then we have

$$v_{j_1, j_2, j_3} + v_{j_3, j_1, j_2} + v_{j_2, j_3, j_1} = 0, \quad (2.3)$$

where

$$v_{j_1, j_2, j_3} = (\beta_{j_1} - \beta_{j_2}) \prod_{m \in S_{\beta_{j_3}}} (x - \alpha_m)^{l_m}.$$

3 The computation

The method of proof by Fuchs and Pethő provides an algorithm to obtain the possible varieties. So we followed the steps described below.

- (i) compute the partitions of $\{1, 2, \dots, n\}$ into $t + 1$ disjoint sets
- (ii) given a partition $S_\infty, S_{\beta_1}, \dots, S_{\beta_t}$ and a vector $(l_1, \dots, l_n) \in \{1, 2, \dots, n\}^n$ compute the corresponding variety $V = \{v_1, \dots, v_r\}$, where v_i is a polynomial in $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_t$ obtained from (2.2) or (2.3)
- (iii) compute Groebner basis using lexicographical ordering (see e.g. [6, 13]) V_G of the ideal generated by the polynomials v_1, \dots, v_r (we note that here Groebner basis makes easier to compare the resulting varieties)
- (iv) test ideal membership for all $\alpha_i - \alpha_j, i, j = 1, 2, \dots, n, i \neq j$ and $\beta_i - \beta_j, i, j = 1, 2, \dots, t, i \neq j$
- (v) if there is no contradiction in step (iv) list the given partition, vector and variety.

We implemented the algorithm in Magma [10], the procedure `CFunc.m` can be downloaded from <http://www.math.unideb.hu/~tengely/CFunc.m>. The function works as follows: `CFunc(t, n, tipus)`, where t denotes the number of roots/poles of g , n denotes the number of roots/poles of f , `tipus` is in $\{0, 1\}$, it is 0 if S_∞ is empty, otherwise it is 1. So one way to call the function is `CFunc(2, 3, 1)`; we obtain a set of cardinality 18, an element of this set is:

$$\begin{aligned} &< [\{1\}, \{2\}, \{3\}], \\ &< 1, 2, 2 >, \\ &[X[1] - X[3] - 1/4 * X[4] + 1/4 * X[5], \\ &X[2] - X[3] - 1/2 * X[4] + 1/2 * X[5]] > \end{aligned}$$

this element is a record having 3 fields

1st: a list containing the partitions of $[1, 2, \dots, n]$

2nd: the exponent vector $\langle l_1, l_2, \dots, l_n \rangle$

3rd: the system of equations defining the variety, where $X[1] = \alpha_1, \dots, X[n] = \alpha_n$ and $X[n+1] = \beta_1, \dots, X[n+t] = \beta_t$.

Other way to use the function is to use the optional parameters `PSet` and `exptup`. `PSet` is a list containing some fixed partitions of $[1, 2, \dots, n]$, `exptup` is a list containing some fixed exponents l_1 up to l_n . In this way we get a system of equations corresponding to the given partition and exponent vector e.g.:

`CFunc(2, 3, 1 : PSet := [[{1}, {2}, {3}]]);`
the output is:

```
{
  < [{1}, {2}, {3}], < 1, 2, 2 >,
  [X[1] - X[3] - 1/4 * X[4] + 1/4 * X[5],
  X[2] - X[3] - 1/2 * X[4] + 1/2 * X[5]] >,
  < [{1}, {2}, {3}], < 2, 2, 1 >,
  [X[1] - X[3] - 1/4,
  X[2] - X[3] + 1/4,
  X[4] - X[5] + 1] >,
  < [{1}, {2}, {3}], < 2, 1, 2 >,
  [X[1] - X[3] - 1/2,
  X[2] - X[3] - 1/4,
  X[4] - X[5] - 1] >
}.
```

That is we obtained all systems with the given partition $[\{1\}, \{2\}, \{3\}]$ and $t = 2, n = 3, S_\infty \neq \emptyset$. In a similar way one can compute all systems with a given exponent vector $[[1, 2, 2]]$: `CFunc(2, 3, 1 : exptup := [[1, 2, 2]])` ;.

Using the above mentioned procedure we computed all systems corresponding to $n \in \{3, 4, 5\}$. Some details of the computations can be found in the following table. Here $\#RS$ denotes the number of remaining systems to be considered, that is those systems which were not eliminated, while $\#TS$ denotes the total number of systems.

n	t	S_∞	$\#RS$	$\#TS$
3	2	$\neq \emptyset$	18	96
3	3	$\neq \emptyset$	0	0
3	3	\emptyset	6	48
4	2	$\neq \emptyset$	264	3888
4	3	$\neq \emptyset$	0	5832
4	4	$\neq \emptyset$	0	0
4	3	\emptyset	24	1944
4	4	\emptyset	24	7776
5	2	$\neq \emptyset$	4644	122880
5	3	$\neq \emptyset$	60	368640
5	4	$\neq \emptyset$	0	491520
5	5	$\neq \emptyset$	0	0
5	3	\emptyset	384	61440
5	4	\emptyset	0	491520
5	5	\emptyset	120	1228800

The above table shows that combinatorial explosion increases the total number of systems very rapidly.

4 Proof of Theorem 1.

We are going to deal with the three possible cases $(n, t, S_\infty) \in \{(3, 2, \neq \emptyset), (3, 3, \neq \emptyset), (3, 3, \emptyset)\}$. We note that the previous table shows that there are no solutions with $(n, t, S_\infty) = (3, 3, \neq \emptyset)$.

4.1 The case $n = 3, t = 2$ and $S_\infty \neq \emptyset$

There are two types of systems here, in the first class one obtains solutions having two parameters, in the second class one has solutions having three parameters. Below we indicate the 12 systems which yield families with two parameters.

$(S_\infty, S_{\beta_1}, S_{\beta_2}), (l_1, l_2, l_3)$	System of equations	Solution $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2)$
$(\{3\}, \{2\}, \{1\})$ (1, 2, 2)	$\alpha_1 - \alpha_3 + 1/4 = 0$ $\alpha_2 - \alpha_3 + 1/2 = 0$ $\beta_1 - \beta_2 + 1 = 0$	$(-1/4 + \alpha_3, -1/2 + \alpha_3, \alpha_3, -1 + \beta_2, \beta_2)$
$(\{2\}, \{1\}, \{3\})$ (2, 2, 1)	$\alpha_1 - \alpha_3 + 1/4 = 0$ $\alpha_2 - \alpha_3 - 1/4 = 0$ $\beta_1 - \beta_2 + 1 = 0$	$(-1/4 + \alpha_3, 1/4 + \alpha_3, \alpha_3, -1 + \beta_2, \beta_2)$
$(\{1\}, \{2\}, \{3\})$ (2, 2, 1)	$\alpha_1 - \alpha_3 - 1/4 = 0$ $\alpha_2 - \alpha_3 + 1/4 = 0$ $\beta_1 - \beta_2 + 1 = 0$	$(1/4 + \alpha_3, -1/4 + \alpha_3, \alpha_3, -1 + \beta_2, \beta_2)$
$(\{1\}, \{3\}, \{2\})$ (2, 1, 2)	$\alpha_1 - \alpha_3 - 1/2 = 0$ $\alpha_2 - \alpha_3 - 1/4 = 0$ $\beta_1 - \beta_2 + 1 = 0$	$(1/2 + \alpha_3, 1/4 + \alpha_3, \alpha_3, -1 + \beta_2, \beta_2)$
$(\{2\}, \{3\}, \{1\})$ (2, 2, 1)	$\alpha_1 - \alpha_3 + 1/4 = 0$ $\alpha_2 - \alpha_3 - 1/4 = 0$ $\beta_1 - \beta_2 - 1 = 0$	$(-1/4 + \alpha_3, 1/4 + \alpha_3, \alpha_3, 1 + \beta_2, \beta_2)$
$(\{3\}, \{1\}, \{2\})$ (2, 1, 2)	$\alpha_1 - \alpha_3 + 1/2 = 0$ $\alpha_2 - \alpha_3 + 1/4 = 0$ $\beta_1 - \beta_2 + 1 = 0$	$(-1/2 + \alpha_3, -1/4 + \alpha_3, \alpha_3, -1 + \beta_2, \beta_2)$
$(\{1\}, \{3\}, \{2\})$ (2, 2, 1)	$\alpha_1 - \alpha_3 - 1/4 = 0$ $\alpha_2 - \alpha_3 + 1/4 = 0$ $\beta_1 - \beta_2 - 1 = 0$	$(1/4 + \alpha_3, -1/4 + \alpha_3, \alpha_3, 1 + \beta_2, \beta_2)$
$(\{1\}, \{2\}, \{3\})$ (2, 1, 2)	$\alpha_1 - \alpha_3 - 1/2 = 0$ $\alpha_2 - \alpha_3 - 1/4 = 0$ $\beta_1 - \beta_2 - 1 = 0$	$(1/2 + \alpha_3, 1/4 + \alpha_3, \alpha_3, 1 + \beta_2, \beta_2)$
$(\{2\}, \{1\}, \{3\})$ (1, 2, 2)	$\alpha_1 - \alpha_3 - 1/4 = 0$ $\alpha_2 - \alpha_3 - 1/2 = 0$ $\beta_1 - \beta_2 - 1 = 0$	$(1/4 + \alpha_3, 1/2 + \alpha_3, \alpha_3, 1 + \beta_2, \beta_2)$
$(\{2\}, \{3\}, \{1\})$ (1, 2, 2)	$\alpha_1 - \alpha_3 - 1/4 = 0$ $\alpha_2 - \alpha_3 - 1/2 = 0$ $\beta_1 - \beta_2 + 1 = 0$	$(1/4 + \alpha_3, 1/2 + \alpha_3, \alpha_3, -1 + \beta_2, \beta_2)$
$(\{3\}, \{2\}, \{1\})$ (2, 1, 2)	$\alpha_1 - \alpha_3 + 1/2 = 0$ $\alpha_2 - \alpha_3 + 1/4 = 0$ $\beta_1 - \beta_2 - 1 = 0$	$(-1/2 + \alpha_3, -1/4 + \alpha_3, \alpha_3, 1 + \beta_2, \beta_2)$
$(\{3\}, \{1\}, \{2\})$ (1, 2, 2)	$\alpha_1 - \alpha_3 + 1/4 = 0$ $\alpha_2 - \alpha_3 + 1/2 = 0$ $\beta_1 - \beta_2 - 1 = 0$	$(-1/4 + \alpha_3, -1/2 + \alpha_3, \alpha_3, 1 + \beta_2, \beta_2)$

As an example consider the system from the sixth row, that is $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{3\}, \{1\}, \{2\})$ and $(l_1, l_2, l_3) = (2, 1, 2)$. Here we obtain the following system of equations

$$\begin{aligned}\alpha_1 - \alpha_3 + 1/2 &= 0, \\ \alpha_2 - \alpha_3 + 1/4 &= 0, \\ \beta_1 - \beta_2 + 1 &= 0.\end{aligned}$$

Therefore one gets the parametric solution $(\alpha_3 - 1/2, \alpha_3 - 1/4, \alpha_3, \beta_2 - 1, \beta_2)$ and with $k_1 = k_2 = 1$ we have

$$\begin{aligned} f(x) &= \frac{(x - \alpha_3 + 1/2)^2(x - \alpha_3 + 1/4)}{(x - \alpha_3)^4}, \\ g(x) &= (x - \beta_2 + 1)(x - \beta_2), \\ h(x) &= \beta_2 - 1 + \frac{(x - \alpha_3 + 1/2)^2}{(x - \alpha_3)^2}. \end{aligned}$$

It is easy to see that $f(x - 1/4)$ is of the form (a) stated in Theorem 1. We note that one gets the same family in case of $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{3\}, \{2\}, \{1\})$ and $(l_1, l_2, l_3) = (1, 2, 2)$. In a similar way we can show that the remaining systems yield equivalent rational functions to the function in part (a) of Theorem 1.

Now we provide the table containing the 6 systems which yield families with three parameters.

$(S_\infty, S_{\beta_1}, S_{\beta_2}), (l_1, l_2, l_3)$	System of equations	Solution $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2)$
$(\{3\}, \{2\}, \{1\})$ (2, 2, 1)	$\alpha_1 - \alpha_2 + 1/2\beta_1 - 1/2\beta_2 = 0$ $\alpha_2 - \alpha_3 - 1/4\beta_1 + 1/4\beta_2 = 0$	$(-\alpha_2 + 2\alpha_3, \alpha_2, \alpha_3, 4\alpha_2 - 4\alpha_3 + \beta_2, \beta_2)$
$(\{1\}, \{3\}, \{2\})$ (1, 2, 2)	$\alpha_1 - \alpha_3 + 1/4\beta_1 - 1/4\beta_2 = 0$ $\alpha_2 - \alpha_3 + 1/2\beta_1 - 1/2\beta_2 = 0$	$(\alpha_1, -\alpha_3 + 2\alpha_1, \alpha_3, -4\alpha_1 + 4\alpha_3 + \beta_2, \beta_2)$
$(\{2\}, \{3\}, \{1\})$ (2, 1, 2)	$\alpha_1 - \alpha_3 + 1/2\beta_1 - 1/2\beta_2 = 0$ $\alpha_2 - \alpha_3 + 1/4\beta_1 - 1/4\beta_2 = 0$	$(2\alpha_2 - \alpha_3, \alpha_2, \alpha_3, -4\alpha_2 + 4\alpha_3 + \beta_2, \beta_2)$
$(\{1\}, \{2\}, \{3\})$ (1, 2, 2)	$\alpha_1 - \alpha_3 - 1/4\beta_1 + 1/4\beta_2 = 0$ $\alpha_2 - \alpha_3 - 1/2\beta_1 + 1/2\beta_2 = 0$	$(\alpha_1, -\alpha_3 + 2\alpha_1, \alpha_3, 4\alpha_1 - 4\alpha_3 + \beta_2, \beta_2)$
$(\{3\}, \{1\}, \{2\})$ (2, 2, 1)	$\alpha_1 - \alpha_2 - 1/2\beta_1 + 1/2\beta_2 = 0$ $\alpha_2 - \alpha_3 + 1/4\beta_1 - 1/4\beta_2 = 0$	$(-\alpha_2 + 2\alpha_3, \alpha_2, \alpha_3, -4\alpha_2 + 4\alpha_3 + \beta_2, \beta_2)$
$(\{2\}, \{1\}, \{3\})$ (2, 1, 2)	$\alpha_1 - \alpha_3 - 1/2\beta_1 + 1/2\beta_2 = 0$ $\alpha_2 - \alpha_3 - 1/4\beta_1 + 1/4\beta_2 = 0$	$(2\alpha_2 - \alpha_3, \alpha_2, \alpha_3, 4\alpha_2 - 4\alpha_3 + \beta_2, \beta_2)$

From the parametrizations one can easily obtain the corresponding rational functions, as an example we take the fourth row of the table. That is, we have

$$(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\}, \{2\}, \{3\}), (l_1, l_2, l_3) = (1, 2, 2)$$

and

$$\begin{aligned} \alpha_1 - \alpha_3 - 1/4\beta_1 + 1/4\beta_2 &= 0, \\ \alpha_2 - \alpha_3 - 1/2\beta_1 + 1/2\beta_2 &= 0. \end{aligned}$$

Thus if $k_1 = k_2 = 1$, then

$$\begin{aligned} f(x) &= \frac{(x - \alpha_3)^2(x - 2\alpha_1 + \alpha_3)^2}{(x - \alpha_1)^2}, \\ g(x) &= (x - 4\alpha_1 + 4\alpha_3 - \beta_2)(x - \beta_2), \\ h(x) &= \beta_2 + \frac{(x - \alpha_3)^2}{x - \alpha_1}. \end{aligned}$$

It is clear that f is equivalent to the rational function in part (b) of Theorem 1. The remaining systems can be handled in a similar way, all of these are equivalent to the rational function indicated in part (b).

4.2 The case $n = 3, t = 3$ and $S_\infty = \emptyset$

In total there are six parametrizations here, these are indicated in the table below.

$(S_{\beta_1}, S_{\beta_2}, S_{\beta_3}), (l_1, l_2, l_3)$	System of equations	Solution $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{1\}, \{3\}, \{2\})$ $(1, 1, 1)$	$\alpha_1\beta_2 - \alpha_1\beta_3 + \alpha_2\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_1 + \alpha_3\beta_3 = 0$	$(-\frac{\alpha_2\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_1 + \alpha_3\beta_3}{\beta_2 - \beta_3},$ $\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{2\}, \{1\}, \{3\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_3 - \alpha_2\beta_2 + \alpha_2\beta_3 - \alpha_3\beta_1 + \alpha_3\beta_2 = 0$	$(\frac{\alpha_2\beta_2 - \alpha_2\beta_3 + \alpha_3\beta_1 - \alpha_3\beta_2}{\beta_1 - \beta_3},$ $\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{3\}, \{1\}, \{2\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_3 - \alpha_2\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_2 + \alpha_3\beta_3 = 0$	$(\frac{\alpha_2\beta_1 - \alpha_2\beta_2 + \alpha_3\beta_2 - \alpha_3\beta_3}{\beta_1 - \beta_3},$ $\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{1\}, \{2\}, \{3\})$ $(1, 1, 1)$	$\alpha_1\beta_2 - \alpha_1\beta_3 - \alpha_2\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_1 - \alpha_3\beta_2 = 0$	$(\frac{\alpha_2\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_1 + \alpha_3\beta_2}{\beta_2 - \beta_3},$ $\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{3\}, \{2\}, \{1\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 - \alpha_3\beta_3 = 0$	$(\frac{\alpha_2\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_3\beta_3}{\beta_1 - \beta_2},$ $\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{2\}, \{3\}, \{1\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_2 + \alpha_2\beta_2 - \alpha_2\beta_3 - \alpha_3\beta_1 + \alpha_3\beta_3 = 0$	$(-\frac{\alpha_2\beta_2 - \alpha_2\beta_3 - \alpha_3\beta_1 + \alpha_3\beta_3}{\beta_1 - \beta_2},$ $\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$

We remark that all these systems correspond to trivial solutions ($\deg h=1$). We have that $\omega_j = \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m}$ is a linear polynomial for all $j \in \{1, 2, 3\}$, since $l_1 = l_2 = l_3 = 1$ and the cardinality of S_{β_j} is 1 for all possible cases. Therefore

$$h(x) = \frac{\beta_{j_1}\omega_{j_2} - \beta_{j_2}\omega_{j_1}}{\omega_{j_2} - \omega_{j_1}}$$

is a linear polynomial. So we do not obtain non-trivial rational function from this case.

As an illustration we provide an example corresponding to the parametrization indicated in the fourth row, that is $(S_{\beta_1}, S_{\beta_2}, S_{\beta_3}) = (\{1\}, \{2\}, \{3\})$ and $(l_1, l_2, l_3) = (1, 1, 1)$. Now let $(\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = (2, 1, -1, 1, 0)$ and $k_1 = k_2 = 1, k_3 = -2$. One has that $\alpha_1 = 0$ and

$$\begin{aligned} f(x) &= \frac{(x-2)x}{(x-1)^2}, \\ g(x) &= \frac{(x-1)(x+1)}{x^2}, \\ h(x) &= x-1. \end{aligned}$$

5 Cases with $n = 4$

In this section we provide some details of the computation corresponding to cases with $n = 4$.

5.1 The case $n = 4, t = 2$ and $S_\infty \neq \emptyset$

There are 264 systems to deal with. We will treat only a few representative examples.

Systems containing two polynomials.

If $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{4\}, \{1, 2\}, \{3\})$ and $(l_1, l_2, l_3, l_4) = (1, 1, 2, 1)$, then we have

$$\begin{aligned}\alpha_1 + \alpha_2 - 2\alpha_3 - \beta_1 + \beta_2 &= 0 \\ \alpha_2^2 - 2\alpha_2\alpha_3 - \alpha_2\beta_1 + \alpha_2\beta_2 + \alpha_3^2 + \alpha_4\beta_1 - \alpha_4\beta_2 &= 0.\end{aligned}$$

Since $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$ if $i \neq j$, we have that

$$\begin{aligned}\alpha_1 &= -\alpha_2 + 2\alpha_3 + \beta_1 - \beta_2, \\ \alpha_4 &= \alpha_2 - \frac{(\alpha_2 - \alpha_3)^2}{\beta_1 - \beta_2}.\end{aligned}$$

For example, if we consider the solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (-2, 1, 0, 2, 0, 1)$, then we get

$$\begin{aligned}f(x) &= \frac{(x-1)x^2(x+2)}{(x-2)^2}, \\ g(x) &= (x-1)x, \\ h(x) &= \frac{(x-1)(x+2)}{x-2}.\end{aligned}$$

Systems containing three polynomials.

If $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\}, \{2, 3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (1, 2, 1, 3)$, then we get

$$\begin{aligned}\alpha_1 + 1/3\alpha_3 - 4/3\alpha_4 &= 0 \\ \alpha_2 + 1/2\alpha_3 - 3/2\alpha_4 &= 0 \\ \alpha_3^2 - 2\alpha_3\alpha_4 + \alpha_4^2 - 4/3\beta_1 + 4/3\beta_2 &= 0.\end{aligned}$$

Thus one obtains the parametrization

$$\begin{aligned}\alpha_1 &= -1/3\alpha_3 + 4/3\alpha_4, \\ \alpha_2 &= -1/2\alpha_3 + 3/2\alpha_4, \\ \beta_1 &= 3/4\alpha_3^2 - 3/2\alpha_3\alpha_4 + 3/4\alpha_4^2 + \beta_2.\end{aligned}$$

Let us take $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (-1/3, -1/2, 1, 0, 1, 1/4)$, then we have

$$\begin{aligned}f(x) &= \frac{(x-1)x^3(x+1/2)^2}{(x+1/3)^2}, \\ g(x) &= (x-1)(x-1/4), \\ h(x) &= \frac{1}{4} + \frac{x^3}{x+1/3}.\end{aligned}$$

Systems containing four polynomials.

Consider the case $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\}, \{2, 3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (3, 1, 1, 3)$. One gets the system

$$\begin{aligned}\alpha_1 - \alpha_4 - 1/3 &= 0 \\ \alpha_2 + \alpha_3 - 2\alpha_4 - 1/3 &= 0 \\ \alpha_3^2 - 2\alpha_3\alpha_4 - 1/3\alpha_3 + \alpha_4^2 + 1/3\alpha_4 + 1/27 &= 0 \\ \beta_1 - \beta_2 - 1 &= 0.\end{aligned}$$

The parametrization is as follows

$$\begin{aligned}\alpha_1 &= \alpha_4 + 1/3, \\ \alpha_2 &= \alpha_4 \mp \frac{\sqrt{-3}}{18} + \frac{1}{6}, \\ \alpha_3 &= \alpha_4 \pm \frac{\sqrt{-3}}{18} + \frac{1}{6}, \\ \beta_1 &= \beta_2 + 1.\end{aligned}$$

As an example we take $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (1/6, -\sqrt{-3}/18, \sqrt{-3}/18, -1/6, 1, 0)$, then we obtain

$$\begin{aligned}f(x) &= \frac{(x - \sqrt{-3}/18)(x + \sqrt{-3}/18)(x + 1/6)^3}{(x - 1/6)^6}, \\ g(x) &= (x - 1)x, \\ h(x) &= \frac{(x + 1/6)^3}{(x - 1/6)^3}.\end{aligned}$$

Systems containing five polynomials.

If $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\}, \{2, 3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (3, 1, 2, 2)$, then we have

$$\begin{aligned}\alpha_1 - 1/3\alpha_2 - 2/3\alpha_3 - 1/3 &= 0 \\ \alpha_2^2 - 2\alpha_2\alpha_4 + 2\alpha_2 + 8\alpha_3^2 - 16\alpha_3\alpha_4 + 6\alpha_3 + 9\alpha_4^2 - 8\alpha_4 + 1 &= 0 \\ \alpha_2 + 7/2\alpha_3 - 9/2\alpha_4 + 1 &= 0 \\ \alpha_3 - \alpha_4 + 8/27 &= 0 \\ \beta_1 - \beta_2 + 1 &= 0.\end{aligned}$$

We get the parametrization

$$\begin{aligned}\alpha_1 &= \alpha_4 + 4/27, \\ \alpha_2 &= \alpha_4 + 1/27, \\ \alpha_3 &= \alpha_4 - 8/27, \\ \beta_1 &= \beta_2 - 1.\end{aligned}$$

As a concrete example we deal with the case

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (4/27, 1/27, -8/27, 0, 0, 1).$$

It easily follows that

$$\begin{aligned} f(x) &= \frac{(x - 1/27)x^2(x + 8/27)^2}{(x - 4/27)^6}, \\ g(x) &= (x - 1)x, \\ h(x) &= 1 + \frac{x^2}{(x - 4/27)^3}. \end{aligned}$$

5.2 The case $n = 4, t = 3$ and $S_\infty = \emptyset$

There are 24 systems to handle in this case. The systems are getting more and more complicated therefore we deal with two typical cases. There are 6 systems having two polynomials in the Groebner basis, one of these is as follows: $(S_{\beta_1}, S_{\beta_2}, S_{\beta_3}) = (\{1, 3\}, \{4\}, \{2\})$ and $(l_1, l_2, l_3, l_4) = (1, 2, 1, 2)$. The system of equations are given by

$$\begin{aligned} \alpha_1\beta_2 - \alpha_1\beta_3 + 2\alpha_2\beta_1 - 2\alpha_2\beta_2 + \alpha_3\beta_2 - \alpha_3\beta_3 - 2\alpha_4\beta_1 + 2\alpha_4\beta_3 &= 0 \\ \alpha_2^2\beta_1 - \alpha_2^2\beta_2 - 2\alpha_2\alpha_3\beta_1 + 2\alpha_2\alpha_3\beta_2 - \alpha_3^2\beta_2 + \alpha_3^2\beta_3 + 2\alpha_3\alpha_4\beta_1 - 2\alpha_3\alpha_4\beta_3 - \alpha_4^2\beta_1 + \alpha_4^2\beta_3 &= 0. \end{aligned}$$

There are four solutions where $\alpha_i = \alpha_j$ or $\beta_i = \beta_j$

$$\begin{aligned} (\alpha_1 &= \alpha_4, \alpha_2 = \alpha_4, \alpha_3 = \alpha_4, \alpha_4, \beta_1, \beta_2, \beta_3), \\ (\alpha_1 &= \alpha_3, \alpha_2 = \alpha_3, \alpha_3, \alpha_4, \beta_1 = \beta_3, \beta_2, \beta_3), \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1 &= \beta_3, \beta_2 = \beta_3, \beta_3), \\ (\alpha_1, \alpha_2 &= \alpha_4, \alpha_3, \alpha_4, \beta_1, \beta_2 = \beta_3, \beta_3). \end{aligned}$$

These solutions do not lead to appropriate rational functions. There is one solution which yield solutions of the original problem

$$\begin{aligned} \alpha_1 &= -\frac{\alpha_2\alpha_3 - 2\alpha_2\alpha_4 + \alpha_3\alpha_4}{\alpha_2 - 2\alpha_3 + \alpha_4}, \\ \beta_2 &= \frac{\alpha_2^2\beta_1 - 2\alpha_2\alpha_3\beta_1 + \alpha_3^2\beta_3 + 2\alpha_3\alpha_4\beta_1 - 2\alpha_3\alpha_4\beta_3 - \alpha_4^2\beta_1 + \alpha_4^2\beta_3}{(\alpha_2 - \alpha_3)^2}, \end{aligned}$$

where $\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_3$ are parameters such that $\alpha_i \neq \alpha_j, \beta_i \neq \beta_j$ and $\alpha_2 - 2\alpha_3 + \alpha_4 \neq 0$. As an example consider the case $(\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_3) = (0, 1, 3, 0, 1)$. We obtain that $\alpha_1 = -3$ and $\beta_2 = 4$. Let $k_1 = k_2 = 1$ and $k_3 = -2$. We get that

$$\begin{aligned} f(x) &= \frac{(x - 3)^2(x - 1)(x + 3)}{x^4}, \\ g(x) &= \frac{(x - 4)x}{(x - 1)^2}, \\ h(x) &= \frac{(x - 1)(x + 3)}{2x - 3}. \end{aligned}$$

There are 18 systems having three polynomials in the Groebner basis, one of these is as follows: $(S_{\beta_1}, S_{\beta_2}, S_{\beta_3}) = (\{1\}, \{2, 3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (2, 1, 1, 2)$. The system of equations is

$$\begin{aligned}\alpha_1\alpha_2 + \alpha_1\alpha_3 - 2\alpha_1\alpha_4 - 2\alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 &= 0 \\ \alpha_1\beta_2 - \alpha_1\beta_3 - 1/2\alpha_2\beta_1 + 1/2\alpha_2\beta_3 - 1/2\alpha_3\beta_1 + 1/2\alpha_3\beta_3 + \alpha_4\beta_1 - \alpha_4\beta_2 &= 0 \\ \alpha_2^2\beta_1 - \alpha_2^2\beta_3 + 2\alpha_2\alpha_3\beta_1 - 4\alpha_2\alpha_3\beta_2 + 2\alpha_2\alpha_3\beta_3 - 4\alpha_2\alpha_4\beta_1 + \\ + 4\alpha_2\alpha_4\beta_2 + \alpha_3^2\beta_1 - \alpha_3^2\beta_3 - 4\alpha_3\alpha_4\beta_1 + 4\alpha_3\alpha_4\beta_2 + 4\alpha_4^2\beta_1 - 4\alpha_4^2\beta_2 &= 0.\end{aligned}$$

The only solution where one can obtain appropriate rational functions is

$$\begin{aligned}\alpha_1 &= \frac{-\alpha_2\alpha_4 - \alpha_3\alpha_4 + 2\alpha_2\alpha_3}{\alpha_2 + \alpha_3 - 2\alpha_4}, \\ \beta_3 &= \frac{\alpha_2^2\beta_1 + 2\alpha_2\alpha_3\beta_1 - 4\alpha_2\alpha_3\beta_2 - 4\alpha_2\alpha_4\beta_1 + 4\alpha_2\alpha_4\beta_2 + \alpha_3^2\beta_1 - 4\alpha_3\alpha_4\beta_1 + 4\alpha_3\alpha_4\beta_2 + 4\alpha_4^2\beta_1 - 4\alpha_4^2\beta_2}{(\alpha_2 - \alpha_3)^2},\end{aligned}$$

where $\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2$ are parameters such that $\alpha_i \neq \alpha_j, \beta_i \neq \beta_j$ and $\alpha_2 + \alpha_3 - 2\alpha_4 \neq 0$. Now we consider the example with $(\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (0, 1, 3, 0, 1)$. We have that $\alpha_1 = 2/3$ and $\beta_3 = -8$. Let $k_1 = k_2 = 1$ and $k_3 = -2$. We have that

$$\begin{aligned}f(x) &= \frac{(x - 2/3)^2(x - 1)x}{(x - 2)^4}, \\ g(x) &= \frac{(x - 1)x}{(x + 8)^2}, \\ h(x) &= \frac{(3x - 2)^2}{-3x + 4}.\end{aligned}$$

5.3 The case $n = 4, t = 4$ and $S_\infty = \emptyset$

Here we have 24 systems to solve. We have the same remark as in case of $t = 3, n = 3$ and $S_\infty = \emptyset$. That is there are only trivial solutions here with $\deg h = 1$.

Since one has 24 very similar systems, we will deal with one of these only. Let $(S_{\beta_1}, S_{\beta_2}, S_{\beta_3}, S_{\beta_4}) = (\{1\}, \{2\}, \{3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (1, 1, 1, 1)$. One gets the system of equations

$$\begin{aligned}\alpha_1\beta_2 - \alpha_1\beta_4 - \alpha_2\beta_1 + \alpha_2\beta_4 + \alpha_4\beta_1 - \alpha_4\beta_2 &= 0 \\ \alpha_1\beta_3 - \alpha_1\beta_4 - \alpha_3\beta_1 + \alpha_3\beta_4 + \alpha_4\beta_1 - \alpha_4\beta_3 &= 0 \\ \alpha_2\beta_3 - \alpha_2\beta_4 - \alpha_3\beta_2 + \alpha_3\beta_4 + \alpha_4\beta_2 - \alpha_4\beta_3 &= 0.\end{aligned}$$

There are three solutions which do not correspond to appropriate rational functions, the remaining solution has

$$\begin{aligned}\alpha_1 &= \frac{\alpha_3\beta_1 - \alpha_3\beta_4 - \alpha_4\beta_1 + \alpha_4\beta_3}{\beta_3 - \beta_4}, \\ \alpha_2 &= \frac{\alpha_3\beta_2 - \alpha_3\beta_4 - \alpha_4\beta_2 + \alpha_4\beta_3}{\beta_3 - \beta_4}.\end{aligned}$$

Now let $(\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) = (0, 1, 3, 2, 1, 0)$ and $k_1 = k_2 = 1, k_3 = k_4 = -1$. One obtains that

$$\begin{aligned} f(x) &= \frac{(x+1)(x+2)}{(x-1)x}, \\ g(x) &= \frac{(x-3)(x-2)}{(x-1)x}, \\ h(x) &= -x + 1. \end{aligned}$$

6 Some examples with $n \in \{5, 6, 7\}$

We computed all the varieties corresponding to $n = 5$, the systems are getting more and more complicated therefore we selected only three examples given below. All systems in case of $n = 5$ can be downloaded from

<http://www.math.unideb.hu/~tengely/CFunc5.txt.tar.gz>. We also consider examples with $n = 6$ and 7.

- Consider the case $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1, 5\}, \{3, 4\}, \{2\})$ and $(l_1, l_2, l_3, l_4, l_5) = (3, 1, 1, 3, 1)$. One gets a system containing 5 equations

$$\begin{aligned} \alpha_1 - 2\alpha_4 + \alpha_5 &= 0 \\ \alpha_2 - 3/2\alpha_4 + 1/2\alpha_5 &= 0 \\ \alpha_3 - 3\alpha_4 + 2\alpha_5 &= 0 \\ \alpha_4^3 - 3\alpha_4^2\alpha_5 + 3\alpha_4\alpha_5^2 - \alpha_5^3 + 1/2 &= 0 \\ \beta_1 - \beta_2 + 1 &= 0. \end{aligned}$$

The solutions of this system of equations are given by

$$(\alpha_1, \alpha_1 + \frac{1}{4}\sqrt[3]{4}\zeta^k, \frac{1}{2}(\sqrt[3]{2}\alpha_1 - 1)\sqrt[3]{4}\zeta^k, \frac{1}{2}(\sqrt[3]{2}\alpha_1 + 1)\sqrt[3]{4}\zeta^k, \frac{1}{2}(\sqrt[3]{2}\alpha_1 + 2)\sqrt[3]{4}\zeta^k, \beta_1, \beta_1 + 1),$$

where $\zeta = \frac{1+i\sqrt{3}}{2}$ and $k = 0, 1, 2$.

- Let $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1, 2, 5\}, \{3\}, \{4\})$ and $(l_1, l_2, l_3, l_4, l_5) = (1, 1, 1, 3, 1)$. We obtain the following system of equations

$$\begin{aligned} \alpha_1 + \alpha_2 - 3\alpha_4 + \alpha_5 &= 0 \\ \alpha_2^2 - 3\alpha_2\alpha_4 + \alpha_2\alpha_5 + 3\alpha_4^2 - 3\alpha_4\alpha_5 + \alpha_5^2 - 1 &= 0 \\ \alpha_3 - \alpha_4^3 + 3\alpha_4^2\alpha_5 - 3\alpha_4\alpha_5^2 + \alpha_5^3 - \alpha_5 &= 0 \\ \beta_1 - \beta_2 - 1 &= 0. \end{aligned}$$

The general solutions are given by

$$\begin{aligned}
&\alpha_1, \\
&\alpha_2, \\
&\alpha_3 = \frac{1}{18} \sqrt{-\alpha_1 + \alpha_2 + 2} \left(4 \sqrt{\alpha_1 - \alpha_2 + 2} \sqrt{3} \alpha_1 \alpha_2 - 2 \sqrt{\alpha_1 - \alpha_2 + 2} \sqrt{3} \alpha_2^2 - \sqrt{\alpha_1 - \alpha_2 + 2} \left(2 \sqrt{3} \alpha_1^2 + \sqrt{3} \right) \right) + \\
&\quad \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2, \\
&\alpha_4 = -\frac{1}{6} \sqrt{-\alpha_1^2 + 2 \alpha_1 \alpha_2 - \alpha_2^2 + 4\sqrt{3}} + \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2, \\
&\alpha_5 = -\frac{1}{2} \sqrt{-\alpha_1 + \alpha_2 + 2} \sqrt{\alpha_1 - \alpha_2 + 2} \sqrt{3} + \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2 \\
&\beta_1, \\
&\beta_2 = \beta_1 - 1.
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_1, \\
&\alpha_2, \\
&\alpha_3 = -\frac{1}{18} \sqrt{-\alpha_1 + \alpha_2 + 2} \left(4 \sqrt{\alpha_1 - \alpha_2 + 2} \sqrt{3} \alpha_1 \alpha_2 - 2 \sqrt{\alpha_1 - \alpha_2 + 2} \sqrt{3} \alpha_2^2 - \sqrt{\alpha_1 - \alpha_2 + 2} \left(2 \sqrt{3} \alpha_1^2 + \sqrt{3} \right) \right) + \\
&\quad \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2, \\
&\alpha_4 = \frac{1}{6} \sqrt{-\alpha_1^2 + 2 \alpha_1 \alpha_2 - \alpha_2^2 + 4\sqrt{3}} + \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2, \\
&\alpha_5 = \frac{1}{2} \sqrt{-\alpha_1 + \alpha_2 + 2} \sqrt{\alpha_1 - \alpha_2 + 2} \sqrt{3} + \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2, \\
&\beta_1, \\
&\beta_2 = \beta_1 - 1.
\end{aligned}$$

- Now, we provide an example where the zeroes and poles of f form an arithmetic progression. Let $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{4, 5\}, \{2, 3\}, \{1\})$ and $(l_1, l_2, l_3, l_4, l_5) = (1, 1, 1, 1, 1)$. We get the following system of equations

$$\begin{aligned}
\alpha_1 - \alpha_3^2 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 - \alpha_3 - \alpha_4 \alpha_5 &= 0 \\
\alpha_2 + \alpha_3 - \alpha_4 - \alpha_5 + 1 &= 0 \\
\beta_1 - \beta_2 + 1 &= 0.
\end{aligned}$$

Now assume that $\alpha_5, \alpha_3, \alpha_1, \alpha_4, \alpha_2$ form an arithmetic progression in this order. We have that

$$\begin{aligned}
\alpha_5 + \alpha_1 - 2\alpha_3 &= 0 \\
\alpha_1 + \alpha_2 - 2\alpha_4 &= 0 \\
\alpha_5 + \alpha_2 - 2\alpha_1 &= 0.
\end{aligned}$$

The two systems of equations above simplify to

$$\begin{aligned}\alpha_1 &= \alpha_5 - 1, \\ \alpha_2 &= \alpha_5 - 2, \\ \alpha_3 &= \alpha_5 - \frac{1}{2}, \\ \alpha_4 &= \alpha_5 - \frac{3}{2}, \\ \beta_1 &= \beta_2 - 1.\end{aligned}$$

Thus the rational functions are given by

$$\begin{aligned}f(x) &= \frac{(x - \alpha_5 + 1/2)(x - \alpha_5 + 1)(x - \alpha_5 + 2)}{(x - \alpha_5)^2(x - \alpha_5 + 3/2)^2} \\ g(x) &= (x - \beta_2)(x - \beta_2 + 1) \\ h(x) &= \beta_2 + \frac{x - \alpha_5 + 1}{(x - \alpha_5)(x - \alpha_5 + 3/2)}.\end{aligned}$$

- Let $n = 6$ and we fix the vector (l_1, l_2, \dots, l_6) to be $(1, 1, 2, 1, 1, 1)$. The procedure `CFunc(3, 6, 1: exptup:=[[1, 1, 2, 1, 1, 1]])`; yields that there are 30 systems to deal with. One of these is as follows. The partition is given by $[\{4\}, \{1, 2\}, \{5, 6\}, \{3\}]$ and the system of equations is

$$\begin{aligned}\alpha_1 - \alpha_4 + \alpha_5 - \alpha_6 - \beta_1 + \beta_2 &= 0, \\ \alpha_2\beta_1 - \alpha_2\beta_3 + 1/4\alpha_4^2 - \alpha_4\alpha_5 + 1/2\alpha_4\alpha_6 - 1/2\alpha_4\beta_2 + 1/2\alpha_4\beta_3 + \alpha_5^2 - \alpha_5\alpha_6 - \\ \alpha_5\beta_1 + \alpha_5\beta_2 + 1/4\alpha_6^2 - 1/2\alpha_6\beta_2 + 1/2\alpha_6\beta_3 + 1/4\beta_2^2 - 1/2\beta_2\beta_3 + 1/4\beta_3^2 &= 0, \\ \alpha_2\beta_2 - \alpha_2\beta_3 + 1/4\alpha_4^2 - 1/2\alpha_4\alpha_6 - 1/2\alpha_4\beta_2 + 1/2\alpha_4\beta_3 + 1/4\alpha_6^2 - 1/2\alpha_6\beta_2 + \\ 1/2\alpha_6\beta_3 + 1/4\beta_2^2 - 1/2\beta_2\beta_3 + 1/4\beta_3^2 &= 0, \\ \alpha_3 - 1/2\alpha_4 - 1/2\alpha_6 + 1/2\beta_2 - 1/2\beta_3 &= 0, \\ \alpha_4^2\beta_1 - \alpha_4^2\beta_2 + 4\alpha_4\alpha_5\beta_2 - 4\alpha_4\alpha_5\beta_3 - 2\alpha_4\alpha_6\beta_1 - 2\alpha_4\alpha_6\beta_2 + \\ 4\alpha_4\alpha_6\beta_3 - 2\alpha_4\beta_1\beta_2 + 2\alpha_4\beta_1\beta_3 + 2\alpha_4\beta_2^2 - 2\alpha_4\beta_2\beta_3 - 4\alpha_5^2\beta_2 + 4\alpha_5^2\beta_3 + \\ 4\alpha_5\alpha_6\beta_2 - 4\alpha_5\alpha_6\beta_3 + 4\alpha_5\beta_1\beta_2 - 4\alpha_5\beta_1\beta_3 - 4\alpha_5\beta_2^2 + 4\alpha_5\beta_2\beta_3 + \\ \alpha_6^2\beta_1 - \alpha_6^2\beta_2 - 2\alpha_6\beta_1\beta_2 + 2\alpha_6\beta_1\beta_3 + 2\alpha_6\beta_2^2 - 2\alpha_6\beta_2\beta_3 + \beta_1\beta_2^2 - \\ 2\beta_1\beta_2\beta_3 + \beta_1\beta_3^2 - \beta_2^3 + 2\beta_2^2\beta_3 - \beta_2\beta_3^2 &= 0.\end{aligned}$$

- Finally, we show an example with $n = 7$. Using our Magma procedure

`CFunc(3, 7, 1:PSet:=[[{1},{2,3},{4,5},{6,7}]], exptup:=[[1,1,1,1,1,1,1]])`;

we get the system of equations

$$\begin{aligned}
\alpha_1\beta_1 - \alpha_1\beta_3 + \alpha_3^2 - \alpha_3\alpha_6 - \alpha_3\alpha_7 - \alpha_3\beta_1 + \alpha_3\beta_3 + \alpha_6\alpha_7 &= 0, \\
\alpha_1\beta_2 - \alpha_1\beta_3 + \alpha_5^2 - \alpha_5\alpha_6 - \alpha_5\alpha_7 - \alpha_5\beta_2 + \alpha_5\beta_3 + \alpha_6\alpha_7 &= 0, \\
\alpha_2 + \alpha_3 - \alpha_6 - \alpha_7 - \beta_1 + \beta_3 &= 0, \\
\alpha_3^2\beta_2 - \alpha_3^2\beta_3 - \alpha_3\alpha_6\beta_2 + \alpha_3\alpha_6\beta_3 - \alpha_3\alpha_7\beta_2 + \alpha_3\alpha_7\beta_3 - \\
\alpha_3\beta_1\beta_2 + \alpha_3\beta_1\beta_3 + \alpha_3\beta_2\beta_3 - \alpha_3\beta_3^2 - \alpha_5^2\beta_1 + \alpha_5^2\beta_3 + \\
\alpha_5\alpha_6\beta_1 - \alpha_5\alpha_6\beta_3 + \alpha_5\alpha_7\beta_1 - \alpha_5\alpha_7\beta_3 + \alpha_5\beta_1\beta_2 - \alpha_5\beta_1\beta_3 - \\
\alpha_5\beta_2\beta_3 + \alpha_5\beta_3^2 - \alpha_6\alpha_7\beta_1 + \alpha_6\alpha_7\beta_2 &= 0, \\
\alpha_4 + \alpha_5 - \alpha_6 - \alpha_7 - \beta_2 + \beta_3 &= 0.
\end{aligned}$$

We note that the above system has a solution

$$\begin{aligned}
(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \beta_1, \beta_2, \beta_3) = \\
(-1, 0, 2, -1 - \sqrt{-3}, -1 + \sqrt{-3}, \frac{1 - \sqrt{-3}}{2}, \frac{1 + \sqrt{-3}}{2}, 0, -4, -1).
\end{aligned}$$

It corresponds to the example given by Ayad and Fleischmann, that is

$$f = \frac{x^4 - 8x}{x^3 + 1}, \quad g = \frac{x^2 + 4x}{x + 1}, \quad h = \frac{x^2 - 2x}{x + 1}.$$

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