Common expansions in noninteger bases

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1. Prologue

The number $z \in \mathbb{R}$ has a *common expansion*

- in real bases $1 < q_1 < \cdots < q_r, r \geq 2$
- with the digit set $A = A_m = \{-m, \ldots, 0, \ldots, m\} \subset \mathbb{Z}$

if there exists $(c_j) \in A^\infty$ satisfying

$$z = \sum_{j=1}^{\infty} \frac{c_j}{q_1^j} = \cdots = \sum_{j=1}^{\infty} \frac{c_j}{q_r^j}.$$

The existence of *common expansions* seems to be a rare event.

2. Simultaneous radix representations

Indlekofer, Kátai and Racskó (1992) called $\mathbf{a} \in \mathbb{Z}^d$ *simultaneously representable* by $\mathbf{q} \in \mathbb{Z}^d$, if there exist integers $0 \leq m_0, \ldots, m_\ell < Q := |q_1 \cdots q_d|$ such that

$$a_i = \sum_{j=0}^{\ell} m_j q_i^j, \quad i = 1, \ldots, d.$$

If $q_1, \ldots, q_d > 0$ then apart from the zero vector no integer vectors are simultaneously representable by $\mathbf{q}$. If some of the base numbers are negative, then simultaneous representations may appear.

For example take $q_1 = -2$ and $q_2 = -3$ then $Q = 2 \cdot 3 = 6$ and

$$(101)_{10} = (1431335045)_{-2} = (1431335045)_{-3}.$$

Changing the sign of the "digits" with odd position we get a common representation of 101 in bases 2 and 3 with digits from $A_6 = \{-6, \ldots, 0, \ldots, 6\}$.

Pethő (2006) gave a criterion of simultaneous representability on the one hand with the Chinese reminder theorem and, on the other hand with CNS polynomials. A similar result was proved by Kane (2006), A. Chen (2008).

3. A construction of common expansions

No results on simultaneous representability of real numbers in noninteger bases seem to have appeared in the literature. Komornik and Pethő (201?) proved recently:

Theorem 1. *Let* r *be a positive integer. There exists a positive integer* m *and an interval* I *such that for all* $z \in I$ *there exist continuum many* $(q_1, \ldots, q_r, (c_j)) \in \mathbb{R}^r \times A_m^\infty$ *such that*
$1 < q_1 < \cdots < q_r$ *and*

$$z = \sum_{j=1}^{\infty} \frac{c_j}{q_1^j} = \cdots = \sum_{j=1}^{\infty} \frac{c_j}{q_r^j}.$$

4. Basic steps of the proof

I. Choose a polynomial $P(x) = p_d x^d + \cdots + p_1 x + p_0 \in \mathbb{Z}[x]$ with r real roots $1 < \alpha_1 < \cdots < \alpha_r$. Put $m = H(P) = \max\{|p_0|, \ldots, |p_d|\}$. Then p_d admits a common expansion in $\alpha_1, \ldots, \alpha_r$ with digit set A_m because $P(\alpha_i)/\alpha_i^d = 0$, i.e.

$$p_d = \sum_{j=1}^{d} -p_{d-j}\alpha_i^{-j}, i = 1 \ldots, r.$$

II. Let $\delta = \frac{\alpha_1 - 1}{2}$, $\alpha_0 = 1, \alpha_{r+1} = \alpha_r + 1$ and

$$M = \min_{j=0}^{r} \max_{i=1}^{r} \left\{ \frac{|P(x)|}{x^d} \;:\; x \in \left[\frac{\alpha_i + \alpha_{i-1}}{2}, \frac{\alpha_i + \alpha_{i+1}}{2} \right] \right\}.$$

III. Choose ℓ large enough such that

$$\frac{M}{4} > m \sum_{j=\ell}^{\infty} \alpha_0^{-j} = \frac{m}{\alpha_0^{\ell-1}(\alpha_0 - 1)}.$$

If $(c_j) \in A_m^\infty$ with $c_1 = \cdots = c_{\ell-1} = 0$ and $\beta \in [\alpha_0, \alpha_{r+1}]$ then

$$\left| \sum_{j=\ell}^{\infty} c_j \beta^{-j} \right| \leq m \sum_{j=\ell}^{\infty} \beta^{-j} \leq m \sum_{j=\ell}^{\infty} \alpha_0^{-j} \leq \frac{M}{4}.$$

IV. Take $I = \left[p_d - \frac{M}{4}, p_d + \frac{M}{4}\right]$, $z \in I$ and $(c_j) \in A_m^\infty$ with $c_1 = \cdots = c_{\ell-1} = 0$ such that III. holds. Let

$$f_{z,(c_j)}(x) = \frac{P(x)}{x^d} - p_d + z - \sum_{j=0}^{\infty} c_j x^{-j}.$$

$\frac{P(x)}{x^d}$ has the real roots $\alpha_1, \ldots, \alpha_r$ in $[\alpha_0, \alpha_{r+1}]$. Thus $\frac{P(x)}{x^d}$ changes its sign in $\left[\frac{\alpha_i+\alpha_{i-1}}{2}, \frac{\alpha_i+\alpha_{i+1}}{2}\right]$. Moreover

$$\max\left\{\frac{|P(x)|}{x^d} \ : \ x \in \left[\frac{\alpha_i+\alpha_{i-1}}{2}, \frac{\alpha_i+\alpha_{i+1}}{2}\right]\right\} \geq M, \ i = 1, \ldots r.$$

By the choice of z and (c_j) we have

$$\left|f_{z,(c_j)}(x) - \frac{P(x)}{x^d}\right| \leq |p_d - z| + \left|\sum_{j=0}^{\infty} c_j x^{-j}\right| \leq \frac{M}{2}$$

for $x \in \left[\frac{\alpha_0+\alpha_1}{2}, \frac{\alpha_r+\alpha_{r+1}}{2}\right]$.

Thus $f_{z,(c_j)}(x)$ changes its sign in $\left[\frac{\alpha_i+\alpha_{i-1}}{2}, \frac{\alpha_i+\alpha_{i+1}}{2}\right]$ for $i = 1, \dots, r$, i.e. it has r real roots, say $q_1, \dots, q_r$. Hence

$$f_{z,(c_j)}(q_i) = \frac{P(q_i)}{q_i^d} - p_d + z - \sum_{j=0}^{\infty} c_j q_i^{-j} = 0,\ i = 1, \dots, r,$$

thus z admits the common expansions

$$z = -\frac{P(q_i)}{q_i^d} + p_d + \sum_{j=0}^{\infty} c_j q_i^{-j} = 0,\ i = 1, \dots, r.$$

There are continuum many possibilities for $(c_j) \in A_m^{\infty}$.

V. It remains to prove that we have for any r an appropriately polynomial $P(x)$. By Narkiewicz (1990) there are for any r (infinitely many) totally real number fields of degree r. Let $P_1(x)$ a defining polynomial of such a field. With appropriate choice of the integer t all roots of $P_1(x-t)$ will be larger than one.

5. Common expansions with small digit sets

The digit set A_m is in the above construction usually large. For example if $P_1(x) = x^2 - x - 1$, which has two real roots $\alpha_1 = \frac{1+\sqrt{5}}{2}, \alpha_2 = \frac{1-\sqrt{5}}{2}$ must translate by 2 to get an appropriate quadratic polynomial, which is $x^2 - 5x + 5$. Thus $m = 5$.

Using interval filling sequences Komornik and Pethő (2014) proved results in the case $r = 2$. In the sequel we use $q_1 = q, q_2 = p$. Let $p > q > 1, m \geq 1$ and denote by $C(p,q)$ the set of sequences $(c_j) \in A_m^\infty$ satisfying

$$\sum_{j=1}^{\infty} \frac{c_j}{q^j} = \sum_{j=1}^{\infty} \frac{c_j}{p^j}. \tag{1}$$

We call $C(p,q)$ *trivial* if its only element is the null sequence.

Theorem 2. *Let* $p > q > 1$.

(i) *If* $q < (1 + \sqrt{8m+1})/2$, *then* $C(p,q)$ *has the cardinality of the continuum.*

(ii) *If* $(1 + \sqrt{8m+1})/2 \le q \le m+1$, *then* $C(p,q)$ *is infinite.*

(iii) *Let* $m + 1 < q \le 2m + 1$.

(a) If

$$p \le \frac{(m+1)(q-1)}{q-m-1}, \tag{2}$$

then $C(p,q)$ *is nontrivial.*

(b) If

$$p > \frac{(m+1)(q-1)}{q-m-1}, \tag{3}$$

then $C(p,q)$ *is trivial.*

(iv) *Let* $2m + 1 < q < m + 1 + \sqrt{m(m + 1)}$.

(a) $C(p, q)$ *is a finite set.*

(b) There is a continuum of values $p > q$ *for which* $C(p, q)$ *is nontrivial.*

(c) If $p > q$ *satisfies* (3), *then* $C(p, q)$ *is trivial.*

(v) *If* $q \geq m + 1 + \sqrt{m(m + 1)}$, *then* $C(p, q)$ *is trivial.*

If e.g. $m = 1$ and $q < 2$ and $p > q$, then there is a continuum of sequences $(c_j) \in \{-1, 0, 1\}^\infty$ satisfying

$$\sum_{j=1}^{\infty} \frac{c_j}{q^j} = \sum_{j=1}^{\infty} \frac{c_j}{p^j}.$$

If $2 \geq q \leq 3$ then this equality has for any $p > q$ infinitely many solutions, and if $q \geq 2+\sqrt{2}$, then only the trivial sequence $(c_j) = 0^\infty$ satisfies the former equality.

6. Outline of the proof of Theorem 2

The basic tool of the proof of Theorem 2 is a variant of a classical theorem of Kakeya (1914).

Proposition 3. *Let* $A = \{-m, \dots, 0, \dots, m\}$ *and let* $\sum_{k=1}^{\infty} r_k$ *be a convergent series of positive numbers, satisfying the inequalities*

$$r_n \le 2m \sum_{k=n+1}^{\infty} r_k \tag{4}$$

for all $n = 1, 2, \dots$. *Then the sums*

$$\sum_{k=1}^{\infty} c_k r_k, \quad (c_k) \in A^{\infty} \tag{5}$$

fill the interval

$$\left[-m \sum_{k=1}^{\infty} r_k, m \sum_{k=1}^{\infty} r_k \right]. \tag{6}$$

We apply Proposition 3 to subsequences of $(q^{-i} - p^{-i})$.

Lemma 4. *If* $1 < q < (1+\sqrt{8m+1})/2$ *and* $p > q$, *then the sequence* $(r_k)_{k\in\mathbb{N}} := (q^{-i} - p^{-i})_{i\in\mathbb{N}\setminus n\mathbb{N}}$ *satisfies*

$$r_n \leq 2m \sum_{k=n+1}^{\infty} r_k$$

for all sufficiently large integers n.

Lemma 5. *Let* $p > q > 1$. *The sequence*

$$\left(\frac{\sum_{i=n+1}^{\infty}(q^{-i} - p^{-i})}{q^{-n} - p^{-n}}\right)_{n=1}^{\infty}$$

is strictly decreasing, and tends to $1/(q-1)$.

Proof of Theorem 2 (i) *If* $q < (1+\sqrt{8m+1})/2$, *then* $C(p,q)$ *has the cardinality of the continuum.*

Fix a large positive integer n such that the sequence $(r_k)_{k\in\mathbb{N}} := (q^{-i}-p^{-i})_{i\in\mathbb{N}\setminus n\mathbb{N}}$ satisfies

$$r_n \le 2m \sum_{k=n+1}^{\infty} r_k.$$

Next we fix a large positive integer N such that

$$\left[-m\sum_{i=N}^{\infty}(q^{-in}-p^{-in}), m\sum_{i=N}^{\infty}(q^{-in}-p^{-in})\right] \subset \left[-m\sum_{i\in\mathbb{N}\setminus n\mathbb{N}}(q^{-in}-p^{-in}), m\sum_{i\in\mathbb{N}\setminus n\mathbb{N}}(q^{-in}-p^{-in})\right]. \quad (7)$$

This is possible because the right side interval contains 0 in its interior.

The sets

$$B := \mathbb{N} \setminus n\mathbb{N},$$
$$C := \{jn \ : \ j = N, N+1, \ldots\},$$
$$D := \{jn \ : \ j = 1, \ldots, N-1\}$$

form a partition of $\mathbb{N}$.

Choose an arbitrary sequence $(c_i)_{i \in C} \in A^C$; there is a continuum of such sequences because C is an infinite set. Since

$$-\sum_{i \in C} c_i (q^{-i} - p^{-i})$$

belongs to the left side interval in (7), applying Proposition 3 there exists a sequence $(c_i)_{i \in B} \in A^B$ such that

$$\sum_{i \in B \cup C} c_i (q^{-i} - p^{-i}) = 0.$$

Setting $c_i = 0$ for $i \in D$ we obtain a sequence $(c_i)_{i \in \mathbb{N}} \in C(p, q)$.

Proof of Theorem 2 (i) *If* $(1+\sqrt{8m+1})/2 \le q \le m+1$, *then* $C(p,q)$ *is infinite.*

Since $q \le m+1$, by Lemma 5 we have

$$0 < q^{-n} - p^{-n} < (q-1)\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i}) \le m\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i}).$$

Since $q \le 2m+1$, Lemma 5 also shows that the condition (4) of Proposition 3 is satisfied for the alphabet $A = \{-m,\dots,m\}$ and the sequence $r_k := q^{-k-n} - p^{-k-n}$, $k = 1,2,\dots$. Hence there exists a sequence $(c_i)_{i=n+1}^{\infty} \in A^{\infty}$ satisfying

$$q^{-n} - p^{-n} = \sum_{i=n+1}^{\infty} c_i(q^{-i}-p^{-i});$$

setting $c_1 = \cdots = c_{n-1} = 0$ and $c_n = -1$ this yields (1).

6. Open problems

1. Find the optimal conditions on p and q in Theorem 2. In particular,

(a) Can $C(p,q)$ be infinite for some $p > q > m+1$?

(b) In case $2m+1 < q < m+1+\sqrt{m(m+1)}$ is $C(p,q)$ nontrivial for *all* $p > q$ sufficiently close to q?

2. Does there exist three (or more) different bases such that a continuum of (or infinitely many) real numbers have identical expansions in all three bases with digit set $\{-1,0,1\}$?

3. Given two bases $p > q > 1$ investigate the set of points of the form

$$\sum_{i=1}^{\infty} c_i(p^{-i} - q^{-i}), \quad (c_i) \in A^{\infty}.$$