Abh. Math. Sem. Univ. Hamburg **65** (1995), 365–383

Complete Solution of a Family of Quartic Thue Equations

By G. Lettl and A. Pethő

1 Introduction

Let $F \in \mathbb{Z}[X, Y]$ be an irreducible binary form of degree ≥ 3 and $0 \neq m \in \mathbb{Z}$. The diophantine equation

$$F = m \tag{1}$$

is called *Thue equation* in honour of the Norwegian mathematician Axel Thue, who proved in 1909 (see [18]) that (1) has only finitely many integral solutions (i. e. $(x, y) \in \mathbb{Z}^2$ with $F(x, y) = m$). In 1968, A. Baker [1] showed that Thue equations can be solved effectively. Since then, much work was done to lower the effective upper bound for the solutions of (1) to some reasonable size, and nowadays it is no problem to calculate all solutions of a Thue equation of low degree with a computer (see e. g. [12], [19], [2]).

On the other hand, it was quite recently that parametrized families of Thue equations were completely solved. For example, one knows all solutions of the Thue equations

$$X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3 = \pm 1 \qquad (n \in \mathbb{Z}) \tag{2.a}$$

and

$$X^4 - aX^3Y - X^2Y^2 + aXY^3 + Y^4 = 1 \qquad (a \in \mathbb{Z}) \tag{2.b}$$

(see [16], [8], [11], [9]).

For the following families the solutions are known for large parameters:

$$X^3 - (n-1)X^2Y + nXY^2 - Y^3 = 1 \qquad (n \geq 3.33 \cdot 10^{23}) \tag{2.c}$$

and

$$X^4 - aX^3Y - 3X^2Y^2 + aXY^3 + Y^4 = \pm 1 \qquad (a > 9.9 \cdot 10^{27}) \tag{2.d}$$

This paper evolved from a visit of the first author to the University of Debrecen, Hungary, which was supported by the Austrian-Hungarian Science Cooperation project Nr. 10-U-3. Research of the second author was partially supported by Hungarian National Foundation for Scientific Research Grant 1641/90.

(see [10], [11]).

All these families of equations are stably solvable in the sense of [17], i. e. up to finitely many exceptions, all solutions are given by finitely many pairs of rational functions of the parameter.

In this paper we consider the family of Thue equations

$$F_a := X^4 - aX^3Y - 6X^2Y^2 + aXY^3 + Y^4 = c \tag{3}$$

with $a \in \mathbb{Z}$ and $c \in \{\pm 1, \pm 4\}$.

If $a \in \{0, \pm 3\}$, F_a factors into two quadratic forms and it is quite elementary to solve (3). For all other values of a, F_a is irreducible. Our main result is the following

Theorem 1. *The only solutions of* (3) *are*

$$F_a(\pm 1, 0) = F_a(0, \pm 1) = 1,$$
$$F_a(\pm 1, 1) = F_a(\pm 1, -1) = -4,$$

(these solutions we call 'trivial solutions'*), and*

$$F_1(\pm 1, \pm 2) = F_1(\mp 2, \pm 1) = F_{-1}(\pm 2, \pm 1) = F_{-1}(\mp 1, \pm 2) = -1,$$
$$F_1(\pm 3, \pm 1) = F_1(\mp 1, \pm 3) = F_{-1}(\pm 1, \pm 3) = F_{-1}(\mp 3, \pm 1) = 4,$$
$$F_4(\pm 2, \pm 3) = F_4(\mp 3, \pm 2) = F_{-4}(\pm 3, \pm 2) = F_{-4}(\mp 2, \pm 3) = 1,$$
$$F_4(\pm 5, \pm 1) = F_4(\mp 1, \pm 5) = F_{-4}(\pm 1, \pm 5) = F_{-4}(\mp 5, \pm 1) = -4.$$

Let α denote a root of the polynomial

$$f_a := F_a(X, 1) = X^4 - aX^3 - 6X^2 + aX + 1 \in \mathbb{Z}[X]. \tag{4}$$

From Theorem 1 we derive the following

Theorem 2. *Let $a \in \mathbb{Z} \setminus \{0, \pm 3\}$. We have $\mathbb{Z}[\alpha] = \mathbb{Z}[\delta]$ with a $\delta \in \mathbb{Z}[\alpha]$ if and only if*

a arbitrary and $\delta = \pm\alpha^{\pm 1} + d$ or

$a = 1$ and $\delta = \pm\theta + d$, where $\theta = -1 + 3\alpha^2 - \alpha^3$ or $\theta = 8 + 25\alpha + 2\alpha^2 - 4\alpha^3$ are roots of the polynomial $X^4 - 19X^3 - 24X^2 - 9X - 1$ or

$a = 4$ and $\delta = \pm\theta + d$, where $\theta = 3 + 9\alpha - 22\alpha^2 + 4\alpha^3$ or $\theta = -13 - 74\alpha - 30\alpha^2 + 9\alpha^3$ are roots of the polynomial $X^4 + 72X^3 - 84X^2 + 32X - 4$.

Here $d \in \mathbb{Z}$ is an arbitrary integer.

Similar results were proved by using a different method in [9] for the order $\mathbb{Z}[\alpha]$, where α is a root of the polynomial associated to (2.b).

At first, we will state some elementary transformation properties of the form F_a.

Lemma 1. *Let* $a, c, x, y \in \mathbb{Z}$.

(i) *If* $F_a(x,y) = c$ *then* $F_a(x-y, x+y) = -4c$.

(ii) *If* $F_a(x,y) = -4c$ *then* $F_a(\frac{x-y}{2}, \frac{x+y}{2}) = c$.

(iii) $F_a(x,y) = F_a(-x,-y) = F_a(y,-x) = F_a(-y,x)$.

(iv) $F_{-a}(x,y) = F_a(y,x)$.

The proof of lemma 1 is done by direct calculation. Note that $F_a(x,y) = -4c$ implies $x \equiv y \bmod (2)$, so $\frac{x \pm y}{2}$ are indeed integers in this case.

Parts (i), (ii) and (iv) of lemma 1 show that there is a bijection between the solutions of $F_a = c$, $F_a = -4c$ and $F_{-a} = c$. If $(x,y) \in \mathbb{Z}^2$ is a solution of (3) with $x = 0$ (or $y = 0$), this implies $y = \pm 1$ (or $x = \pm 1$) and $c = 1$. It is easy to check that for $y = \pm 1$ or $x = \pm 1$ one obtains just solutions stated in theorem 1. By (iii), we can also restrict to positive y.

Therefore, to prove theorem 1 we have only to show the following:

For $a \in \mathbb{N} \backslash \{3\}$ *the only solutions* $(x,y) \in \mathbb{Z}^2$ *of the equation* $F_a = \pm 1$ *with* $|x| \geq 2$ *and* $y \geq 2$ *are* $F_4(2,3) = F_4(-3,2) = 1$.

2 Simplest Quartic Number Fields

The polynomials $f_a = F_a(X, 1)$, which are given in (4), are irreducible for each $a \in \mathbb{N} \setminus \{3\}$. If $f_a(\alpha) = 0$, one can calculate that also $f_a(\frac{\alpha-1}{\alpha+1}) = 0$. Thus the rational map $x \mapsto \frac{x-1}{x+1}$ permutes the roots of f_a and, for $a \neq 3$, $K := \mathbb{Q}(\alpha)$ is a real quartic number field with cyclic Galois group $G = \langle \tau \rangle$, generated by the automorphism $\tau : \alpha \mapsto \frac{\alpha-1}{\alpha+1}$.

These socalled "simplest" quartic number fields were investigated by M. N. Gras in [5], I.5 (see also [6]). They have very similar properties as the "simplest" cubic number fields, which were called so by D. Shanks [14] and are generated by the polynomials associated to the Thue equation (2.a).

We order the four real roots of f_a by their size and denote them by

$$\alpha_4 < \alpha_3 < \alpha_2 < \alpha_1 .$$

For the size of the roots we have the following estimations, which hold either for all $a \in \mathbb{N}$ or for the indicated values of a:

$$a + \frac{1}{a} < a + \frac{5}{a} - \frac{1}{a^2} \underset{(a>20)}{<} \alpha_1 = \alpha < a + \frac{5}{a} \tag{5.a}$$

$$1 - \frac{2}{a} < \tau(\alpha) = \alpha_2 = \frac{\alpha - 1}{\alpha + 1} < 1 - \frac{2}{a} + \frac{3}{a^2} \tag{5.b}$$

$$-\frac{1}{a} < -\frac{1}{a} + \frac{4}{a^3} \underset{(a>5)}{<} \tau^2(\alpha) = \alpha_3 = -\frac{1}{\alpha} < -\frac{1}{a} + \frac{5}{a^3} \tag{5.c}$$

$$-1 - \frac{2}{a} - \frac{2}{a^2} < \tau^3(\alpha) = \alpha_4 = -\frac{\alpha+1}{\alpha-1} \underset{(a>9)}{<} -1 - \frac{2}{a} - \frac{1}{a^2} < -1 - \frac{1}{a} \tag{5.d}$$

These inequalities can be obtained by considering the sign of f_a at the points $a+\frac{5}{a}, a+\frac{5}{a}-\frac{1}{a^2}, \dots$

Now let $a \in \mathbb{N} \setminus \{3\}$, $\alpha = \alpha_1 > 1$ be the largest root of f_a, $K = \mathbb{Q}(\alpha)$ and $\beta := \frac{\alpha+1}{\alpha-1} = -\tau^3(\alpha) > 1$. We will repeatedly use that

$$\tau(\alpha) = \frac{1}{\beta}, \quad \tau(\beta) = -\alpha, \quad \tau^2(\alpha) = -\frac{1}{\alpha}, \quad \tau^2(\beta) = -\frac{1}{\beta}. \tag{6}$$

For the algebraic part of the proof of our theorem we will work with the order $\mathfrak{O} := \mathbb{Z}[\alpha, \beta]$ and its group of units, $\mathfrak{O}^\times$.

Lemma 2. *The order $\mathfrak{O}$ is invariant under $G = \mathrm{Gal}(K/\mathbb{Q})$ and has discriminant $D_\mathfrak{O} = (16+a^2)^3$.*

Proof. By (6), the conjugates of α are $\frac{1}{\beta}, -\frac{1}{\alpha}, -\beta$. Thus $-\frac{1}{\alpha} = \alpha^3 - a\alpha^2 - 6\alpha + a$ and $\frac{1}{\beta} = -\beta^3 - a\beta^2 + 6\beta + a$ show the invariance of $\mathfrak{O}$ under G.

Expressing β, β^2, β^3 with respect to the basis $\{1, \alpha, \alpha^2, \alpha^3\}$ shows that $\{\tilde{\alpha}, \alpha, \alpha^2, \alpha^3\}$ is a $\mathbb{Z}$-basis for $\mathfrak{O}$, where

$$\tilde{\alpha} = \begin{cases} \frac{1+\alpha^3}{2} & \text{if } a \text{ is odd,} \\ \frac{1+\alpha+\alpha^2+\alpha^3}{2} & \text{if } a \text{ is even.} \end{cases}$$

Therefore $(\mathfrak{O} : \mathbb{Z}[\alpha]) = 2$ and $D_\mathfrak{O} = \frac{1}{4} D_{\mathbb{Z}[\alpha]} = \frac{1}{4} D(f_a) = (16+a^2)^3$, where $D(f_a)$ denotes the discriminant of the polynomial f_a.

Obviously, α and β are units of $\mathfrak{O}$. The following proposition is the quartic analogue of theorem 3.10 in [15], which deals with simplest cubic fields. Furthermore, our proof shows that $\{\alpha, \beta\}$ is a basis for the relative units of $\mathfrak{O}$.

Let k be the quadratic subfield of K and $\varepsilon > 1$ be the fundamental unit of the quadratic order $\mathfrak{o} := \mathfrak{O} \cap k$.

Proposition 1. *We have $\mathfrak{O}^\times = \langle -1, \alpha, \beta, \eta \rangle$ with $\eta \in \{\varepsilon, \sqrt{\varepsilon}, \sqrt{\alpha\beta\varepsilon}\}$.*

For the proof we will need the following result from [7].

Lemma 3. *Let γ be an algebraic integer of degree d and*

$$M(\gamma) := \prod_{k=1}^{d} \max\{1, |\gamma^{(k)}|\},$$

where $\gamma^{(k)}$ $(1 \le k \le d)$ are the conjugates of γ. Then

$$|D_{\mathbb{Z}[\gamma]}| \le d^d M(\gamma)^{2(d-1)}.$$

Proof of Proposition 1. Let R_0 be the regulator of the group $\langle\varepsilon, \alpha, \beta\rangle$. Then $R_0 = 2\log\varepsilon(\log^2\alpha - \log^2\beta) \neq 0$ shows that $\{\varepsilon, \alpha, \beta\}$ are multiplicatively independent.

First we will show that

$$\{\alpha, \beta\} \textit{ can be extended to a basis of } \mathfrak{O}^\times.$$

Let us assume on the contrary that there exists a $\gamma \in \mathfrak{O}^\times$ and k, l, $n \in \mathbb{Z}$ with $n \geq 2$, $(k, l, n) = 1$ and

$$\gamma^n = \alpha^k \beta^l. \tag{7}$$

Without restriction, we suppose that $|k|, |l| \leq \frac{n}{2}$ and $\gamma > 0$.

From $\tau(\gamma)^n = (-1)^l\alpha^l\beta^{-k}$ and $\tau^{-1}(\gamma)^n = \alpha^{-l}(-1)^k\beta^k$ we see that an even n also forces l and k to be even, which contradicts $(k, l, n) = 1$. Thus n must be odd, and we have $n \geq 3$ and $|k|, |l| < \frac{n}{2}$.

If $\frac{2}{3}n < |k| + |l|$, we consider $\gamma' := \alpha^{\operatorname{sgn}(k)}\beta^{\operatorname{sgn}(l)}\gamma^{-2}$ instead of γ, which yields $(\gamma')^n = \alpha^{k'}\beta^{l'}$ with $k' = n\operatorname{sgn}(k) - 2k$, $l' = n\operatorname{sgn}(l) - 2l$ and $0 < |k'| + |l'| \leq \frac{2}{3}n$. Therefore we can assume that we have a unit $\gamma \in \mathfrak{O}^\times$ with $\gamma = \alpha^{\frac{k}{n}}\beta^{\frac{l}{n}}$ and $0 < |\frac{k}{n}| + |\frac{l}{n}| \leq \frac{2}{3}$.

Let $M(\gamma)$ be defined as in lemma 3. From (5.a-d) we obtain $M(\alpha) = M(\beta) = \alpha \cdot \beta \leq (a + \frac{5}{a})(1 + \frac{2}{a} + \frac{2}{a^2})$. Observe that $M(\gamma_1\gamma_2) \leq M(\gamma_1) \cdot M(\gamma_2)$, thus $M(\gamma) \leq M(\alpha)^{|\frac{k}{n}|}M(\beta)^{|\frac{l}{n}|} \leq M(\alpha)^{\frac{2}{3}}$. Now lemma 3 yields

$$(16 + a^2)^3 = D_{\mathfrak{O}} \leq D_{\mathbb{Z}[\gamma]} \leq 4^4 M(\gamma)^6 \leq 4^4 M(\alpha)^4.$$

But $(16 + a^2)^3 \leq 4^4\left((a + \frac{5}{a})(1 + \frac{2}{a} + \frac{2}{a^2})\right)^4$ does not hold for $a \geq 20$, which contradicts the existence of γ and proves our assertion for $a \geq 20$.

By [13], 6.22, p. 366, we have the following lower bound for the regulator R of $\mathfrak{O}^\times$

$$R \geq \frac{\log^3(D_{\mathfrak{O}}/16)}{80\sqrt{10}}.$$

The quotient R_0/R gives the index of $\langle\varepsilon, \alpha, \beta\rangle$ in $\mathfrak{O}^\times$, for which we can compute an upper bound by the above inequality for each $a \neq 3$ with $1 \leq a \leq 19$. Hence, if (7) holds, we have $3 \leq n \leq \frac{R_0}{R}$, $|k|, |l| \leq \frac{n}{2}$ and $0 < |k| + |l| \leq \frac{2}{3}n$. We checked all triplets (n, k, l) for $1 \leq a \leq 19$ and $a \neq 3$ by computer and never found a solution of (7), i. e. our assertion holds for all positive integers $a \neq 3$.

Now let $\eta \in \mathfrak{O}^\times$ with $\mathfrak{O}^\times = \langle -1, \alpha, \beta, \eta\rangle$, and consider the absolute value of the relative norm map from K to k

$$\mathcal{N} : \mathfrak{O}^\times \to \langle\varepsilon\rangle$$
$$\gamma \mapsto |N_{K/k}(\gamma)|$$

Since $N_{K/k}(\alpha) = N_{K/k}(\beta) = -1$, we have $\langle -1, \alpha, \beta\rangle \subset \ker(\mathcal{N})$. On the other hand, $\mathcal{N}(\varepsilon) = \varepsilon^2$ shows that $(\langle\varepsilon\rangle : \mathcal{N}(\mathfrak{O}^\times)) \leq 2$, $\mathcal{N}(\eta) \neq 1$ and $\ker(\mathcal{N}) = \langle -1, \alpha, \beta\rangle$.

Therefore either $\mathcal{N}(\eta) = \varepsilon^2$, which yields $\eta\varepsilon^{-1} \in \ker(\mathcal{N})$ and $\mathfrak{O}^\times = \langle -1, \alpha, \beta, \varepsilon \rangle$, or $\mathcal{N}(\eta) = \varepsilon$, which yields $\eta^2\varepsilon^{-1} \in \ker(\mathcal{N})$. In the latter case, we can choose η such that $\eta^2 = \varepsilon\alpha^k\beta^l$ with $k, l \in \{0,1\}$. Considering the signs of the conjugates, we can see that either $\mathfrak{O}^\times = \langle -1, \alpha, \beta, \sqrt{\varepsilon} \rangle$ (and $N_{k/\mathbb{Q}}(\varepsilon) = 1$) or $\mathfrak{O}^\times = \langle -1, \alpha, \beta, \sqrt{\alpha\beta\varepsilon} \rangle$ (and $N_{k/\mathbb{Q}}(\varepsilon) = -1$).

3 Preparations for the Proof of Theorem 1

We will use the notations introduced in the last chapter. Let $a \in \mathbb{N} \setminus \{3\}$ and suppose that there exist $x, y \in \mathbb{Z}$ with $|x| \geq 2$, $y \geq 2$ and

$$F_a(x, y) = \pm 1 .$$

Since $F_a(x, y) = \prod_{i=1}^4 (x - \alpha_i y)$ is just the norm from $K = \mathbb{Q}(\alpha)$ to $\mathbb{Q}$ of

$$\gamma_1 := x - \alpha_1 y \in \mathfrak{O}^\times,$$

proposition 1 yields that there exist $u_1, u_2, u_3 \in \frac{1}{2}\mathbb{Z}$ with

$$|\gamma_1| = \varepsilon^{u_1}\alpha^{u_2}\beta^{u_3} \tag{8.a}$$

Using (6), we obtain for the conjugates $\gamma_{i+1} := \tau(\gamma_i) = x - \alpha_{i+1} y$ $(1 \leq i \leq 3)$:

$$|\gamma_2| = \varepsilon^{-u_1}\beta^{-u_2}\alpha^{u_3} \tag{8.b}$$
$$|\gamma_3| = \varepsilon^{u_1}\alpha^{-u_2}\beta^{-u_3} \tag{8.c}$$
$$|\gamma_4| = \varepsilon^{-u_1}\beta^{u_2}\alpha^{-u_3} \tag{8.d}$$

From $\prod_{i=1}^4 |\frac{x}{y} - \alpha_i| = \frac{1}{y^4} \leq \frac{1}{16}$ one sees that the rational $\frac{x}{y}$ is close to one of the zeroes of f_a. Let $j \in \{1, \ldots, 4\}$ with $|x - \alpha_j y| = \min\{|x - \alpha_i y| \mid 1 \leq i \leq 4\}$. At first we will show that

it suffices to consider the cases $j = 1$ and $j = 4$.

Let $1 \leq i \leq 4$ and $i \neq j$. Then

$$|y||\alpha_i - \alpha_j| \leq |x - \alpha_i y| + |x - \alpha_j y| \leq 2|x - \alpha_i y|,$$

which implies

$$|x - \alpha_j y| \leq \frac{8}{\prod_{i=1, i\neq j}^4 (|\alpha_i - \alpha_j||y|)} = \frac{8}{|f_a'(\alpha_j)|}|y|^{-3}. \tag{9}$$

The inequalities (5a)–(5d) and a direct calculation for some small values of a show that $\frac{8}{|f_a'(\alpha_j)|} \leq 2$ holds for all $a \geq 1$. Thus

$$\left|\alpha_j - \frac{x}{y}\right| \leq \frac{1}{2y^2},$$

which means that $\frac{x}{y}$ is a convergent to α_j. Now $|x| \leq 2|\alpha_j y|$ follows from the inequality

$$|x| \leq |x - \alpha_j y| + |\alpha_j y| \leq |\alpha_j y| + \frac{1}{y^2}\,.$$

We have

$$\left|-\frac{1}{\alpha_j} + \frac{y}{x}\right| = \left|\frac{\alpha_j y - x}{\alpha_j x}\right| \leq \frac{1}{2|y||\alpha_j x|} < \frac{1}{|yx|} \leq \frac{1}{4}\,.$$

As $|\alpha_i - \alpha_j| > 0.5$ for all $1 \leq i < j \leq 4$, the last inequality means that

$$\left|-\frac{1}{\alpha_j}x + y\right| = \min_{1\leq i\leq 4}\{|\alpha_i x - y|\}\,.$$

As $-1/\alpha_2 = \alpha_4$ and $-1/\alpha_3 = \alpha_1$, the claim is confirmed.

We call (x, y) a solution of type I (resp. type II) if $j = 1$ (resp. $j = 4$). In the following we will show that in both cases we obtain non vanishing integers N_1, $N_2 \in \mathbb{Z}$ with

$$\log|N_1 \log\alpha + N_2 \log\beta| < -c \max\{|N_1|, |N_2|\} \log\alpha\,.$$

Using a lower bound for this linear form in two logarithms, we will arrive at a contradiction to the existence of a solution (x, y) for sufficiently large a.

Case 1: Solutions of Type I

We suppose that the solution (x, y) is of type I, i. e.

$$\left|\frac{x}{y} - \alpha_1\right| = \min\left\{\left|\frac{x}{y} - \alpha_i\right| \,\middle|\, 1 \leq i \leq 4\right\}\,.$$

Then obviously $0 < \gamma_2 < \gamma_3 < \gamma_4$ and $\operatorname{sgn}(\gamma_1) = \operatorname{sgn}(F_a(x, y))$.

Our next aim is to obtain estimations for the γ_i. We have $f_a'(\alpha_1) > a^3$ for $a > 20$ by (5a) and a simple calculation. The same is true for $a \geq 1$, which can be easily seen by computing the roots. Thus by (9) we obtain

$$|\gamma_1| < \frac{8}{a^3 y^3} \leq \frac{1}{a^3}\,. \tag{10.a}$$

Therefore $\alpha_1 - \frac{1}{2a^3} < \frac{x}{y} < \alpha_1 + \frac{1}{2a^3}$ and $y(\alpha_1 - \alpha_i - \frac{1}{2a^3}) < \gamma_i < y(\alpha_1 - \alpha_i + \frac{1}{2a^3})$ for $2 \leq i \leq 4$. Using (5.a-c) and calculating for some small values of a with a computer yields

$$y(a-1) < y\left(a - 1 + \frac{7}{a} - \frac{5}{a^2}\right) \underset{(a>8)}{<} \gamma_2 \tag{10.b}$$

$$ya < \gamma_3 < y\left(a + \frac{6}{a}\right) \tag{10.c}$$

Eliminating x and y from the equations

$$\gamma_i = x - \alpha_i y \qquad (i = 1, 2, 4)$$

yields via the socalled *Siegel's identity* the following:

$$0 < \frac{\gamma_4(\alpha_1 - \alpha_2)}{\gamma_2(\alpha_1 - \alpha_4)} = 1 - \frac{\gamma_1(\alpha_2 - \alpha_4)}{\gamma_2(\alpha_1 - \alpha_4)}.$$

Using $\frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_4} = \beta^{-1}$, $\frac{\alpha_2 - \alpha_4}{\alpha_1 - \alpha_4} = \frac{2}{\alpha + 1}$ and (8.b,d) we obtain

$$\alpha^{-2u_3}\beta^{2u_2 - 1} = 1 - \frac{\gamma_1}{\gamma_2}\frac{2}{\alpha + 1}. \tag{11}$$

Since we deal with solutions of type I, $|\gamma_1|$ is very small and the above expression very near to 1.

Lemma 4. *For $a > 0$ we have*

$$|2u_3 \log \alpha - (2u_2 - 1)\log \beta| < \frac{|\gamma_1|}{\gamma_2}\frac{2}{\alpha}.$$

Proof. If $\gamma_1 > 0$, (11) yields

$$|2u_3 \log \alpha - (2u_2 - 1)\log \beta| = -\log\left(1 - \frac{\gamma_1}{\gamma_2}\frac{2}{\alpha + 1}\right).$$

Using $-\log(1 - x) < x + x^2$, which holds for $x < 0.68$, and $\frac{\gamma_1}{\gamma_2} < \frac{1}{2a^3(a-1)}$ for $a > 1$, one easily shows the assertion. For $a = 1$ a direct calculation is used.
If $\gamma_1 < 0$, the assertion follows with $\log(1 + x) \leq x$.

Since $\log \alpha > \log a$, $\log \beta \sim \frac{2}{a}$ and $\frac{|\gamma_1|}{\gamma_2}\frac{2}{\alpha} < \frac{1}{a^5 - a^4}$ we can deduce that $2u_3 \neq 0$, $2u_2 - 1 \neq 0$, $\operatorname{sgn}(2u_3) = \operatorname{sgn}(2u_2 - 1)$ and $M := |2u_2 - 1| = \max\{|2u_3|, |2u_2 - 1|\}$. Furthermore, $|\frac{\gamma_1}{\gamma_3}| = \alpha^{2u_2}\beta^{2u_3} < \frac{1}{2a^4}$ shows that $u_2 < 0$, $\quad u_3 < 0$.

Next we will derive bounds for $|2u_2 - 1|$ and $|2u_3|$:

Lemma 5. *If $a > 1$ then*

$$M := |2u_2 - 1| > \frac{a}{2}\log a$$

and

$$|2u_2 - 1| + |2u_3| < M\left(1 + \frac{2}{a \log a}\right).$$

Proof. From lemma 4 we obtain

$$|2u_2 - 1|\log \beta > |2u_3|\log \alpha - \frac{|\gamma_1|}{\gamma_2}\frac{2}{\alpha} \geq \log \alpha - \frac{|\gamma_1|}{\gamma_2}\frac{2}{\alpha},$$

so it suffices to prove

$$\log\alpha - \frac{|\gamma_1|}{\gamma_2}\frac{2}{\alpha} \geq \frac{a}{2}\log a \log\beta\,.$$

We have the estimation

$$\frac{\alpha\gamma_2}{|\gamma_1|} > a\,2(a-1)\,a^3 = 2(a^5 - a^4) \tag{12}$$

and use

$$\log(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

to obtain

$$\log\beta < \log\left(1 + \frac{2}{a} + \frac{2}{a^2}\right) < \frac{2}{a} - \frac{1}{a^3}\,, \tag{13}$$

where the last inequality holds for $a > 6$, but $\log\beta < \frac{2}{a} - \frac{1}{a^3}$ holds for $a \geq 1$, too.

To prove the first inequality of our lemma, it suffices to show that

$$\log a - \frac{1}{a^5 - a^4} \geq \frac{a}{2}\left(\frac{2}{a} - \frac{1}{a^3}\right)\log a\,,$$

but this is true for $a \geq 2$.

For the proof of the second inequality we start with

$$|2u_3|\log\alpha < |2u_2 - 1|\log\beta + \frac{|\gamma_1|}{\gamma_2}\frac{2}{\alpha}\,.$$

Using again (12) and (13), we arrive at

$$|2u_3| < M\left(\frac{2}{a} - \frac{1}{a^3}\right)\frac{1}{\log a} + \frac{2}{(a^5 - a^4)\log a} < M\frac{2}{a\log a}\,, \tag{14}$$

which immediately yields our assertion.

Lemma 6. *If* $a \geq 902$ *then*

$$\log|2u_3\log\alpha - (2u_2 - 1)\log\beta| < -0.98|2u_2 - 1|\log\alpha\,,$$

and if $a \geq 1$ *then*

$$\log|2u_3\log\alpha - (2u_2 - 1)\log\beta| < -0.8|2u_2 - 1|\log\alpha\,.$$

Proof. Taking logarithms of the equations (8.b-d) we get a regular system of linear equations for the u_i. Using Cramer's rule and $\prod_{i=1}^{4}|\gamma_i| = 1$, we obtain

$$u_2 = -\frac{\log\alpha\log\frac{\gamma_3}{|\gamma_1|} - \log\beta\log\frac{\gamma_4}{\gamma_2}}{2(\log^2\alpha + \log^2\beta)}$$

and

$$|2u_2 - 1| = \frac{\log\alpha \log\frac{\gamma_3}{|\gamma_1|} - \log\beta \log\frac{\gamma_4}{\gamma_2}}{\log^2\alpha + \log^2\beta} + 1 < \frac{1}{\log\alpha}\log\frac{\gamma_3}{|\gamma_1|} + 1\,. \qquad (15)$$

From lemma 4 we get

$$\log|2u_3\log\alpha - (2u_2 - 1)\log\beta| < -\log\Big(\frac{\gamma_2}{|\gamma_1|}\frac{\alpha}{2}\Big)\,,$$

thus we have to show that

$$0.98|2u_2 - 1|\log\alpha < \log\Big(\frac{\gamma_2}{|\gamma_1|}\frac{\alpha}{2}\Big)\,.$$

Using (15), it remains to show

$$0.98\Big(\log\frac{\gamma_2}{|\gamma_1|} + \log\frac{\gamma_3}{\gamma_2} + \log\alpha\Big) < \log\frac{\gamma_2}{|\gamma_1|} + \log\alpha - \log 2\,,$$

or equivalently,

$$0.98\log\frac{\gamma_3}{\gamma_2} + \log 2 < 0.02\log\frac{\gamma_2\alpha}{|\gamma_1|}\,.$$

With (10.b,c) we get for $a > 8$

$$\frac{\gamma_3}{\gamma_2} < \frac{a + \frac{6}{a}}{a - 1 + \frac{7}{a} - \frac{5}{a^2}} < 1 + \frac{1}{a} \qquad \text{and} \qquad \log\frac{\gamma_3}{\gamma_2} < \frac{1}{a}\,.$$

A direct calculation shows that also for $1 \le a \le 8$ we have $\frac{\gamma_3}{\gamma_2} < 1 + \frac{1}{a}$ and $\log\frac{\gamma_3}{\gamma_2} < \frac{1}{a}$.

Using this and (12), the first part of lemma 6 follows from

$$\frac{0.98}{a} + \log 2 < 0.02\log\big(2(a^5 - a^4)\big)\,,$$

which holds for $a \ge 902$.

The proof of the second inequality is analogous for $a \ge 3$, and for $a = 1$, 2 a direct calculation is used.

Case 2: Solutions of Type II

Now we suppose that the solution (x, y) is of type II, i. e.

$$\left|\frac{x}{y} - \alpha_4\right| = \min\left\{\left|\frac{x}{y} - \alpha_i\right| \;\middle|\; 1 \le i \le 4\right\}.$$

This case differs from case 1 just by applying the Galois automorphism τ^{-1}. Thus the algebraic relations can be transformed directly, whereas another archimedian valuation makes the metrical results of this case similar, but not identical to those of case 1.

Obviously we now have $\gamma_1 < \gamma_2 < \gamma_3 < 0$ and $\operatorname{sgn}(\gamma_4) = -\operatorname{sgn}(F_a(x,y))$. Again we derive estimations for the γ_i. Using the estimations (5d) one can easily check that $|f_a'(\alpha_4)| > 2a$ for $a \geq 1$. Hence by (9) we obtain

$$|\gamma_4| < \frac{8}{2ay^3} \leq \frac{1}{2a}. \tag{10.d'}$$

We obtain $\alpha_4 - \frac{1}{4a} < \frac{x}{y} < \alpha_4 + \frac{1}{4a}$ and $y(\alpha_i - \alpha_4 - \frac{1}{4a}) < |\gamma_i| < y(\alpha_i - \alpha_4 + \frac{1}{4a})$ for $1 \leq i \leq 3$. Using (5.a-c) and calculating for some small values of a yields

$$y\left(a+1+\frac{2}{a}\right) < |\gamma_1| < y\left(a+1+\frac{7.25}{a}+\frac{2}{a^2}\right) \tag{10.a'}$$

$$y\left(2-\frac{0.25}{a}\right) < |\gamma_2| < y\left(2+\frac{0.25}{a}+\frac{5}{a^2}\right) \tag{10.b'}$$

$$y\left(1+\frac{0.45}{a}\right) < |\gamma_3| < y\left(1+\frac{1.25}{a}+\frac{3}{a^2}\right) \tag{10.c'}$$

Applying τ^{-1} to (11) yields

$$\alpha^{-2u_2+1}\beta^{-2u_3} = 1 + \frac{\gamma_4}{\gamma_1}(\alpha - 1). \tag{11'}$$

Since we deal with solutions of type II, $|\gamma_4|$ is small and the above expression near to 1.

Lemma 4'. *For $a \geq 1$ we have*

$$|(2u_2 - 1)\log\alpha + 2u_3 \log\beta| < \left|\frac{\gamma_4}{\gamma_1}\right|\alpha.$$

Proof. The proof is analogue to that of lemma 4.

Since $\log\alpha > \log a$, $\log\beta > \frac{1}{a} - \frac{1}{2a^2}$ and $|\frac{\gamma_4}{\gamma_1}|\alpha < \frac{1}{2a}$, we can deduce that $2u_3 \neq 0$, $2u_2 - 1 \neq 0$, $\operatorname{sgn}(2u_3) = -\operatorname{sgn}(2u_2-1)$ and $M := |2u_3| = \max\{|2u_3|, |2u_2-1|\}$. Furthermore, $\alpha^{-2u_3}\beta^{2u_2} = |\frac{\gamma_4}{\gamma_2}| < \frac{1}{7a}$ shows that $u_2 < 0$ and $u_3 > 0$.

For $|2u_2 - 1|$ and $|2u_3|$ we obtain nearly the same bounds as in case 1:

Lemma 5'. *For $a \geq 2$ we have*

$$M := |2u_3| > \frac{a}{2}\log a - \frac{1}{4}$$

and

$$|2u_2 - 1| + |2u_3| < M\left(1 + \frac{2}{a\log a}\right) + \frac{1}{2a\log a}.$$

Proof. From lemma 4' we obtain

$$|2u_3|\log\beta > |2u_2-1|\log\alpha - \left|\frac{\gamma_4}{\gamma_1}\right|\alpha \geq \log\alpha - \left|\frac{\gamma_4}{\gamma_1}\right|\alpha,$$

so using (13), it suffices to prove

$$\log\alpha - \left|\frac{\gamma_4}{\gamma_1}\right|\alpha \geq \left(\frac{a}{2}\log a - \frac{1}{4}\right)\left(\frac{2}{a} - \frac{1}{a^3}\right).$$

Estimating

$$\left|\frac{\gamma_4}{\gamma_1}\right|\alpha < \frac{a+\frac{5}{a}}{4a(a+1)}, \tag{12'}$$

it suffices to show that

$$\log a - \frac{a+\frac{5}{a}}{4a(a+1)} > \left(\frac{a}{2}\log a - \frac{1}{4}\right)\left(\frac{2}{a} - \frac{1}{a^3}\right),$$

which is easily shown to be true for $a \geq 2$.

With (12') and (13) we obtain for the second inequality:

$$\begin{aligned}
|2u_2-1| &< M\frac{\log\beta}{\log\alpha} + \left|\frac{\gamma_4}{\gamma_1}\right|\frac{\alpha}{\log\alpha} \\
&< M\left(\frac{2}{a} - \frac{1}{a^3}\right)\frac{1}{\log a} + \frac{a+\frac{5}{a}}{4a(a+1)\log a} \\
&< M\frac{2}{a\log a} - \frac{\frac{a}{2}\log a - \frac{1}{4}}{a^3\log a} + \frac{a+\frac{5}{a}}{4a(a+1)\log a} \\
&< M\frac{2}{a\log a} + \frac{1}{2a\log a}.
\end{aligned} \tag{14'}$$

Lemma 6'. *If* $a \geq 131111$ *then*

$$\log|(2u_2-1)\log\alpha + 2u_3\log\beta| < -0.95|2u_3|\log\alpha,$$

and if $a \geq 1$ *then*

$$\log|(2u_2-1)\log\alpha + 2u_3\log\beta| < -0.8|2u_3|\log\alpha.$$

Proof. Like in lemma 6, we use Cramer's rule to obtain

$$|2u_3| = \frac{\log\alpha\log\left|\frac{\gamma_2}{\gamma_4}\right| + \log\beta\log\left|\frac{\gamma_1}{\gamma_3}\right|}{\log^2\alpha + \log^2\beta} < \frac{1}{\log\alpha}\log\left|\frac{\gamma_2}{\gamma_4}\right| + \frac{\log\beta}{\log^2\alpha}\log\left|\frac{\gamma_1}{\gamma_3}\right|. \tag{15'}$$

Using lemma 4', we have to show that

$$0.95|2u_3|\log\alpha < \log\left(\left|\frac{\gamma_1}{\gamma_4}\right|\frac{1}{\alpha}\right).$$

Using (15'), it remains to show

$$0.95\frac{\log\beta}{\log\alpha}\log\left|\frac{\gamma_1}{\gamma_3}\right| + \log\alpha < 0.05\log\left|\frac{\gamma_2}{\gamma_4}\right| + \log\left|\frac{\gamma_1}{\gamma_2}\right|.$$

If we estimate α, β and γ_i, it remains to show

$$0.95\left(\frac{2}{a} - \frac{1}{a^3}\right)\frac{\log(a+1+\frac{6}{a})}{\log(a+\frac{1}{a})} + \log\left(2+\frac{9}{a^2}\right) < 0.05\log(8a-1)\,.$$

This inequality holds for $a \geq 131111$. Similarly,

$$0.8\left(\frac{2}{a} - \frac{1}{a^3}\right)\frac{\log(a+1+\frac{6}{a})}{\log(a+\frac{1}{a})} + \log\left(2+\frac{9}{a^2}\right) < 0.2\log(8a-1)$$

holds for $a \geq 11$.

For $1 \leq a \leq 10$ we calculate the roots α_i of f_a, use the expression giving $|2u_3|$ in (15') and estimate the γ_i to show that

$$0.8|2u_3|\log\alpha < \log\left(\left|\frac{\gamma_1}{\gamma_4}\right|\frac{1}{\alpha}\right)$$

holds.

4 Proof of Theorem 1 for $a \geq 488050$

Now we need a lower bound for the absolute value of the linear form

$$\Lambda := N_1\log\alpha + N_2\log\beta\,, \qquad N_1, N_2 \in \mathbb{Z}\setminus\{0\}\,.$$

We use the following result of M. Laurent [20], appendix p. A-2, adopted for our application:

Proposition 2. *Let a_1, $a_2 \geq e$ with*

$$h(\alpha) \leq \log a_1\,, \qquad h(\beta) \leq \log a_2\,,$$

where $h(\cdot)$ denotes the logarithmic absolute height of an algebraic number. Let

$$b' := \frac{|N_1|}{4\log a_2} + \frac{|N_2|}{4\log a_1} \qquad \text{and} \qquad B \geq 0.5 + \log b'\,.$$

If $4B \geq 25$ then

$$\log|\Lambda| \geq -87\cdot 4^4 B^2 \log a_1 \log a_2\,.$$

Remark. By the remark of M. Mignotte in [9], one may assume the weaker condition $4B \geq 25$ instead of Laurent's condition $\log b' \geq 25$.

Since

$$h(\alpha) = h(\beta) = \frac{1}{4}(\log\alpha + \log\beta) < \frac{1}{4}\log\big((a+\frac{5}{a})(1+\frac{2}{a}+\frac{2}{a^2})\big) < \frac{1}{4}\log(a+2.1)$$

for $a \geq 72$, we can put

$$a_1 = a_2 = \sqrt[4]{a+2.1}\,.$$

$a_i \geq e$ holds for $a \geq 53$.

With lemma 5 (lemma 5' resp.) we can put

$$B := 0.5 + \log\left(\frac{M(1+\frac{2}{a\log a})}{\log(a+2.1)}\right) \qquad \text{in case 1} \qquad (16)$$

$$B := B' = 0.5 + \log\left(\frac{M(1+\frac{2}{a\log a}) + \frac{1}{2a\log a}}{\log(a+2.1)}\right) \qquad \text{in case 2.} \qquad (16')$$

Using the lower bound for M, which is given in lemma 5 (lemma 5', resp.), one checks, that $4B \geq 25$ is satisfied for $a \geq 629$.

Combining proposition 2 with lemma 6 (lemma 6' resp.) yields for $a \geq 131111$

$$0.98M\log\alpha < 1392B^2\log^2(a+2.1) \qquad \text{in case 1} \qquad (17)$$

$$0.95M\log\alpha < 1392B'^2\log^2(a+2.1) \qquad \text{in case 2.} \qquad (17')$$

Considered as a function of M, the difference of the two sides of these inequalities are monotonous for $M > \frac{a}{2}\log a$ ($M > \frac{a}{2}\log a - \frac{1}{4}$, resp.), if $a > 64000$. Inserting the lower bound for M, we arrive at

$$0.98\,\frac{a}{2}\log a\log\big(a+\frac{5}{a}-\frac{1}{a^2}\big) < 1392\Big(0.5+\log\frac{\frac{a}{2}\log a+1}{\log(a+2.1)}\Big)^2\log^2(a+2.1) \quad (18)$$

$$0.95\,\big(\frac{a}{2}\log a-\frac{1}{4}\big)\log\big(a+\frac{5}{a}-\frac{1}{a^2}\big) < 1392\Big(0.5+\log\frac{\frac{a}{2}\log a+\frac{3}{4}}{\log(a+2.1)}\Big)^2\log^2(a+2.1)\,. \quad (18')$$

But (18) does not hold for $a \geq 470415$ and (18') does not hold for $a \geq 488050$, which concludes the proof of the theorem.

5 Proof of Theorem 1 for $1 \leq a < 488050$

The rest of the proof was done by using numerical techniques. We divided the interval $1 \leq a \leq 5\cdot 10^5$ into three pieces: $I_1 = [1,100]\setminus\{3\}$, $I_2 = [100, 1.32\cdot 10^5]$

and $I_3 = [1.32 \cdot 10^5, 5 \cdot 10^5]$. In order to save computing time we used a different method for I_3.

1. For $a \in I_3$ the method of MIGNOTTE [8] was adopted. By the lemmata 5, 5', 6 and 6' and by (14) and (14') we have to solve the inequality

$$\log|A \log \alpha + M \log \beta| < -0.95|M| \log \alpha \tag{19}$$

in integers A, M with

$$0 < A < |M| \frac{2}{a \log a} + \frac{1}{2a \log a}.$$

Putting $B = B'' := -1.9673 + \log(1.01|M|)$ we see that the conditions of Proposition 2 are fulfilled, thus

$$0.95|M| \log(1.32 \cdot 10^5) < 1392\left(\log(1.01|M|) - 1.9673\right)^2 \log^2(5 \cdot 10^5 + 2.1).$$

This implies $|M| < 3.712 \cdot 10^6$ and $A \leq 4$.

By the lemmata 5 and 5' we have $|M| > 10^5$, and so

$$\left| A \frac{\log \alpha}{\log \beta} + M \right| < 10^{-20}.$$

(Much more is true, but we do not need it.)

Using the computer algebra system MAPLE we get

$$\left| \frac{\log \alpha}{\log \beta} - \left(\frac{a \log a}{2} + \frac{7 \log a}{3a} + \frac{5}{2a} \right) \right| < 10^{-12}.$$

If (19) has a solution then

$$\left\| A \left(\frac{a \log a}{2} + \frac{7 \log a}{3a} + \frac{5}{2a} \right) \right\| < 10^{-11},$$

where $\|\cdot\|$ denotes the distance to the nearest integer. We checked this inequality for all values of a, A with $1.32 \cdot 10^5 \leq a \leq 5 \cdot 10^5$ and $1 \leq A \leq 4$, and found that it cannot hold. For this computation we used the computational number theory system PARI.

2. For $a \in I_1 \cup I_2$ we start with

$$\log|A \log \alpha + M \log \beta| < -0.8|M| \log \alpha. \tag{20}$$

For $a \in I_1$ we use the explicit values of α, β for $a = 1$ ($a = 100$, resp.) and see that we can use proposition 2 with $a_1 = a_2 = 3.18$ and $B = 0.5 + \log\left(\frac{2M}{4.63}\right)$. With $1.06 < \log \alpha$ we obtain

$$0.8 \cdot 1.06M < 1392\left(0.5 + \log\left(\frac{2M}{4.63}\right)\right)^2 (4.63)^2$$

which holds only for $M < 8.6 \cdot 10^6$.

For $a \in I_2$ one can check that $\log(\alpha\beta) > 4.625$ and $\frac{\log\alpha}{\log(\alpha\beta)} > 0.995$. Using Proposition 2 with $B = 0.5 + \log(0.22\,M)$ we obtain

$$0.8 \cdot 0.995M < 1392\Big(0.5 + \log(0.22\,M)\Big)^2 \log(1.32 \cdot 10^5 + 2.1)$$

which holds only for $M < 4.18 \cdot 10^6$.

If $(A, M) \in \mathbb{Z}$ is a solution of (20) then

$$\left|\frac{\log\beta}{\log\alpha} + \frac{A}{M}\right| < \frac{1}{|M|\log\alpha}\exp(-0.8|M|\log\alpha) < \frac{1}{2M^2},$$

hence $\frac{A}{M}$ is a convergent of $\frac{\log\beta}{\log\alpha}$. We tested for $1 \le a \le 1.32 \cdot 10^5$ whether the convergents of $\frac{\log\beta}{\log\alpha}$ with denominators less then 10^8 satisfy (20), and found as only solutions $a = 1$, $A = -2$, $M = 3$ and $a = 4$, $A = -1$, $M = 4$. Thus the proof of Theorem 1 is complete.

6 Proof of Theorem 2

In this section we are using results of Gaál, Pethő and Pohst [3], [4], which enables the reformulation of index form equations over quartic number fields to finitely many Thue equations over the same field. To formulate the relevant result we have to introduce some notation.

Let $\mathbb{K} = \mathbb{Q}(\xi)$ be a quartic number field and denote by

$$f(X) = X^4 + a_1X^3 + a_2X^2 + a_3X + a_4 \in \mathbb{Z}[X]$$

the minimal polynomial of ξ. The discriminant of an integral $\alpha \in \mathbb{Z}_{\mathbb{K}}$ will be denoted by $D(\alpha)$.

Let $\mathcal{O}$ be an order in $\mathbb{K}$ with an integer basis $\omega_1 = 1, \omega_2, \omega_3, \omega_4$, which is presented in the form

$$\omega_i = \frac{1}{d}\sum_{j=1}^{4}\omega_{ji}\,\xi^{j-1}, \quad (i = 1, \ldots, 4),$$

where ω_{ji}, $d \in \mathbb{Z}$. Denotes $D_{\mathcal{O}}$ the discriminant of $\mathcal{O}$ and put $n = (D(\xi)/D_{\mathcal{O}})^{\frac{1}{2}}$.

The index of $\alpha \in \mathcal{O}$ is defined by $I(\alpha)^2 = D(\alpha)/D_{\mathcal{O}}$. It is an integer by the properties of the discriminant. The elements $1, \alpha, \alpha^2, \alpha^3$ build an integral basis of $\mathcal{O}$, with other words, $\mathcal{O} = \mathbb{Z}[\alpha]$ if and only if $|I(\alpha)| = 1$. With this introduction we have the following

Lemma 7. ([3]) *Let $m \in \mathbb{Z}$ and $i_m = d^6m/n$. The element $\alpha = x_0\omega_2 + y_0\omega_3 + z_0\omega_4 \in \mathcal{O}$ satisfies*

$$I(\alpha) = m$$

if and only if there is a solution $(u, v) \in \mathbb{Z}^2$ *of the cubic Thue equation*

$$F(U,V) = U^3 - a_2U^2V + (a_1a_3 - 4a_4)UV^2 + (4a_2a_4 - a_3^2 - a_1^2a_4)V^3 = \pm i_m$$

such that if

$$\begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} \\ \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} \\ \omega_{41} & \omega_{42} & \omega_{43} & \omega_{44} \end{pmatrix} \begin{pmatrix} 0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} a_\alpha \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

then for the integers x_1, y_1, z_1 *the equations*

$$\begin{aligned} Q_1(x_1,y_1,z_1) &= x_1^2 - a_1x_1y_1 + a_2y_1^2 + (a_1^2 - 2a_2)x_1z_1 \\ &\quad + (a_3 - a_1a_2)y_1z_1 + (-a_1a_3 + a_2^2 + a_4)z_1^2 = u \\ Q_2(x_1,y_1,z_1) &= y_1^2 - x_1z_1 - a_1y_1z_1 + a_2z_1^2 \qquad\qquad = v \end{aligned}$$

hold.

Proof of Theorem 2. Let $\mathbb{K} = \mathbb{Q}(\alpha_1)$ and $\mathcal{O} = \mathbb{Z}[\alpha_1]$. The minimal polynomial of α_1 is $f_a(x)$. As $\{1, \alpha_1, \alpha_1^2, \alpha_1^3\}$ is a basis of $\mathcal{O}$, we may take $d = \omega_{ii} = 1$, $i = 1, \dots, 4$ and $\omega_{ij} = 0$, $1 \le i, j \le 4$, $i \ne j$. Hence $n = 1$. As we mentioned above, $\mathbb{Z}[\delta] = \mathcal{O}$ holds if and only if $|I(\delta)| = 1$. To find all $\delta \in \mathcal{O}$ with this property, we have to solve first, by lemma 7, the cubic Thue equation

$$\begin{aligned} F(U,V) &= U^3 + 6U^2V - (a^2+4)UV^2 - (2a^2+24)V^3 \\ &= (U+2V)\big((U+2V)^2 - V^2(a^2+16)\big) = \pm 1 . \end{aligned}$$

The only solution of it is $(u, v) = (\pm 1, 0)$.

In the next step we have to solve the system of ternary cubic forms

$$\begin{aligned} Q_1(x_1,y_1,z_1) &= x_1^2 + ax_1y_1 - 6y_1^2 + (a^2+12)x_1z_1 - 5ay_1z_1 + (a^2+37)z_1^2 \\ &= \pm 1 \\ Q_2(x_1,y_1,z_1) &= y_1^2 - x_1z_1 + ay_1z_1 - 6z_1^2 \\ &= 0 \end{aligned} \tag{21}$$

in integers x_1, y_1, z_1.

Let us assume that $(x, y, z) \in \mathbb{Z}^3$ is a solution of (21). $Q_2(x, y, z) = 0$ yields $x = \frac{y^2}{z} + ay - 6z$. From $Q_1(x, y, z) = \pm 1$ we see that x, y, z are relatively prime, thus each prime p dividing z must not divide $\frac{y^2}{z}$. Thus there exist relatively prime integers $r, s \in \mathbb{Z}$ with $y = rs$ and $z = s^2$, which yields $x = r^2 + ars - 6s^2$. But $Q_1(r^2 + ars - 6s^2, rs, s^2) = F_a(r + as, s) = \pm 1$, and theorem 1 solves this equation. Thus we obtain

$$\begin{array}{llll} & (r,s)=(\pm1,0) & (x,y,z)=(1,0,0) & \delta=\alpha \\ & (r,s)=(\pm a,\mp1) & (x,y,z)=(-6,-a,1) & \delta=-a-1/\alpha \\ a=1: & (r,s)=(\pm1,\mp2) & (x,y,z)=(-25,-2,4) & \delta=-25\alpha-2\alpha^2+4\alpha^3 \\ & (r,s)=(\pm3,\mp1) & (x,y,z)=(0,-3,1) & \delta=-3\alpha^2+\alpha^3 \\ a=4: & (r,s)=(\pm10,\mp3) & (x,y,z)=(-74,-30,9) & \delta=-74\alpha-30\alpha^2+9\alpha^3 \\ & (r,s)=(\pm11,\mp2) & (x,y,z)=(9,-22,4) & \delta=9\alpha-22\alpha^2+4\alpha^3 \end{array}$$

which completes the proof of theorem 2.

Remark. By using the same method one can easily show that the order $\mathbb{Z}[\alpha,\beta]$, in which we worked in the proof of Theorem 1, does not have a power integral basis for $a \neq 1$, 5.

References

[1] A. Baker. Contribution to the theory of Diophantine equations I. On the representation of integers by binary forms. *Philos. Trans. Roy. Soc. Londen* **263** (1968), 173–191.

[2] I. Gaál. On the resolution of some diophantine equations. In: *Computational Number Theory*, Colloquium on Computational Number Theory, Debrecen, 1989. Eds.: A. Pethő, M. Pohst, H.C. Williams, H.G. Zimmer. De Gruyter 1991, 261–280.

[3] I. Gaál, A. Pethő and M. Pohst. On the resolution of a family of index form equations in quartic number fields. *J. Symbolic Comp.* **16** (1993), 563–584.

[4] I. Gaál, A. Pethő and M. Pohst. Simultaneous representation of integers by a pair of ternary quadratic forms with an application to index form equations in quartic number fields. *J. Number Theory*, to appear.

[5] M. N. Gras. Table numerique du nombre de classes et des unites des extensions cycliques reelles de degré 4 de $\mathbb{Q}$. *Publ. Math. Fac. Sci. Besançon* (1977–1978), fasc. 2.

[6] A. J. Lazarus. On the class number and unit index of simplest quartic fields. *Nagoya Math. J.* **121** (1991), 1–13.

[7] K. Mahler. An inequality for the discriminant of a polynomial. *Michigan Math. J.* **11** (1964), 257–262.

[8] M. Mignotte. Verification of a Conjecture of E. Thomas. *J. Number Theory* **44** (1993), 172–177.

[9] M. Mignotte, A. Pethő and R. Roth. Complete solutions of quartic Thue and index form equations. *Math. Comp.*, to appear.

[10] M. Mignotte and N. Tzanakis. On a family of cubics. *J. Number Theory* **39** (1991), 41–49.

[11] A. Pethő. Complete solutions to families of quartic Thue equations. *Math. Comp.* **57** (1991), 777–798.

[12] A. Pethő and R. Schulenberg. Effektives Lösen von Thue Gleichungen. *Publ. Math. Debrecen* **34** (1987), 189–196.

[13] M. Pohst and H. Zassenhaus. *Algorithmic Algebraic Number Theory*. Cambridge Univ. Press 1989.

[14] D. Shanks. The simplest cubic fields. *Math. Comp.* **28** (1974), 1134–1152.

[15] E. Thomas. Fundamental units for orders in certain cubic number fields. *J. reine angew. Math.* **310** (1979), 33–55.

[16] E. Thomas. Complete solutions to a family of cubic diophantine equations. *J. Number Theory* **34** (1990), 235–250.

[17] E. Thomas. Solutions to certain families of Thue equations. *J. Number Theory* **43** (1993), 319–369.

[18] A. Thue. Über Annäherungswerte algebraischer Zahlen. *J. reine angew. Math.* **135** (1909), 284–305.

[19] N. Tzanakis and B. M. M. de Weger. On the practical solution of the Thue equation. *J. Number Theory* **31** (1989), 99–132.

[20] M. Waldschmidt. *Linear independence of logarithms of algebraic numbers*. IMSc. Report **116**. The Institute of Math. Sciences, Madras 1992.

Eingegangen am: 13.05.1994

Authors' addresses: Günter Lettl, Institut für Mathematik, Karl-Franzens-Universität, Heinrichstraße 36, 8010 Graz, Austria.
E-mail address: `lettl@edvz.kfunigraz.ac.at`

Attila Pethő, Laboratory of Informatics, University of Medicine, Nagyerdei krt. 98, 4032 Debrecen, Hungary.
E-mail address: `pethoe@peugeot.dote.hu`