GENERALIZED RADIX REPRESENTATIONS AND DYNAMICAL SYSTEMS IV

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ABSTRACT. For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ given by

$$\tau_{\mathbf{r}}(a_1, \ldots, a_d) = (a_2, \ldots, a_d, -\lfloor r_1 a_1 + \cdots + r_d a_d \rfloor)$$

where $\lfloor \cdot \rfloor$ denotes the floor function, is called a *shift radix system* if for each $\mathbf{a} \in \mathbb{Z}^d$ there exists an integer $k > 0$ with $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$. As shown in Part I of this series of papers, shift radix systems are intimately related to certain well-known notions of number systems like β-expansions and canonical number systems. After characterization results on shift radix systems in Part II of this series of papers and the thorough investigation of the relations between shift radix systems and canonical number systems in Part III, the present part is devoted to further structural relationships between shift radix systems and β-expansions. In particular we establish the distribution of Pisot polynomials with and without the finiteness property (F).

1. INTRODUCTION

This is the fourth part of a series of papers that is devoted to the systematic study of so-called *shift radix systems*. Shift radix systems are dynamical systems that are strongly related to well-known notions of number systems. First of all, let us recall their exact definition.

Definition 1.1. Let $d \geq 1$ be an integer and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$. To $\mathbf{r}$ we associate the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ in the following way: For $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ let[1]

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \ldots, a_d, -\lfloor \mathbf{r}\mathbf{a} \rfloor),$$

where $\mathbf{r}\mathbf{a} = r_1 a_1 + \cdots + r_d a_d$, *i.e.*, the inner product of the vectors $\mathbf{r}$ and $\mathbf{a}$. We call $\tau_{\mathbf{r}}$ a *shift radix system* (*SRS* for short) if for all $\mathbf{a} \in \mathbb{Z}^d$ we can find some $k > 0$ with $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$.

In Part I [2] (*cf.* also [1], where some preliminary studies are contained) of this series we proved that SRS form a common generalization of canonical number systems in residue class rings of polynomial rings (see [9] for a definition) as well as β-expansions of real numbers (which were first studied in [10] and are defined below). Furthermore, some partial results are given that point out the difficulty of characterizing all SRS parameters. A thorough study of the SRS parameters in dimension $d = 2$ is done in Part II [3], while Part III [4] shows that CNS polynomials can be used in order to approximate the set of SRS parameters. The present paper is devoted to the relation between β-expansions and SRS.

The following classes of sets are needed for our studies. For $d \in \mathbb{N}$, $d \geq 1$ let

$$(1.1)\quad \begin{aligned} \mathcal{D}_d &:= \left\{ \mathbf{r} \in \mathbb{R}^d \,:\, \forall \mathbf{a} \in \mathbb{Z}^d \text{ the sequence } (\tau_{\mathbf{r}}^k(\mathbf{a}))_{k \geq 0} \text{ is ultimately periodic} \right\} \quad \text{and} \\ \mathcal{D}_d^0 &:= \left\{ \mathbf{r} \in \mathbb{R}^d \,:\, \forall \mathbf{a} \in \mathbb{Z}^d\, \exists k > 0 : \tau_{\mathbf{r}}^k(\mathbf{a}) = 0 \right\}. \end{aligned}$$

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[1]The function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ denotes the *floor function* which is defined by $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$.

$\mathcal{D}_d$ is strongly related to the set of contracting polynomials. In particular, let

$$\mathcal{E}_d(r) := \left\{(r_1,\dots,r_d) \in \mathbb{R}^d \,:\, X^d + r_d X^{d-1} + \cdots + r_1 \text{ has only roots } y \in \mathbb{C} \text{ with } |y| < r\right\}.$$

In [2, Lemmas 4.1, 4.2 and 4.3] we proved that

$$\mathrm{int}\,(\mathcal{D}_d) = \mathcal{E}_d(1). \tag{1.2}$$

$\mathcal{D}_d^0$ is the set of all parameters $\mathbf{r} \in \mathbb{R}^d$ that give rise to an SRS.

As β-expansions form a central object in the investigations done in the present paper we give their exact definition (*cf.* for instance [6, 8, 10]). Before that we recall the definition of Pisot and Salem numbers.

Let $P(X) = X^d - b_1X^{d-1} - \cdots - b_d \in \mathbb{Z}[X]$ be an irreducible polynomial over $\mathbb{Z}$.

- If P has a real root greater than one and all other roots are located in the open unit disk then P is called a *Pisot polynomial*. The dominant root is called a *Pisot number*.
- If P has a real root greater than one and all other roots are located in the closed unit disk and at least one of them has modulus 1 then P is called a *Salem polynomial*. The dominant root is called a *Salem number*.

Let $\beta > 1$ be a non-integral real number and let $\mathcal{A} = \{0, 1, \dots, \lfloor\beta\rfloor\}$ be the set of digits. Then each $\gamma \in [0,\infty)$ can be represented uniquely as a *β-expansion* by

$$\gamma = a_m\beta^m + a_{m-1}\beta^{m-1} + \cdots \tag{1.3}$$

with $a_i \in \mathcal{A}$ such that

$$0 \le \gamma - \sum_{i=n}^{m} a_i\beta^i < \beta^n \tag{1.4}$$

holds for all $n \le m$. Since by condition (1.4) the digits a_i are selected as large as possible, the representation in (1.3) is often called the *greedy expansion* of γ with respect to β.

Schmidt [11] proved that in order to get ultimately periodic expansions for all $\gamma \in \mathbb{Q} \cap (0,1)$ it is necessary for β to be a Pisot or a Salem number. We are interested in base numbers β which give rise to finite β-expansions for large classes of numbers. Let $\mathrm{Fin}(\beta)$ be the set of positive real numbers having finite greedy expansion with respect to β. We say that $\beta > 1$ has *property (F)* if

$$\mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0,\infty),$$

that is, all reasonable candidates admit finite β-expansions. It is shown in [6, Lemma 1] that (F) can hold only for Pisot numbers β. In [2, Theorem 2.1] property (F) is related to the SRS property. We recall this in more detail.

Associated to Pisot and Salem numbers with periodic β-expansions and with property (F), respectively, we define for each $d \in \mathbb{N}$, $d \ge 1$ the sets

$$\begin{aligned}
\mathcal{B}_d &:= \{(b_1,\dots,b_d) \in \mathbb{Z}^d \,:\, X^d - b_1X^{d-1} - \cdots - b_d \text{ is a Pisot or Salem polynomial}\} \quad \text{and}\\
\mathcal{B}_d^0 &:= \{(b_1,\dots,b_d) \in \mathbb{Z}^d \,:\, X^d - b_1X^{d-1} - \cdots - b_d \text{ is a Pisot polynomial with property (F)}\}.
\end{aligned}$$

We obviously have $\mathcal{B}_d^0 \subseteq \mathcal{B}_d$. Let us consider the map $\psi : \mathcal{B}_d \to \mathbb{R}^{d-1}$ defined as follows. If $(b_1,\dots,b_d) \in \mathcal{B}_d$ then let β be the dominant root of the (Pisot or Salem) polynomial

$$X^d - b_1X^{d-1} - \cdots - b_d.$$

Now let

$$\psi(b_1,\dots,b_d) = (r_d,\dots,r_2),$$

where

$$r_j = b_j\beta^{-1} + b_{j+1}\beta^{-2} + \cdots + b_d\beta^{j-d-1} \quad (2 \le j \le d).$$

In other words, the numbers $r_2,\dots,r_d$ are defined in a way that they satisfy the relation

$$X^d - b_1X^{d-1} - \cdots - b_d = (X-\beta)(X^{d-1} + r_2X^{d-2} + \cdots + r_d).$$

As $(b_1,\dots,b_d) \in \mathcal{B}_d$, the polynomial $X^{d-1} + r_2X^{d-2} + \cdots + r_d$ has all its roots in the closed unit circle. Together with (1.2) this implies that

$$\psi(\mathcal{B}_d) \subseteq \overline{\mathcal{D}_{d-1}}.$$

Moreover Theorem 2.1 of [2] implies that

$$\psi(\mathcal{B}_d^0) \subseteq \mathcal{D}_{d-1}^0.$$

We push the relation between Pisot numbers, property (F) and SRS further in the present paper and show that $\psi(\mathcal{B}_d)$ and $\psi(\mathcal{B}_d^0)$ are excellent approximations of $\mathcal{D}_{d-1}$ and $\mathcal{D}_{d-1}^0$, respectively. To formulate our main results we need some notation. For $M \in \mathbb{N}_{>0}$ we set

$$\mathcal{B}_d(M) := \left\{(b_2, \dots, b_d) \in \mathbb{Z}^{d-1} \,:\, (M, b_2, \dots, b_d) \in \mathcal{B}_d\right\} \tag{1.5}$$

and

$$\mathcal{B}_d^0(M) := \left\{(b_2, \dots, b_d) \in \mathbb{Z}^{d-1} \,:\, (M, b_2, \dots, b_d) \in \mathcal{B}_d^0\right\}. \tag{1.6}$$

With these notations we are able to state the following theorem.

Theorem 1.2. *Let $d \geq 2$. We have*

$$\left|\frac{|\mathcal{B}_d(M)|}{M^{d-1}} - \lambda_{d-1}(\mathcal{D}_{d-1})\right| = O(M^{-1/(d-1)}), \tag{1.7}$$

where λ_{d-1} denotes the $(d-1)$-dimensional Lebesgue measure.

As we do not have enough information about the structure of $\mathcal{D}_d^0$, we are not able to prove an asymptotic estimate for the error term for $\mathcal{B}_d^0(M)$. However, we are able to establish the main term, more precisely we prove:

Theorem 1.3. *Let $d \geq 2$. We have*

$$\lim_{M \to \infty} \frac{|\mathcal{B}_d^0(M)|}{M^{d-1}} = \lambda_{d-1}(\mathcal{D}_{d-1}^0). \tag{1.8}$$

These results are analogous to [4, Theorems 4.1 and 6.1], where corresponding results are proved for canonical number system polynomials. Notice that $\lambda_d(\mathcal{D}_d)$ is equal to $2, 4, \frac{16}{3}, \frac{64}{9}$ and $\frac{1024}{135}$ for $d = 1, 2, 3, 4$ and 5, respectively. This is trivial for $d = 1$ and 2, while the other three values were computed by Paul Surer, to whom we are much indebted for this information.

2. Properties of two auxiliary mappings

In all what follows let $d \geq 2$. For $M \in \mathbb{Z}$ define the mapping $\chi_M : \mathbb{R}^{d-1} \mapsto \mathbb{Z}^d$ such that if $\mathbf{r} = (r_d, \dots, r_2) \in \mathbb{R}^{d-1}$ then let $\chi_M(\mathbf{r}) = \mathbf{b} = (b_1, \dots, b_d)$, where $b_1 = M$, $b_i = \lfloor r_i(M + r_2) - r_{i+1} + \frac{1}{2}\rfloor$, $i = 2, \dots, d-1$, $b_d = \lfloor r_d(M + r_2) + \frac{1}{2}\rfloor$. It is easy to check that if $\mathbf{b} = (b_1, \dots, b_d) \in \mathcal{B}_d$, then $\chi_{b_1}(\psi(\mathbf{b})) = \mathbf{b}$, *i.e.*, χ_{b_1} is one of the left inverses of the mapping ψ. We pointed out above that $\psi(\mathcal{B}_d) \subseteq \overline{\mathcal{D}_{d-1}}$ and $\psi(\mathcal{B}_d^0) \subseteq \overline{\mathcal{D}_{d-1}^0}$. To prove the main theorem we need some properties of the sets

$$\mathcal{S}_d(M) := \chi_M(\overline{\mathcal{D}_{d-1}}) \quad \text{and} \quad \mathcal{S}_d^0(M) := \chi_M(\overline{\mathcal{D}_{d-1}^0})$$

as well as

$$\mathcal{S}_d := \bigcup_{M \in \mathbb{Z}} \mathcal{S}_d(M) \quad \text{and} \quad \mathcal{S}_d^0 := \bigcup_{M \in \mathbb{Z}} \mathcal{S}_d^0(M).$$

Our first lemma shows that if $|M|$ is large enough then the polynomials associated to the elements of $\mathcal{S}_d$ behave in some sense similar as Pisot or Salem polynomials. However, the example $\mathbf{r} = (-0.9999, 2.99970001, -2.9998)$ shows that the polynomial associated to $\chi_M(\mathbf{r})$ is not necessarily a Pisot or Salem polynomial if $\mathbf{r} \in \mathcal{D}_d$. Indeed, we have $\chi_{1800}(\mathbf{r}) = (1800, -5394, 5391, -1797)$ and the polynomial $X^4 - 1800X^3 + 5394X^2 - 5391X + 1797$ has two real roots 1.084..., 1796.9999997..., which are larger than one.

Lemma 2.1. *There exist constants $M_0 > 0$, $c_1 = c_1(d)$ and $c_2 = c_2(d)$ such that the following is true: Let $M \in \mathbb{Z}$, $(b_1, \dots, b_d) \in \mathcal{S}_d(M)$ and $P(X) = X^d - b_1X^{d-1} - \cdots - b_d$. If $|b_1| = |M| \geq M_0$ then $P(X)$ has a real root β for which the inequalities*

$$|\beta - b_1| < c_1 \quad \textit{and} \tag{2.1}$$

$$\left|\beta - b_1 - \frac{b_2}{b_1}\right| < \frac{c_2}{|b_1|} + O\left(\frac{1}{b_1^2}\right) \tag{2.2}$$

hold.

Proof. In this proof the constants implied by the O-notation depend only on d.

There exists $(r_d,\dots,r_2)\in\overline{\mathcal{D}}_{d-1}$ such that $\mathbf{b}=(b_1,\dots,b_d)=\chi_M(r_d,\dots,r_2)$. It is easy to see that $|r_i|\le 2^{d-1}$. Thus $b_i=Mr_i+O(1), i=2,\dots,d$. Put $Q(X)=b_2X^{d-2}+\cdots+b_d$, *i.e.*, let $P(X)=X^d-MX^{d-1}-Q(X)$. Then $P(M)=-Q(M)$ and

$$P(M+t)=t(M+t)^{d-1}-Q(M+t). \tag{2.3}$$

Assume that $M>0$ and $Q(M)>0$ for M large enough. As $|Q(M+t)|\le d2^dM(M+t)^{d-2}$ we have $P(M+t)>0$ provided that $t\ge d2^d$. Thus $P(X)$ has a real root in the interval $(M,M+d2^d)$. Now we assume that $Q(M)<0$ for all large enough M. By (2.3) we have $P(M+t)<0$ if $t\le -d2^d$. Thus $P(X)$ has again a real root, this time in the interval $(M-d2^d,M)$.

The cases $M<0$ can be handled similarly. Thus we proved (2.1) with $c_1=d2^d$.

The relation $P(\beta)=0$ implies

$$\beta=b_1+\frac{b_2}{\beta}+\frac{b_3}{\beta^2}+\cdots+\frac{b_d}{\beta^{d-1}}.$$

Thus

$$\beta-b_1-\frac{b_2}{b_1}=\frac{(b_1-\beta)b_2}{b_1\beta}+\frac{b_3}{\beta^2}+\cdots+\frac{b_d}{\beta^{d-1}}.$$

Using this expression, inequality (2.1), $b_1=M$ and the estimates $|b_i|\le 2^d|M|, i=2,\dots,d$ we get

$$\begin{aligned}\left|\beta-b_1-\frac{b_2}{b_1}\right| &\le \frac{c_12^{d-1}}{|b_1|-c_1}+\frac{2^d|b_1|}{(|b_1|-c_1)^2}+\sum_{j=3}^{d-1}\frac{2^d|b_1|}{(|b_1|-c_1)^j}\\ &< \frac{c_2}{|b_1|}+O\left(\frac{1}{b_1^2}\right),\end{aligned}$$

which proves the second assertion of the lemma. □

Now we are in the position to extend the definition of ψ from the set $\mathcal{B}_d$ to $\mathcal{S}_d$. Indeed, for $(b_1,\dots,b_d)\in\mathcal{S}_d\setminus\mathcal{B}_d$ consider the polynomial

$$P(X)=X^d-b_1X^{d-1}-\cdots-b_d.$$

Select a real root β of $P(X)$ in the following way:

- if $|b_1|<M_0$ then choose β to be some root of $P(X)$,
- otherwise choose β to be a root of $P(X)$ that satisfies (2.1) and (2.2) of Lemma 2.1.

Then let

$$\psi(b_1,\dots,b_d)=(r_d,\dots,r_2),$$

where the real numbers $r_2,\dots,r_d$ are defined in a way that they satisfy the relation

$$X^d-b_1X^{d-1}-\cdots-b_d=(X-\beta)(X^{d-1}+r_2X^{d-2}+\cdots+r_d).$$

We also introduce another mapping, which yields vectors with rational coordinates approximating $\psi(b_1,\dots,b_d)$ good enough provided that $(b_1,\dots,b_d)\in\mathcal{S}_d$. It is easy to see that there exists a constant $M_1=M_1(d)>0$ such that $b_2\ne -b_1^2$ holds for each

$$(b_1,\dots,b_d)\in\tilde{\mathcal{S}}_d$$

where

$$\tilde{\mathcal{S}}_d:=\{(b_1,\dots,b_d)\in\mathcal{S}_d\,:\,|b_1|\ge M_1\}.$$

Let $\tilde\psi\,:\,\tilde{\mathcal{S}}_d\mapsto\mathbb{Q}^{d-1}$ be defined by

$$\tilde\psi(b_1,\dots,b_d)=\left(\frac{b_d}{b_1+\frac{b_2}{b_1}},\frac{b_{d-1}}{b_1+\frac{b_2}{b_1}}+\frac{b_d}{b_1^2},\dots,\frac{b_2}{b_1+\frac{b_2}{b_1}}+\frac{b_3}{b_1^2}\right).$$

The next lemma shows that if $(b_1,\dots,b_d)\in\mathcal{S}_d$ with $|b_1|$ large enough then $\tilde\psi(b_1,\dots,b_d)$ is a good approximation of $\psi(b_1,\dots,b_d)$. We actually prove

Lemma 2.2. *Let $(b_1,\dots,b_d)\in\mathcal{S}_d$ and assume that $|b_1|$ is large enough. Then*

$$\left|\tilde{\psi}(b_1,\dots,b_d)-\psi(b_1,\dots,b_d)\right|_\infty<\frac{c_3}{b_1^2}+O\left(\frac{1}{|b_1|^3}\right),$$

where c_3 as well as the implied constant depend only on d.

Proof. Using Lemma 2.1 we estimate the distance of the coordinates starting by the first one.

$$\begin{aligned}\left|\frac{b_d}{\beta}-\frac{b_d}{b_1+\frac{b_2}{b_1}}\right| &= \frac{|b_d|}{|\beta|}\frac{1}{\left|b_1+\frac{b_2}{b_1}\right|}\left|\beta-b_1-\frac{b_2}{b_1}\right|\\ &< \frac{2^d|b_1|}{(|b_1|-c_1)(|b_1|-2^d)}\left(\frac{c_2}{|b_1|}+O\left(\frac{1}{b_1^2}\right)\right)\\ &< \frac{c_{31}}{b_1^2}+O\left(\frac{1}{|b_1|^3}\right).\end{aligned}$$

We proceed with the second coordinate and get

$$\begin{aligned}\left|\frac{b_{d-1}}{\beta}+\frac{b_d}{\beta^2}-\frac{b_{d-1}}{b_1+\frac{b_2}{b_1}}-\frac{b_d}{b_1^2}\right| &< \frac{|b_{d-1}|}{|\beta|}\frac{1}{\left|b_1+\frac{b_2}{b_1}\right|}\left|\beta-b_1-\frac{b_2}{b_1}\right|+\frac{|b_d|}{\beta^2b_1^2}\left|\beta^2-b_1^2\right|\\ &< \frac{2^d|b_1|}{(|b_1|-c_1)(|b_1|-2^d)}\left(\frac{c_2}{|b_1|}+O\left(\frac{1}{b_1^2}\right)\right)+\frac{2^d}{|\beta|b_1^2}c_1(2|\beta|-c_1))\\ &< \frac{c_{32}}{b_1^2}+O\left(\frac{1}{|b_1|^3}\right).\end{aligned}$$

Finally we turn to the general case. In the next inequalities we have $2\le j\le d-2$.

$$\begin{aligned}&\left|\frac{b_j}{\beta}+\frac{b_{j+1}}{\beta^2}+\dots+\frac{b_d}{\beta^{d+1-j}}-\frac{b_j}{b_1+\frac{b_2}{b_1}}-\frac{b_{j+1}}{b_1^2}\right|\\ &< \frac{|b_j|}{|\beta|(|b_1|-c_1)}\left|\beta-b_1-\frac{b_2}{b_1}\right|+\frac{|b_{j+1}|}{\beta^2b_1^2}\left|\beta^2-b_1^2\right|+\frac{|b_{j+2}|}{|\beta|^3}+\sum_{k=j+3}^{d}\frac{|b_k|}{|\beta|^{k+1-j}}\\ &< \frac{2^d}{|b_1|-c_1}\left(\frac{c_2}{|b_1|}+O\left(\frac{1}{b_1^2}\right)\right)+\frac{c_12^d(2|\beta|-c_1)}{|\beta|b_1^2}+\frac{2^d}{(|b_1|-c_1)^2}+O(|b_1|^{-3})\\ &< \frac{c_{33}}{b_1^2}+O\left(\frac{1}{|b_1|^3}\right).\end{aligned}$$

Putting $c_3=\max\{c_{31},c_{32},c_{33},M_1\}$ we proved the statement. □

In the next lemma we show that the set $\tilde{\psi}(\tilde{\mathcal{S}}_d)$ is lattice-like. More precisely we prove

Lemma 2.3. *Let $\mathbf{b}=(b_1,\dots,b_d),\mathbf{b}'=(b_1',\dots,b_d')\in\tilde{\mathcal{S}}_d$ such that there exists a $j\in\{1,\dots,d\}$ such that $b_i=b_i', i\ne j$ and $b_j'=b_j+1$. Then*

$$|\tilde{\psi}(b_1,\dots,b_d)_k-\tilde{\psi}(b_1',\dots,b_d')_k|=\begin{cases}0, & \text{if } j>2 \text{ and } k\ne d-j+1,d-j+2,\\ \frac{1}{|b_1|}+O(b_1^{-2}), & \text{if } j>2,\ k=d-j+1 \text{ or } j=2,k=d-1,\\ O(b_1^{-2}), & \text{if } j>2,\ k=d-j+2 \text{ or } j=2,k<d-1,\\ |b_{d-k+1}|\left(\frac{1}{b_1^2}+O(|b_1|^{-3}\right), & \text{if } j=1.\end{cases}$$

Here $\mathbf{v}_k$ denotes the k-th coordinate of the vector $\mathbf{v}$. The implied constants depend only on d.

Proof. If $j > 2$ then $\tilde{\psi}(\mathbf{b})$ and $\tilde{\psi}(\mathbf{b}')$ differ only in the $(d-j+1)$-st and $(d-j+2)$-nd coordinates. Comparing these coordinates we obtain

$$\begin{aligned} \left|\frac{b_j'}{b_1+\frac{b_2}{b_1}} - \frac{b_j}{b_1+\frac{b_2}{b_1}}\right| &= \left|\frac{1}{b_1+\frac{b_2}{b_1}}\right| = \frac{1}{|b_1|} + O(b_1^{-2}) \quad \text{for the } (d-j+1)\text{-st coordinate and} \\ \frac{b_j'}{b_1^2} - \frac{b_j}{b_1^2} &= \frac{1}{b_1^2} \qquad\qquad\qquad\qquad \text{for the } (d-j+2)\text{-nd coordinate.} \end{aligned}$$

In contrast, if $j \in \{1,2\}$ then all coordinates are changing. Consider first the case $j = 2$, *i.e.*, $b_2' = b_2 + 1$ and $b_j' = b_j, j \neq 2$. If $k = 1$ then we get

$$\begin{aligned} \left|\frac{b_d'}{b_1'+\frac{b_2'}{b_1'}} - \frac{b_d}{b_1+\frac{b_2}{b_1}}\right| &= \left|\frac{b_d}{b_1+\frac{b_2+1}{b_1}} - \frac{b_d}{b_1+\frac{b_2}{b_1}}\right| \\ &= \frac{b_d}{|b_1|\left(b_1+\frac{b_2}{b_1}\right)\left(b_1+\frac{b_2+1}{b_1}\right)} \\ &= O(b_1^{-2}). \end{aligned}$$

If $1 < k < d-1$ then

$$\begin{aligned} \left|\frac{b_k'}{b_1'+\frac{b_2'}{b_1'}} + \frac{b_{k+1}'}{b_1'^2} - \frac{b_k}{b_1+\frac{b_2}{b_1}} - \frac{b_{k+1}}{b_1^2}\right| &= \left|\frac{b_k}{b_1+\frac{b_2+1}{b_1}} - \frac{b_k}{b_1+\frac{b_2}{b_1}}\right| \\ &= O(b_1^{-2}). \end{aligned}$$

Finally, if $k = d-1$ then

$$\begin{aligned} \left|\frac{b_2'}{b_1'+\frac{b_2'}{b_1'}} + \frac{b_3'}{b_1'^2} - \frac{b_2}{b_1+\frac{b_2}{b_1}} - \frac{b_3}{b_1^2}\right| &= \left|\frac{b_2+1}{b_1+\frac{b_2+1}{b_1}} - \frac{b_2}{b_1+\frac{b_2}{b_1}}\right| \\ &= \frac{|b_1|}{\left|b_1+\frac{b_2}{b_1}\right|\left|b_1+\frac{b_2+1}{b_1}\right|} \\ &= \frac{1}{|b_1|} + O(b_1^{-2}). \end{aligned}$$

If $j = k = 1$ then we have

$$\begin{aligned} \left|\frac{b_d'}{b_1'+\frac{b_2'}{b_1'}} - \frac{b_d}{b_1+\frac{b_2}{b_1}}\right| &= \left|\frac{b_d}{b_1+1+\frac{b_2}{b_1+1}} - \frac{b_d}{b_1+\frac{b_2}{b_1}}\right| \\ &= \frac{b_d}{\left|b_1+\frac{b_2}{b_1}\right|\left|b_1+1+\frac{b_2}{b_1+1}\right|}\left|\frac{b_2}{b_1} - 1 - \frac{b_2}{b_1+1}\right| \\ &= |b_d|\left(\frac{1}{b_1^2} + O(|b_1|^{-3})\right). \end{aligned}$$

The estimates for the other coordinates in the case $j = 1$ are obtained in the same way as in the case $j = 2$. □

3. Proof of Theorem 1.3

We start with the proof of Theorem 1.3. An essential fact is that the region $\mathcal{D}_d^0$ can be approximated by a finite union of rectangles

$$\Delta = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_d, b_d)$$

with arbitrarily small error from above and from below. This means that $\mathcal{D}_d^0$ is Jordan measurable. Before we prove this fact we recall that a set X in $\mathbb{R}^d$ is *Jordan measurable* if for any positive ε there exists finite set of rectangles P_i $(i = 1, \dots, p)$ and Q_j $(j = 1, \dots, q)$ satisfying

$$\bigcup_j Q_j \subset X \subset \bigcup_i P_i$$

and $\mu_d((\bigcup_i P_i) \setminus (\bigcup_j Q_j)) < \varepsilon$. Here μ_d is the Jordan measure, a finitely additive measure that satisfies

$$\mu_d(\Delta) = \prod_{k=1}^{d} (b_k - a_k).$$

Obviously Jordan measurability of X implies Lebesgue measurability and $\mu_d(X) = \lambda_d(X)$ holds where λ_d is the d-dimensional Lebesgue measure. It is well known that X is Jordan measurable if and only if $\partial(X)$ is Jordan measurable and $\mu_d(\partial(X)) = 0$, *i.e.*, the boundary of X has measure zero. It is easy to prove the following result

Lemma 3.1. *$\mathcal{D}_d^0$ is Jordan measurable,* i.e., *$\mu_d(\partial(\mathcal{D}_d^0)) = 0$.*

Proof. We use the same terminology as in [4]:

$$\mathcal{D}_{d,\varepsilon} = \left\{ \mathbf{r} \in \mathbb{R}^d \ : \ \rho(\mathbf{r}) < 1 - \varepsilon \right\}$$

for $\varepsilon \in (0, 1)$ where $\rho(\mathbf{r})$ is the maximal modulus of all the roots of $X^d + r_d X^{d-1} + \dots + r_1$ with $\mathbf{r} = (r_1, r_2, \dots, r_d)$. We put $\mathcal{D}_{d,\varepsilon}^0 = \mathcal{D}_{d,\varepsilon} \cap \mathcal{D}_d^0$. Then we have essentially shown in [2] that $\mathcal{D}_{d,\varepsilon}^0$ is Jordan measurable. In fact, $\partial(\mathcal{D}_{d,\varepsilon})$ is a finite union of algebraic sets and therefore $\mathcal{D}_{d,\varepsilon}$ is Jordan measurable. $\mathcal{D}_{d,\varepsilon}^0$ is also Jordan measurable because it is a subset of $\mathcal{D}_{d,\varepsilon}$ that emerges from $\mathcal{D}_{d,\varepsilon}$ by removing finitely many convex polyhedra. Since $\mathcal{E}_d(1)$ is Jordan measurable, for a given positive ε, there is a finite union Δ of rectangles such that $\partial(\mathcal{E}_d(1)) \subset \operatorname{int}(\Delta)$ with $\mu_d(\Delta) < \varepsilon/2$. There exists a positive κ such that

$$\mathcal{E}_d(1 + \kappa) \setminus \mathcal{E}_d(1 - \kappa) \subset \Delta.$$

Now

$$\mathcal{D}_d^0 = \mathcal{D}_{d,1-\kappa}^0 \cup (\mathcal{D}_d^0 \setminus \mathcal{D}_{d,1-\kappa}^0)$$

implies

$$\partial(\mathcal{D}_d^0) \subset \partial(\mathcal{D}_{d,1-\kappa}^0) \cup \Delta.$$

Therefore there is a finite union Δ' of rectangles such that $\partial(\mathcal{D}_d^0) \subset \Delta'$ with $\mu_d(\Delta') < \varepsilon$. $\square$

Now we are in a position to prove Theorem 1.3. For some rectangle $\Delta = [u_2, v_2) \times [u_3, v_3) \times \dots \times [u_d, v_d)$ in $\mathbb{R}^{d-1}$ let

$$A_M := \{(b_2, \dots, b_d) \in \mathbb{Z}^d \ : \ \psi(M, b_2, \dots, b_d) \in \Delta\}.$$

To prove (1.8) it is enough to show that

$$\lim_{M \to \infty} \frac{|A_M|}{M^{d-1}} = \prod_{i=2}^{d} (v_i - u_i),$$

when $\Delta \subset \mathcal{E}(\frac{3}{2})$. Indeed, by the above Lemma 3.1 one can approximate $\mathcal{D}_d^0$ by a finite disjoint union of rectangles $\Delta \subset \mathcal{E}(\frac{3}{2})$ from below and from above with an arbitrarily small error.

We note that $\psi : \mathcal{B}_d \to \mathbb{R}^{d-1}$ is obviously injective. Thus when we fix $b_1 = M > 0$, each lattice point $(\frac{b_2}{M}, \dots, \frac{b_d}{M})$ with $(M, b_2, \dots, d_d) \in \mathcal{B}_d$ is in one to one correspondence with a point $\psi(M, b_2, \dots, b_d)$ which is close to $(\frac{b_2}{M}, \dots, \frac{b_d}{M})$. Therefore instead of counting the points of the shape $\psi(M, b_2, \dots, b_d)$ contained in Δ we count the number of points in $\Delta \cap \frac{1}{M}\mathbb{Z}^{d-1}$. Lemma 2.2 implies that $r_j = \frac{b_j}{b_1} + O(\frac{1}{b_1}), j = 2, \dots, d$, we see that this causes an error $O(M^{d-2})$ and we get

$$|A_M| = \left| \left\{ (b_2, \dots, b_d) \in \mathbb{Z}^{d-1} \ : \ \left(\frac{b_2}{M}, \dots, \frac{b_d}{M} \right) \in \Delta \right\} \right| + O(M^{d-2}).$$

Indeed, the term $O(M^{d-2})$ estimates the number of lattice points of the form $(\frac{b_2}{M},\dots,\frac{b_d}{M})$ which have distances $O(M^{-1})$ from $\partial\Delta$. Here the implied constants depend only on d.

Combining this with the trivial observation

$$\lim_{M\to\infty}\frac{1}{M^{d-1}}\left|\left\{(b_2,\dots,b_d)\in\mathbb{Z}^{d-1}\ :\ \left(\frac{b_2}{M},\dots,\frac{b_d}{M}\right)\in\Delta\right\}\right| = \prod_{i=2}^{d}(v_i-u_i),$$

we get the assertion.

4. Auxiliary lemmata

In order to show Theorem 1.2 we need two preliminary lemmata which are stated and proved in the present section. We start with a lemma that quantifies the continuous dependence of the roots of a polynomials from its coefficients.

Lemma 4.1. *Let $d\in\mathbb{N}$ and $\rho,\varepsilon\in\mathbb{R}_{>0}$. Then there exists a constant $c_4>0$ depending only on d and ρ with the following property: if all roots $\alpha\in\mathbb{C}$ of the polynomial $P(X)=X^d+p_{d-1}X^{d-1}+\dots+p_0\in\mathbb{R}[X]$ satisfy $|\alpha|<\rho$ and $Q(X)=X^d+q_{d-1}X^{d-1}+\dots+q_0\in\mathbb{R}[X]$ is chosen such that $|p_i-q_i|<\varepsilon, i=0,\dots,d-1$ then for each root β of $Q(X)$ there exists a root α of $P(X)$ satisfying*

$$|\beta-\alpha|<c_4\varepsilon^{1/d}. \tag{4.1}$$

In particular, all roots β of $Q(X)$ satisfy $|\beta|<\rho+c_4\varepsilon^{1/d}$.

Proof. Let $\alpha_1,\dots,\alpha_d$ denote the roots of $P(X)$ and fix an arbitrary root β of $Q(X)$. Let $k\in\{1,\dots,d\}$. Then

$$|p_{d-k}|\le\sum_{1\le i_1<\dots<i_k\le d}|\alpha_{i_1}|\dots|\alpha_{i_k}|\le\rho^k\binom{d}{k}.$$

This implies

$$|q_{d-k}|\le\rho^k\binom{d}{k}+\varepsilon.$$

By a well-known theorem of Cauchy ([7, Corollary 2.5.4]) we get

$$|\beta|\le\max_{1\le k\le d}\left\{(d|q_{d-k}|)^{1/k}\right\}\le c_5\rho$$

with a constant c_5 depending only on d.

Choose the root α of $P(X)$ such that $|\beta-\alpha|$ is minimal among the differences $|\beta-\alpha_j|$. Then on one hand

$$|Q(\beta)-P(\beta)|\le\sum_{j=0}^{d}\varepsilon|\beta|^j\le\varepsilon\sum_{j=0}^{d}(c_5\rho)^j\le c_6\varepsilon.$$

On the other hand

$$|Q(\beta)-P(\beta)|=\prod_{j=1}^{d}|\beta-\alpha_j|\ge|\beta-\alpha|^d.$$

Comparing the last two inequalities we get (4.1) with $c_4=c_6^{1/d}$ for the chosen β. But since β was an arbitrary root of $Q(X)$ this proves the result. □

The next lemma contains a refinement of Lemma 4.7 of [4] for $\mathcal{D}_d$.

Lemma 4.2. *Let $0<\eta<1$. Then we have*

$$\lambda_d\left(\mathcal{E}_d\left(1+\eta\right)\setminus\mathcal{D}_d\right)\le2^{d(d+1)/2}\lambda_d(\mathcal{E}_d(1))\eta$$

and

$$\lambda_d\left(\mathcal{D}_d\setminus\mathcal{E}_d\left(1-\eta\right)\right)\le2^{d(d+1)/2}\lambda_d(\mathcal{E}_d(1))\eta.$$

Proof. First we express $\mathcal{E}_d(t)$ with the help of $\mathcal{E}_d(1)$ for any positive real number t. Indeed let $(r_1,\dots,r_d)\in\mathcal{E}(1)$. This means that the roots of the polynomial $P(X)=X^d+r_dX^{d-1}+\cdots+r_1$ lie in the open unit circle. Then the roots of $P(\frac{X}{t})t^d=X^d+\frac{r_d}{t}X^{d-1}+\cdots+\frac{r_1}{t^d}$ are of absolute value at most t, *i.e.*, the point $(r_1t^d,\dots,r_dt)$ belongs to $\mathcal{E}_d(t)$. Obviously this mapping is bijective, which means we have $\mathcal{E}_d(t)=\operatorname{diag}(t^d,\dots,t)\mathcal{E}_d(1)$, where $\operatorname{diag}(v_1,\dots,v_d)$ denotes the d-dimensional diagonal matrix with entries $v_1,\dots,v_d$. This implies

$$\lambda_d(\mathcal{E}_d(t))=t^{d(d+1)/2}\lambda_d(\mathcal{E}_d(1)). \tag{4.2}$$

We prove only the first relation, because the second one can be done similarly. Let $0<\eta<1$. Setting $t=1+\eta$ in (4.2) we get immediately

$$\begin{aligned}\lambda_d(\mathcal{E}_d(1+\eta)\setminus\mathcal{D}_d) &= \lambda_d(\mathcal{E}_d(1+\eta)\setminus\mathcal{E}_d(1))\\ &= \lambda_d(\mathcal{E}_d(1+\eta))-\lambda_d(\mathcal{E}_d(1))\\ &= \left((1+\eta)^{d(d+1)/2}-1\right)\lambda_d(\mathcal{E}_d(1))\\ &\le 2^{d(d+1)/2}\lambda_d(\mathcal{E}_d(1))\eta.\end{aligned}$$

This proves the first assertion. The second one follows similarly. □

5. Proof of Theorem 1.2

It is possible to prove Theorem 1.2 without error term following the line of the Section 3. However, we are able to give a bound for the error term in this case. This makes the proof of Theorem 1.2 much more involved. Before starting with this proof we introduce some notation. Let $M>0$ and put

$$W(\mathbf{x},s)=\{\mathbf{y}\in\mathbb{R}^{d-1}\,:\,|\mathbf{x}-\mathbf{y}|_\infty\le s/2\}\qquad(\mathbf{x}\in\mathbb{R}^{d-1},s\in\mathbb{R})$$

and

$$\mathcal{W}_{d-1}(M)=\bigcup_{\mathbf{x}\in\mathcal{B}_d(M)}W(\psi(\mathbf{x}),M^{-1}).$$

Then we claim

$$\lambda_{d-1}(\mathcal{W}_{d-1}(M))=\frac{|\mathcal{B}_d(M)|}{M^{d-1}}\left(1+O\left(\frac{1}{M}\right)\right). \tag{5.1}$$

Indeed, let M be large enough and $\mathbf{x},\mathbf{y}\in\mathcal{B}_d(M)$ such that $\mathbf{x}-\mathbf{y}=\mathbf{e}_j$ for some $j\in\{2,\dots,d\}$. Then by Lemmata 2.2 and 2.3

$$\begin{aligned}|\psi(\mathbf{x})_k-\psi(\mathbf{y})_k| &= |\psi(\mathbf{x})_k-\tilde\psi(\mathbf{x})_k+\tilde\psi(\mathbf{x})_k-\tilde\psi(\mathbf{y})_k+\tilde\psi(\mathbf{y})_k-\psi(\mathbf{y})_k|\\ &= \begin{cases}\frac{1}{M}+O\left(\frac{1}{M^2}\right), & \text{if } j\ge 2, k=d-j+1\\ O\left(\frac{1}{M^2}\right), & \text{otherwise.}\end{cases}\end{aligned}$$

Thus

$$\lambda_{d-1}(W(\psi(\mathbf{x}),M^{-1})\cap W(\psi(\mathbf{y}),M^{-1}))=O\left(\frac{1}{M^d}\right). \tag{5.2}$$

As $\mathbf{x}$ has at most 2^d neighbors we get

$$\lambda_{d-1}\left(\bigcup_{\substack{\mathbf{x},\mathbf{y}\in\mathcal{B}_d(M)\\ \mathbf{x}\ne\mathbf{y}}}\left(W(\psi(\mathbf{x}),M^{-1})\cap W(\psi(\mathbf{y}),M^{-1})\right)\right)=O\left(\frac{|\mathcal{B}_d(M)|}{M^d}\right)$$

and the claim is proved.

Now we are in the position to give a lower bound for $\lambda_{d-1}(\mathcal{D}_{d-1})$. Let $\mathbf{x}\in\mathcal{B}_d(M)$ such that $\psi(\mathbf{x})\in\mathcal{E}_{d-1}\left(1-c_4(2M)^{-1/(d-1)}\right)\subseteq\mathcal{D}_{d-1}$. Let $\mathbf{y}\in W(\psi(\mathbf{x}),M^{-1})$. Then $\rho(\psi(\mathbf{x}))<1-c_4(2M)^{-1/(d-1)}$ and as $|\psi(\mathbf{x})-\mathbf{y}|_\infty\le\frac{1}{2M}$ we get $\rho(\mathbf{y})<1$ by Lemma 4.1. Thus

$$\bigcup_{\substack{\mathbf{x}\in\mathcal{B}_d(M)\\ \rho(\psi(\mathbf{x}))<1-c_4(2M)^{-1/(d-1)}}}W(\psi(\mathbf{x}),M^{-1})\subseteq\mathcal{D}_{d-1}. \tag{5.3}$$

Putting $\eta = c_4(2M)^{-1/(d-1)}$, Lemma 4.2 implies that the measure of the set

$$\mathcal{D}_{d-1} \setminus \mathcal{E}_{d-1}\left(1 - c_4(2M)^{-1/(d-1)}\right)$$

is bounded by $O(M^{-1/(d-1)})$. Moreover this set satisfies the conditions of the Theorem of H. Davenport [5]. Observe that h and V_m of [5] are in the actual application independent from M. Thus the number of $\mathbf{x} \in \mathcal{B}_d(M)$ such that $1 - c_4(2M)^{-1/(d-1)} \leq \rho(\psi(\mathbf{x})) \leq 1$ is at most $O(M^{d-1-1/(d-1)})$. Combining this with (5.2) and (5.3) we obtain the desired lower bound

$$\lambda_{d-1}(\mathcal{D}_{d-1}) \geq \frac{|\mathcal{B}_d(M)|}{M^{d-1}}\left(1 - c_7 M^{-1/(d-1)}\right). \tag{5.4}$$

Here, $c_7 > 0$ is a constant.

To prove an upper bound we need some preparation, more precisely we will construct for every $\mathbf{r} = (r_d, \dots, r_2) \in \mathcal{D}_{d-1}$ and M a large enough integer vector $\mathbf{b} = (b_1, \dots, b_d) \in \mathbb{Z}^d$ such that $\psi(\mathbf{b})$ is located near enough to $\mathbf{r}$.

Indeed put $\mathbf{b} = \chi_M(\mathbf{r})$ and consider

$$\tilde{\psi}(\mathbf{b}) = \left(\frac{b_d}{b_1 + \frac{b_2}{b_1}}, \frac{b_{d-1}}{b_1 + \frac{b_2}{b_1}} + \frac{b_d}{b_1^2}, \dots, \frac{b_2}{b_1 + \frac{b_2}{b_1}} + \frac{b_3}{b_1^2}\right).$$

We estimate the distance of the coordinates of this vector to $\mathbf{r}$. Putting $k = \frac{r_2^2 - r_3}{M}$ for $d > 2$ and $k = \frac{r_2^2 + 1}{M}$ for $d = 2$ we have $k = O(M^{-1})$ and

$$r_2 + k - \frac{1}{2M} < \frac{b_2}{b_1} = \frac{b_2}{M} \leq r_2 + k + \frac{1}{2M}.$$

As

$$(M + r_2)r_d - \frac{1}{2} < b_d \leq (M + r_2)r_d + \frac{1}{2}$$

we obtain

$$\left|\frac{b_d}{b_1 + \frac{b_2}{b_1}} - r_d\right| \leq \max\left\{\left|\frac{(M + r_2)r_d + 1/2}{M + r_2 + k - 1/(2M)} - r_d\right|, \left|\frac{(M + r_2)r_d - 1/2}{M + r_2 + k + 1/(2M)} - r_d\right|\right\}.$$

This implies after a short computation

$$\left|\frac{b_d}{b_1 + \frac{b_2}{b_1}} - r_d\right| \leq \frac{1}{2M} + O\left(\frac{1}{M^2}\right).$$

A similar calculation proves

$$\left|\frac{b_j}{b_1 + \frac{b_2}{b_1}} + \frac{b_{j+1}}{b_1^2} - r_j\right| \leq \frac{1}{2M} + O\left(\frac{1}{M^2}\right)$$

for $j = 2, \dots, d-1$. This means

$$|\tilde{\psi}(\mathbf{b}) - \mathbf{r}|_\infty \leq \frac{1}{2M} + O\left(\frac{1}{M^2}\right).$$

Applying now Lemma 2.2 we obtain

$$|\psi(\mathbf{b}) - \mathbf{r}|_\infty \leq |\tilde{\psi}(\mathbf{b}) - \mathbf{r}|_\infty + |\psi(\mathbf{b}) - \tilde{\psi}(\mathbf{b})|_\infty \leq \frac{1}{2M} + O\left(\frac{1}{M^2}\right).$$

Thus by Lemma 4.1 (note that $\psi(\mathbf{b}), \mathbf{r} \in \mathcal{E}_{d-1}(2)$) we get

$$\rho(\psi(\mathbf{b})) \leq \rho(\mathbf{r}) + 2c_4(2M)^{-1/(d-1)} \leq 1 + 2c_4(2M)^{-1/(d-1)}$$

for large enough M. This means that if M is large enough then all but one root of $X^d - b_1X^{d-1} - \cdots - b_d$ have absolute value at most $1 + 2c_4(2M)^{-1/(d-1)}$ and one root is close to M. We have further

$$\begin{aligned}\mathcal{D}_{d-1} &\subseteq \bigcup_{\substack{\mathbf{x}\in\mathbb{Z}^d\\ \psi(\mathbf{x})\in\mathcal{E}_{d-1}(1+2c_4(2M)^{-1/(d-1)})}} W(\psi(\mathbf{x}), M^{-1})\\ &= \bigcup_{\mathbf{x}\in\mathcal{B}_d(M)} W(\psi(\mathbf{x}), M^{-1}) \cup \bigcup_{\substack{\mathbf{x}\in\mathbb{Z}^d\\ \psi(\mathbf{x})\in\mathcal{E}_{d-1}(1+2c_4(2M)^{-1/(d-1)})\setminus\mathcal{E}_{d-1}(1)}} W(\psi(\mathbf{x}), M^{-1}).\end{aligned}$$

This time we apply Lemma 4.2 with $\eta = 2c_4(2M)^{-1/(d-1)}$ and conclude that the volume of the set $\mathcal{E}_{d-1}(1 + 2c_4(2M)^{-1/(d-1)}) \setminus \mathcal{D}_{d-1}$ is at most $O(M^{-1/(d-1)})$. As the conditions of the Theorem of Davenport [5] hold again we get that the number of $\mathbf{x} \in \mathbb{Z}^d$ such that $\psi(\mathbf{x})$ lies in $\mathcal{E}_{d-1}\left(1 + 2c_4(2M)^{-1/(d-1)}\right) \setminus \mathcal{D}_{d-1}$ is at most $O(M^{d-1-1/(d-1)})$. Thus there is a constant $c_8 > 0$ such that

$$\lambda_{d-1}(\mathcal{D}_{d-1}) \le \frac{|\mathcal{B}_d(M)|}{M^{d-1}}\left(1 + c_8M^{-1/(d-1)}\right).$$

Comparing this inequality with (5.4) we obtain Theorem 1.2.

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