GENERALIZED RADIX REPRESENTATIONS AND DYNAMICAL SYSTEMS II

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Abstract. For $\mathbf{r} \in \mathbb{R}^d$ define $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ by setting

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{r}\mathbf{a} \rfloor) \qquad (\mathbf{a} = (a_1, \dots, a_d)).$$

We call $\tau_{\mathbf{r}}$ a shift radix system if for each $\mathbf{a} \in \mathbb{Z}^d$ there exists an integer $k > 0$ with $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$. Shift radix systems have been defined in the first part of this series of papers. It turns out that they are intimately related to certain well known notions of number systems like β-expansions and canonical number systems.

It seems to be a hard problem to characterize all $\mathbf{r} \in \mathbb{R}^d$ giving rise to a shift radix system. In the present paper we give partial characterization results. After proving some general theorems we are mainly concerned with the characterization of two dimensional shift radix systems.

1. Introduction

In the first part [4] of this series of papers we introduced the notion of shift radix system and described its basic properties as well as its relations to β-expansions and canonical number systems[1]. Specifically, let $d \geq 1$ be an integer and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$. To $\mathbf{r}$ we associate the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ in the following way: For $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ let[2]

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{r}\mathbf{a} \rfloor),$$

where $\mathbf{r}\mathbf{a} = r_1 a_1 + \cdots + r_d a_d$, i.e. the inner product of the vectors $\mathbf{r}$ and $\mathbf{a}$. We call $\tau_{\mathbf{r}}$ a *shift radix system* (*SRS* for short) if for all $\mathbf{a} \in \mathbb{Z}^d$ we can find some $k > 0$ with $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$ [3]. In [4] we have started the investigation of the following sets which are closely connected with the orbits of $\tau_{\mathbf{r}}$:

$$\begin{aligned} \mathcal{D}_d^0 &:= \left\{ \mathbf{r} \in \mathbb{R}^d \,|\, \forall \mathbf{a} \in \mathbb{Z}^d \, \exists k > 0 : \tau_{\mathbf{r}}^k(\mathbf{a}) = 0 \right\} \quad \text{and} \\ \mathcal{D}_d &:= \left\{ \mathbf{r} \in \mathbb{R}^d \,|\, \forall \mathbf{a} \in \mathbb{Z}^d \text{ the sequence } (\tau_{\mathbf{r}}^k(\mathbf{a}))_{k \geq 0} \text{ is ultimately periodic} \right\}. \end{aligned}$$

It has turned out that the description of these sets is almost trivial for $d = 1$, whereas considerable difficulties occur already in dimension 2.

Despite of its simple shape, the SRS system gives a unified understanding of number systems and its related dynamics. For example, if the β-expansion by a Pisot number base corresponds to an SRS system, one can construct a tiling of the Euclidean space which provides a concrete Markov partition of the dynamical system, that is often almost conjugate to a toral automorphism (cf. [2, 8, 21, 25]). This is essentially due to the fact that a tile contains the origin as an inner point. The same fact is valid for tilings associated with canonical number systems. Therefore characterizing SRS systems is to make an atlas of good number systems from a dynamical point of view.

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[1] For a definition of β-expansion and canonical number system we refer the reader to [4] (see also [13, 19, 20, 22]).

[2] $\lfloor \dots \rfloor$ denotes the floor function.

[3] For simplicity, we write $0 = (0, \dots, 0)$.

In the present paper we are mainly concerned with the characterization of quadratic SRS. This is tantamount to the characterization of the set $\mathcal{D}_2^0$. The results on the characterization of $\mathcal{D}_2^0$ are summarized in Figure 1. Note that by the correspondence between SRS and β-expansions as well as canonical number systems[4] our characterization results of $\mathcal{D}_2^0$ imply the characterization of property (F) [5] for β-expansions with respect to cubic Pisot units (cf. [1]) as well as the characterization of quadratic canonical number systems (cf. [10, 12, 15, 16]). Moreover, our results imply new characterization results for property (F) for β-expansions of large classes of cubic Pisot numbers.

Figure 1 has to be interpreted as follows. $\mathcal{D}_2^0$ is a subset of the large trapezium. All the white regions are proved to be contained in $\mathcal{D}_2^0$ in the present paper. The label "T. n.m" means that the corresponding region is proved to belong to $\mathcal{D}_2^0$ in Theorem n.m ("L. n.m" means "Lemma n.m"). The dark grey regions are known to be outside $\mathcal{D}_2^0$. The light grey regions of Figure 1 are regions where $\mathcal{D}_2^0$ has a very complicated structure. It has been proved in [4, Section 6-7] that in these regions there exist infinitely many different small polygons which do not belong to $\mathcal{D}_2^0$. Some of them are visualized in [4, Figure 1] (this figure gives an impression of the difficulty of the structure of $\mathcal{D}_2^0$ in these regions).

The characterization problem becomes harder and harder the nearer we get to the line $x = 1$ or to the line $y = x + 1$. For this reason, the proofs of Theorems 4.6, 4.8 and 4.27 are the most involved ones in this paper.

The paper is organized as follows. In Section 2 we give some results on $\mathcal{D}_2$. Most of $\mathcal{D}_2$ is easy to characterize, however, it turns out to be a hard problem to decide which part of $\partial\mathcal{D}_2$ belongs to $\mathcal{D}_2$. For some parts of $\partial\mathcal{D}_2$ we give a solution of this problem. In Section 3 we describe some important subsets of $\mathcal{D}_d^0$ by generalizing results of Hollander [14], Kovács and Pethő [17] as well as Pethő [20] to our new setting. In particular we present applications of these results for the characterization of $\mathcal{D}_2^0$. In the next two sections we are aiming at further concrete statements for two dimensional SRS. In Section 4 we concentrate on the investigation of points of $\mathcal{D}_2$ which lie near its boundary. We apply two different methods for the characterization of elements of $\mathcal{D}_2^0$: In Section 4.1 we investigate the purely periodic elements of $\tau_{\mathbf{r}}$ in order to get an SRS region near to the upper boundary of Figure 1 (Theorem 4.8). In Section 4.2 we exploit a certain "structural stability" of the mapping $\tau_{\mathbf{r}}$; we illustrate this remarkable property of $\tau_{\mathbf{r}}$ by some numerical examples (see Figures 5 and 6 below). This leads to a characterization result for SRS regions with parameters close to the point $(1, -1)$ located on the right lower vertex of the trapezium in Figure 1. This proof also gives as a byproduct that the point $(1, -1)$ is not a critical point in the sense of [4, Definition 7.1] (Theorem 4.21). Section 5 is devoted to the characterization of $\mathcal{D}_2^0$ in regions which are far from the boundary of $\mathcal{D}_2$. For these regions a powerful algorithm (presented in [4, Theorem 5.2]) allows to derive many results with help of extensive computer calculations. The combination of these results with the results of the previous sections yields Theorems 5.6 and 5.8. Both of them characterize all SRS in quite large regions.

We conclude this paper with some conjectures (Section 6).

2. On the set $\mathcal{D}_2$

We will frequently need the set

$$\mathcal{E}_d = \mathcal{E}_d(1) := \left\{(r_1, \dots, r_d) \in \mathbb{R}^d \mid X^d + r_d X^{d-1} + \cdots + r_1 \text{ has only roots } y \in \mathbb{C} \text{ with } |y| < 1\right\}.$$

In [4, Lemmas 4.1 and 4.2] it is shown that up to the boundary the set $\mathcal{D}_d$ is equal to the set $\mathcal{E}_d$. In particular, for $d = 2$ the set $\mathcal{D}_2$ is (again apart from the boundary) equal to the isosceles rectangular triangle

$$\mathcal{E}_2 = \{(x, y) \in \mathbb{R}^2 \mid x < 1, -x - 1 < y < x + 1\}.$$

Deciding whether a point of $\partial\mathcal{E}_2$ belongs to $\mathcal{D}_2$ or not seems to be a very difficult problem. In this section we give a partial solution. In particular, we will show the following result.

[4] This correspondence is established in [4, Theorems 2.1 and 3.1].

[5] cf. [4] for a definition.

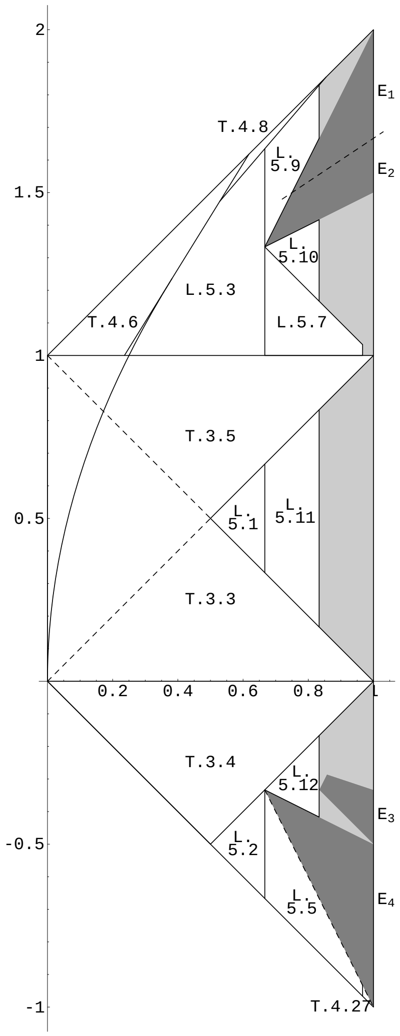

FIGURE 1. Overview over the results of the present paper

Theorem 2.1. *Let*

$$\begin{aligned} D \quad := \quad & \{(x,y) \in \mathbb{R}^2 \mid x \le 1, -x-1 \le y \le x+1,\ (x,y) \ne (1,-2),(1,2)\} \\ & \setminus \{(x,-x-1) \in \mathbb{R}^2 \mid 0 < x < 1\} \end{aligned}$$

and

$$L = D \setminus \{(1,y) \in \mathbb{R}^2 \mid 0 < |y| < 1 \text{ or } 1 < |y| < 2\}.$$

Then

$$L \subseteq \mathcal{D}_2 \subseteq D,$$

the Lebesgue measure of $\mathcal{D}_2$ equals 4, and $\mathcal{D}_2$ is neither open nor closed.

Lemma 2.2. *If $-1 \le x \le 0$ then $(x,-x-1),(x,x+1) \in \mathcal{D}_2 \setminus \mathcal{D}_2^0$.*

Proof. For $x \in \{-1, 0\}$ the assertions are easy to check. Let now $-1 < x < 0$.

Firstly, we consider $\tau = \tau_{(x,-x-1)}$, thus $\tau(a,b) = (b, b - \lfloor (a-b)x \rfloor)$ for $a, b \in \mathbb{Z}$. Observe that for $n \in \mathbb{N}$ we have $\tau(n,n) = (n,n)$, hence $(x,-x-1) \notin \mathcal{D}_2^0$. Now, it suffices to show that for all $\mathbf{z} \in \mathbb{Z}^2$ we have $\| \tau(\mathbf{z}) \|_\infty \leq \| \mathbf{z} \|_\infty$. Let therefore $n, m \in \mathbb{N}$ with $(n,m) \neq (0,0)$. If $n < m$ then $\tau(n,m) = (m,p)$ with $n+1 \leq p \leq m$, and if $n \geq m$ then $\tau(n,m) = (m,p)$ with $m \leq p \leq n$. Further, $\tau(n,-m) = (-m,p)$ with $-m+1 \leq p \leq n$. Similarly, $\tau(-n,m) = (m,-p)$ with $-m \leq p \leq n-1$. Finally, $\tau(-n,-m) = (-m,-p)$ with $0 \leq p \leq \max\{n,m\}$.

Secondly, let $\tau = \tau_{(x,x+1)}$, thus $\tau(a,b) = (b, -b - \lfloor (a+b)x \rfloor)$ for $a, b \in \mathbb{Z}$. We note $\tau^2(-n,n) = (-n,n)$ for $n \in \mathbb{N}$, hence $(x, x+1) \notin \mathcal{D}_2^0$. Again, we confine ourselves to showing $\| \tau(\mathbf{z}) \|_\infty \leq \| \mathbf{z} \|_\infty$ for all $\mathbf{z} \in \mathbb{Z}^2$. Let therefore $n, m \in \mathbb{N}$. Then $\tau(n,m) = (m,p)$ with $-m+1 \leq p \leq n$. Further, $\tau(n,-m) = (-m,p)$ and $\tau(-n,m) = (m,-p)$ with $0 \leq p \leq \max\{n,m\}$. Finally, $\tau(-n,-m) = (-m,p)$ with $-n+1 \leq p \leq m$. □

For $\mathbf{r} \in \mathbb{R}^d$ we denote by $S(\mathbf{r})$ the set of elements $\mathbf{z} \in \mathbb{Z}^d$ such that the sequence $(\tau_{\mathbf{r}}^k(\mathbf{z}))_{k \in \mathbb{N}}$ is ultimately periodic.

Lemma 2.3. *If $0 < x < 1$ then $(x, x+1) \in \mathcal{D}_2 \setminus \mathcal{D}_2^0$.*

Proof.
(i) Let $\tau = \tau_{(x,x+1)}$, thus $\tau(a,b) = (b, -b - \lfloor (a+b)x \rfloor)$ for $a, b \in \mathbb{Z}$. Observe that for $n \in \mathbb{N}$ we have $\tau^2(-n,n) = (-n,n)$, hence $(x, x+1) \notin \mathcal{D}_2^0$.
(ii) First we show that $M = \{(-n,m) \in \mathbb{Z}^2 \mid m \geq n \geq 0\}$ is contained in $S(x, x+1)$ by using induction on $\delta(-n,m) = m - n$. By (i) this assertion is clear if $\delta(-n,m) = 0$. Let $a = (-n,m) \in M$ with $\delta(a) > 0$ then $\tau(a) = (m,-p)$ with $m \leq p \leq 2m - n - 1$ and $\tau^2(a) = (-p, p+k) \in M$ with $0 \leq k \leq p - m$. As $\delta(\tau^2(a)) < \delta(a)$ we conclude that $a \in S(x, x+1)$.
(iii) From (ii) we immediately derive $(-\mathbb{N})^2 \subset S(x, x+1)$ because $\tau(-n,-m) = (-m, m+l) \in M$ with some $l \in \mathbb{N}$.
(iv) We now show that $L = \{(a,b) \in \mathbb{Z}^2 \mid ab \leq 0\} \cup \mathbb{N}^2$ is contained in $S(x, x+1)$ by using induction on $\| \cdot \|_1$. The induction start is trivial because $(0,0) \in S(x, x+1)$. Take $\mathbf{z} \in L \setminus \{0\}$.

Case I $\mathbf{z} = (n,-m)$ with $n, m \in \mathbb{N}$. Then $\tau(\mathbf{z}) = (-m, s)$ with $s = m - \lfloor (n-m)x \rfloor$.

Case I.1 $s \leq 0$. Then $\tau(\mathbf{z}) \in (-\mathbb{N})^2$ and we are done by (iii).

Case I.2 $s > 0$.

Case I.2.1 $s \geq m$. Then $\tau(\mathbf{z}) \in M$ and we are done by (ii).

Case I.2.2 $s < m$. Then $n > m$, $\tau(\mathbf{z}) \in L$ and $\| \tau(\mathbf{z}) \|_1 = m + s < \| \mathbf{z} \|_1$, hence we are done by induction hypothesis.

Case II $\mathbf{z} = (-n,m)$ with $n, m \in \mathbb{N}$.

Case II.1 $m \geq n$. Then $\mathbf{z} \in M$ and we are done by (ii).

Case II.2 $m < n$.

Case II.2.1 $m = 0$. Then $\mathbf{z} \in (-\mathbb{N})^2$ and we are done by (iii).

Case II.2.2 $m > 0$. Then $\tau(\mathbf{z}) = (m,s)$ with $s = l - m$, $l = -\lfloor -(n-m)x \rfloor$ and $1 \leq l \leq n - m$. Clearly, $\tau(\mathbf{z}) \in L$ and $-m+1 \leq s \leq n - 2m$.

Case II.2.2.1 $s \geq 0$. Then $\| \tau(\mathbf{z}) \|_1 = m + s < \| \mathbf{z} \|_1$ and we are done by induction hypothesis.

Case II.2.2.2 $s < 0$. Then $\| \tau(\mathbf{z}) \|_1 = m - s < \| \mathbf{z} \|_1$ and we are done.

Case III $z = (n,m)$ with $n, m \in \mathbb{N}$. Then $\tau(\mathbf{z}) = (m,-p)$ with $p \geq m$ and $\tau^2(\mathbf{z}) = (-p, q)$ with $q \geq p$. Thus $\tau^2(\mathbf{z}) \in M$ and our assertion follows from (ii).
(v) By (iii) and (iv) we finally see $\mathbb{Z}^2 \subseteq S(x, x+1)$ thereby completing the proof of the lemma. □

Lemma 2.4. *If $0 < x < 1$ then $(x, -x-1) \notin \mathcal{D}_2$.*

Proof. If $m > n > 0$ then $\tau_{(x,-x-1)}(n,m) = (m,p)$ with $p > m$. Thus the sequence

$$(\tau_{(x,-x-1)}^k(1,2))_{k \in \mathbb{N}}$$

is strictly monotonously increasing with respect to the norm $\| \cdot \|_1$. □

Proof of Theorem 2.1. (i) By the Schur-Cohn criterion [6] and ([4, Lemmas 4.1 and 4.2]) we know that $\mathcal{E}_2 \subseteq \mathcal{D}_2 \subseteq \overline{\mathcal{E}_2}$.

[6]See e. g. [18].

(ii) Let $(x,y)\in L$. We are going to show that (x,y) belongs to $\mathcal{D}_2$.
Case I $x<1$.
Case I.1 $|y|<1+x$. Then we are done by (i).
Case I.2 $|y|=1+x$.
Case I.2.1 $y<0$. Then $-y=1+x, x\le 0$ and we are done by Lemma 2.2.
Case I.2.2 $y\ge 0$. Thus $y=1+x$.
Case I.2.2.1 $x\le 0$. We are done by Lemma 2.2.
Case I.2.2.2 $x>0$. Our assertion drops out of Lemma 2.3.
Case II $x=1$. Then $y\in\{-1,0,1\}$ and the assertion can easily be checked.

(iii) Finally, let $(x,y)\in\mathcal{D}_2$. By (i) we know $|x|\le 1$ and $|y|\le 1+x$. We have to show that $(x,y)\in D$.
Case I $x<1$.
Case I.1 $|y|<1+x$. Then clearly $(x,y)\in D$.
Case I.2 $|y|=1+x$.
Case I.2.1 $y\ge 0$. Then $y=1+x$ and $(x,y)\in D$.
Case I.2.2 $y<0$. Then $-y=1+x$ and by Lemma 2.4 we have $x\le 0$, hence $(x,y)\in D$.
Case II $x=1$. Then $(1,2)\notin\mathcal{D}_2$ because it is easily seen by induction that for all $k\in\mathbb{N}$ there exists some $n\in\mathbb{N}, n>k$ such that $\tau_{(1,2)}^k(-1,2)\in\{(n,-(n+1)),(-n,n+1)\}$.
Similarly $(1,-2)\notin\mathcal{D}_2$ because for $a,b\in\mathbb{Z}$ we find $\tau_{(1,-2)}(a,b)=(b,2b-a)$ yielding

$$\parallel \tau_{(1,-2)}(a,b)\parallel_\infty > \parallel (a,b)\parallel_\infty$$

for $b>a>0$.

The result now follows easily. $\square$

Corollary 2.5. *We have*

$$\mathcal{D}_2^0\subset\mathcal{E}_2.$$

Proof. Since $\mathcal{D}_2^0\subset\mathcal{D}_2\subset\overline{\mathcal{E}_2}$ we have to show that $\mathcal{D}_2^0\cap\partial\mathcal{E}_2=\emptyset$. In view of Lemmas 2.2, 2.3 and 2.4 it remains to show

$$\{(1,y)\in\mathbb{R}^2 | -2\le y\le 2\}\cap\mathcal{D}_2^0=\emptyset.$$

In fact, assume $(1,y)\in\mathcal{D}_2^0$ for some $y\in\mathbb{R}$. Pick $z\in\mathbb{Z}^2\setminus\{0\}$ and choose the minimal $m\in\mathbb{N}$ with $\tau_{(1,y)}^m(z)=0$. Then $\tau_{(1,y)}^{m-1}(z)=(a,0)$ with some $a\in\mathbb{Z}\setminus\{0\}$. However, the relation $\tau_{(1,y)}(a,0)=0$ is impossible. $\square$

Remark 2.6. Let $\mathbf{r}=(1,y)\in\mathbb{R}^2$ with $0<y<1$. The first few elements of the sequence $(\tau_{\mathbf{r}}^k(\mathbf{z}))_{k\in\mathbb{N}}$ may grow considerably for certain $\mathbf{z}\in\mathbb{Z}^2\setminus\{0\}$. Thus it seems to be difficult to show that, as we conjecture, for each fixed $\mathbf{z}\in\mathbb{Z}^2\setminus\{0\}$ all elements of this sequence remain in a bounded region. Only some minor examples can be given here. Set $N:=\max\left\{n\in\mathbb{N}\mid n<(1-y)^{-1}\right\}$. Then

(i) $\{(a,b)\in\mathbb{Z}^2 \mid \parallel (a,b)\parallel_\infty\le 2,\ \ |a+\lfloor by\rfloor|\le 2\}\subset S(\mathbf{r})$.
(ii) If $0\le n\le N$ then $(0,n)\in S(\mathbf{r})$ with period length $6n+1$.
(iii) $\{(-2,3),(N+1,1)\}\cup\{(N+1,-k),(k,-(N+1))\in\mathbb{Z}^2\mid 0\le k\le N\}$

$$\cup\bigcup_{n=0}^{N}(\{(k,-n),(n,-k)\in\mathbb{Z}^2\mid 0\le k\le n\}\cup\{(0,n),(n+1,-n),(n+1,-(n-1)\})\subset S(\mathbf{r}).$$

You find partial results concerning this question in [5].

3. Several subsets of $\mathcal{D}_d^0$

In this section we give unified versions of results of Hollander [14], Kovács and Pethő [17], Pethő [20] (see also [3, Theorem 2.3]) as well as Frougny and Solomyak [11]. These results will be stated in the language of SRS and can be transformed to characterization results on β-expansions and canonical number systems, respectively, by applying the correspondence results in [4, Theorem 2.1 and Theorem 3.1].

In the following, we employ Hollander's framework and additionally make use of the idea of the "set of witnesses" invented independently by Brunotte [10] and Scheicher and Thuswaldner [24]. This makes the proofs substantially simpler and we shall have a way to describe what happens on the boundary of the region that Hollander gave (see Corollary 3.6).

Before stating the results, we review the algorithms contained in [4, Theorem 5.1 and Theorem 5.2] in a convenient form. For $i \in \{1, \dots, d\}$ let e_i be the i-th canonical basis vector of $\mathbb{R}^d$ and set $r_{d+1} = 1$. For a given $\mathbf{r} = (r_1, \dots, r_d)$, we say that a set $\mathcal{V} \subset \mathbb{Z}^d$ is a *set of witnesses* if $\pm e_i \in \mathcal{V}$ $(1 \le i \le d)$ and if for each $(z_1, \dots, z_d) \in \mathcal{V}$, the element $(z_2, \dots z_{d+1})$ belongs to $\mathcal{V}$ provided that

$$-1 < r_1 z_1 + \cdots + r_{d+1} z_{d+1} < 1. \tag{3.1}$$

Let $\mathcal{G}(\mathcal{V})$ be a graph with vertices $\mathcal{V}$ and edges defined by $(z_1, \dots, z_d) \to (z_2, \dots, z_{d+1})$ if and only if

$$0 \le r_1 z_1 + \cdots + r_{d+1} z_{d+1} < 1. \tag{3.2}$$

We say that $(a_1, a_2, \dots, a_d); a_{d+1}, \dots, a_L$ is a period of length L in the graph $\mathcal{G}(\mathcal{V})$ if there are edges

$$(a_i, \dots, a_{i+d-1}) \to (a_{i+1}, \dots, a_{i+d})$$

for each $i \in \mathbb{Z}$. Here a_i $(i \in \mathbb{Z})$ is naturally defined by periodicity $a_i = a_{i+L}$.

By definition, for each vertex there exists exactly one outgoing edge. The result in [4, Theorem 5.1] states the following.

Lemma 3.1. *If every infinite walk in the graph $\mathcal{G}(\mathcal{V})$ ends up in the trivial cycle $0 \to 0$ then $\mathbf{r} \in \mathcal{D}_d^0$.*

Suppose that $\pi = (a_1, a_2, \dots, a_d); a_{d+1}, \dots, a_L$ is a period of length L. Obviously, π is a period of $\tau_{\mathbf{r}}$ if and only if $\mathbf{r} \in \mathcal{D}_d$ satisfies

$$0 \le r_1 a_i + \cdots + r_{d+1} a_{d+i} < 1 \qquad (i \in \mathbb{N}).$$

Since by periodicity this is a finite set of inequalities it determines a (possibly degenerate) polyhedron $P(\pi) \subset \mathbb{R}^d$. We call this polyhedron the *cutout polyhedron* corresponding to π. Note that if $(a_1, \dots, a_d) \ne 0$ then $P(\pi) \cap \mathcal{D}_d^0 = \emptyset$ since each $\mathbf{r} \in P(\pi)$ has period π and is therefore not an SRS. So each nontrivial period π "cuts out" a polyhedron from $\mathcal{D}_d$.

In [4, Theorem 5.2] it is shown that a similar algorithm even works for the convex hull H of finitely many points $\mathbf{r}_1, \dots, \mathbf{r}_k \in \mathcal{D}_d$. In particular, the following result was proved.

Lemma 3.2. *Let H be as above. If the diameter of H is sufficiently small then there is an algorithm for the construction of a graph $(\mathcal{V}, E)$ having the following properties.*

1. $\pm e_1, \dots, \pm e_d \in \mathcal{V}$
2. *If $= (z_1, \dots, z_d) \in \mathcal{V}$, then $(z_2, \dots z_{d+1}) \in \mathcal{V}$ if and only if*
$$z_{d+1} \in \left[\min_{1 \le i \le k}\{\lfloor -\mathbf{r}_i \mathbf{z} \rfloor\}, \max_{1 \le i \le k}\{-\lfloor \mathbf{r}_i \mathbf{z} \rfloor\}\right] \cap \mathbb{Z}.$$
Furthermore, we put an edge $(z_1, \dots, z_d) \to (z_2, \dots z_{d+1}) \in E$ if we even have
$$z_{d+1} \in \left[\min_{1 \le i \le k}\{-\lfloor \mathbf{r}_i \mathbf{z} \rfloor\}, \max_{1 \le i \le k}\{-\lfloor \mathbf{r}_i \mathbf{z} \rfloor\}\right] \cap \mathbb{Z}.$$
3. $H \cap \mathcal{D}_d^0 = H \setminus \bigcup_\pi P(\pi)$, *where the union is taken over all nonzero primitive cycles π of $(\mathcal{V}, E)$.*

$(\mathcal{V}, E)$ can be constructed by the following algorithm. Start with $V_0 := \{\pm e_1, \dots, \pm e_d\}$. Given V_i $(i \ge 0)$ we construct V_{i+1} by (2). This is done until $V_i = V_{i+1} =: \mathcal{V}$. The edges E between the vertices $\mathcal{V}$ are defined by (2).

This lemma is a slight improvement of [4, Theorem 5.2] since the number of edges in the graph is diminished. However, the proof remains the same. We only have to note that the edges occurring in the present lemma are enough to guarantee that each graph given by a point $\mathbf{r} \in H$ according to Lemma 3.1 is a subgraph of $(\mathcal{V}, E)$.

If the algorithm given in Lemma 3.2 does not converge, we have to subdivide H into several parts and perform the algorithm for each of these parts. For further details on this algorithm we refer to [4, Theorem 5.2] and the discussion after its proof.

The next theorem together with its proof is a slight modification of [7, Corollary 1].

Theorem 3.3. *If $\sum_{i=1}^{d} |r_i| \le 1$ then $\mathcal{U}_d = \{(z_1, \dots, z_d) \mid z_i \in \{0, \pm 1\}\}$ is a set of witnesses for $\mathbf{r}$. Further if $r_i \ge 0$ for $i = 1, \dots, d$ and $\sum_{i=1}^{d} r_i < 1$ then $\mathbf{r} \in \mathcal{D}_d^0$.*

Proof. By $\sum_{i=1}^{d} |r_i| \le 1$, we have $|\sum_{i=1}^{d} r_i z_i| \le \sum_{i=1}^{d} |r_i| \le 1$. Thus $z_{d+1} \in \{0, \pm 1\}$ by (3.1) which shows that $\mathcal{U}_d$ is a set of witnesses. Further if $\sum_{i=1}^{d} |r_i| < 1$ then $\mathbf{r} \in \mathcal{E}_d$, i.e $\tau_{\mathbf{r}}$ is contracting. Thus it suffices to show that the only period in $\mathcal{G}(\mathcal{U}_d)$ is the 0-cycle if $r_i \ge 0$ for $i = 1, \dots, d$ and $\sum_{i=1}^{d} r_i < 1$. Suppose that $(a_1, \dots, a_d); a_{d+1}, \dots, a_L$ is a period in $\mathcal{G}(\mathcal{U}_d)$. Assume that there exists an index i such that $a_i = -1$. Then shifting indices, we have

$$0 \le r_1 a_1 + \cdots + r_{d+1} a_{d+1} < 1.$$

with $a_{d+1} = -1$. This implies that $1 \le \sum_{i=1}^{d} r_i a_i \le \sum_{i=1}^{d} r_i < 1$ which is a contradiction. Thus $a_i \ge 0$ for each i. Assume that there exists i that $a_i = 1$. Shifting indices again, we have $0 \le r_1 a_1 + \cdots + r_{d+1} a_{d+1} < 1$ with $a_{d+1} = 1$. But this implies another contradiction $0 \le \sum_{i=1}^{d} r_i a_i < 0$. The result now follows from Lemma 3.1. □

We can also generalize [6, Theorem 3.5].

Theorem 3.4. *If $\sum_{i=1}^{d} |r_i| < 1$ and there exists exactly one index k in $\{1, 2, \dots, d\}$ such that $r_{d+1-k} < 0$. Then $\mathbf{r} \in \mathcal{D}_d^0$ if and only if $\sum_{1 \le j \le d/k} r_{d+1-kj} \ge 0$.*

Proof. By Theorem 3.3, $\mathcal{U}_d$ is a set of witnesses. First if $\sum_{0 \le j \le d/k} r_{d+1-kj} < 0$ then the period $0, 0, \dots, 0, 1$ of length k is in $\mathcal{G}(\mathcal{U}_d)$. Thus $\mathbf{r} \notin \mathcal{D}_d^0$ which shows the necessity of the condition. Let us show the sufficiency. Assume that there exists a non-zero period

$$(a_1, \dots, a_d); a_{d+1}, \dots, a_L$$

in $\mathcal{G}(\mathcal{U}_d)$. By the same discussion as in the proof of Theorem 3.3, $a_i \ge 0$ for all i. Shifting indices, we have $0 \le r_1 a_1 + \cdots + r_{d+1} a_{d+1} < 1$ with $a_{d+1} = 1$. This shows that $a_{d+1-k} = 1$ since $d+1-k$ is the only index such that $r_{d+1-k} < 0$. Repeating this, we have $a_{d+1-kj} = 1$ for all $j = 0, 1, 2, \dots$. However this shows that

$$\sum_{i=1}^{d+1} r_i a_i \ge 1 + \sum_{1 \le k \le d/k} r_{d+1-kj} \ge 1$$

which contradicts (3.2). The result now follows from Lemma 3.1. □

We say that $(z_1, \dots, z_d) \in \mathcal{U}_d$ is *sign alternating*, if $z_i z_j \le 0$ holds for any pair of positive integers $i < j$ having the property that $z_k = 0$ for each $i < k < j$. In other words, ignoring 0 the numbers 1 and -1 occur alternatively[7]. Define a set

$$\mathcal{W}_d = \{\, (z_1, \dots, z_d) \in \mathcal{U}_d \mid \ (z_1, \dots, z_d) \text{ is sign alternating} \}\,.$$

For example,

$$\begin{aligned}
\mathcal{W}_1 &= \{-1, 0, 1\},\\
\mathcal{W}_2 &= \{(-1,0), (-1,1), (0,-1), (0,0), (0,1), (1,-1), (1,0)\},\\
\mathcal{W}_3 &= \{(-1,0,0), (-1,0,1), (-1,1,-1), (-1,1,0), (0,-1,0), (0,-1,1), (0,0,-1), (0,0,0),\\
&\qquad (0,0,1), (0,1,-1), (0,1,0), (1,-1,0), (1,-1,1), (1,0,-1), (1,0,0)\}.
\end{aligned}$$

An easy induction argument shows that the cardinality of $\mathcal{W}_d$ is $2^{d+1} - 1$. We say that a period $(a_1, a_2, \dots a_d); a_{d+1}, \dots, a_L$ is sign alternating if each $(a_i, \dots, a_{i+d-1})$ is sign alternating for all $i \in \mathbb{N}$.

[7]This sign alternating set first appeared in Scheicher [23].

Theorem 3.5. *If* $0 \le r_1 \le r_2 \le \cdots \le r_d \le 1$ *then* $\mathcal{W}_d$ *is a set of witnesses for* $\mathbf{r}$. *Further if* $0 \le r_1 \le r_2 \le \cdots \le r_d < 1$ *then* $\mathbf{r} \in \mathcal{D}_d^0$.

Proof. Note that $\pm e_i \in \mathcal{W}_d$ $(1 \le i \le d)$. Assume that $(z_1, \dots, z_d) \ne 0$. Let j be the maximum index in $\{1, 2, \dots, d\}$ for which $z_j \ne 0$. Then by the sign alternating property, the sum $\sum_{i=1}^d r_i z_i$ takes a value between 0 and $\mathrm{sign}(z_j) r_j$. Thus (3.2) implies $z_j z_{d+1} \le 0$. This shows that $\mathcal{W}_d$ is a set of witnesses. Further if $0 \le r_1 \le r_2 \le \cdots \le r_d < 1$ then $\mathbf{r} \in \mathcal{E}_d$, i.e. $\tau_{\mathbf{r}}$ is contracting (cf. [9]). Thus by Lemma 3.1 it suffices to show that the only period in $\mathcal{G}(\mathcal{W}_d)$ is the trivial 0-cycle. Note that each period in $\mathcal{G}(\mathcal{W}_d)$ is sign alternating by definition. Suppose that $(a_1, \dots, a_d); a_{d+1}, \dots, a_L$ is a nonzero period in $\mathcal{G}(\mathcal{W}_d)$. By shifting indices and (3.2), we have

$$0 \le r_1 a_1 + \cdots + r_{d+1} a_{d+1} < 1$$

with $a_{d+1} = -1$. By the left inequality, there must be an index $j \in \{1, \dots, d\}$ such that $a_j = 1$. Take the maximal j with $a_j > 0$. Then $a_k = 0$ for $j < k < d+1$ and $0 \le \sum_{i=1}^d r_i a_i \le r_j < 1$ by the sign alternating property. This gives a contradiction. Thus $a_i \ge 0$ for all i. Assume that there exists an index i that $a_i = 1$. Shifting indices again we have $0 \le r_1 a_1 + \cdots + r_{d+1} a_{d+1} < 1$. with $a_{d+1} = 1$. By the right inequality, there must exist an index $j \in \{1, \dots, d\}$ with $a_j = -1$ which is a contradiction. ☐

Theorems 3.3, 3.4 and 3.5 give a pretty large SRS region in $\mathcal{D}_d^0$. Moreover one can discuss the boundary of these regions. In fact, Theorems 3.3 and 3.5 give a set of witnesses when $0 \le r_1 \le \cdots \le r_d \le 1$ or $\sum_{i=1}^d |r_i| \le 1$. Thus we can describe the sets $\left\{ \mathbf{r} \in \mathcal{D}_d^0 \cap \mathcal{E}_d \;\middle|\; 0 \le r_1 \le \cdots \le r_d \le 1 \right\}$ and $\left\{ \mathbf{r} \in \mathcal{D}_d^0 \cap \mathcal{E}_d \;\middle|\; \sum_{i=1}^d |r_i| \le 1 \right\}$ including their boundary explicitly by using the algorithm in Lemma 3.1.

The remaining part of the paper is devoted to the characterization of $\mathcal{D}_2^0$. It is clear that $\mathcal{D}_2^0$ is a subset of $\mathcal{D}_2$. However, by [4, Example 4.7] and Corollary 2.5 we even see that

$$\mathcal{D}_2^0 \subset \mathcal{D}_2'$$

where $\mathcal{D}_2'$ is the trapezium

$$\mathcal{D}_2' = \{(x, y) \mid 0 \le x < 1,\ -x < y < x + 1\}.$$

depicted in Figure 1. Theorems 3.3, 3.4 and 3.5 imply that

$$\begin{gathered} \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le y < 1\}, \\ \{(x, y) \in \mathbb{R}^2 \mid x \ge 0,\ 0 \le x + y < 1,\ y > x - 1\} \end{gathered}$$

are contained in $\mathcal{D}_2^0$ (cf. Figure 1). We now give the characterization result for the boundary of these regions. In the following we frequently denote by $\Delta(a, b, c)$ the plane closed triangle with vertices $a, b, c \in \mathbb{R}^2$.

Corollary 3.6. *Let*

$$\begin{aligned} F_1 &= \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le y \le 1\}, \\ F_2 &= \{(x, y) \in \mathbb{R}^2 \mid x \ge 0,\ 0 \le x + y \le 1,\ y \ge x - 1\} \quad \textit{and} \\ F &= (F_1 \cup F_2) \setminus \{(0, 1), (1, 0), (1, 1)\}. \end{aligned}$$

Then $F \subset \mathcal{D}_2^0$, *and* $(0, 1), (1, 0), (1, 1) \in \mathcal{D}_2 \setminus \mathcal{D}_2^0$.

Proof. Note that $(F_1 \cup F_2) \setminus \mathcal{E}_2 = \{(0, 1), (1, 0), (1, 1)\}$. For these exceptional points we have $(0, 1), (1, 0), (1, 1) \in \mathcal{D}_2 \setminus \mathcal{D}_2^0$ by Theorem 2.1. Define three triangles $\Delta_1 = \Delta((0, 0), (0, 1), (1, 1))$, $\Delta_2 = \Delta((0, 0), (0, 1), (1, 0))$ and $\Delta_3 = \Delta((0, 0), (1, 0), (1, -1))$. Lemma 3.1 can be applied to each point of these triangles. It is easy to check that at each point this algorithm yields exactly the set $\mathcal{V}$ given by Theorems 3.5, 3.4 and 3.3, respectively, as set of witnesses. We just draw all possible edges according to Lemma 3.1 and depict their graphs. For Δ_1, we get the graph given in Figure 2. The trivial cycle $0 \to 0$ and the incoming edges of it (indicated by wavy arrows) are removed. After this removal, there are 6 primitive cycles and we can directly show $\Delta_1 \setminus \{(0, 1), (1, 1)\} \subset \mathcal{D}_2^0$ from these by calculating the related cutout polygons. In fact, it can be done even simpler. The two broken arrows appear only for the points $(0, 1)$ or $(1, 0)$ which have been excluded by Theorem 2.1.

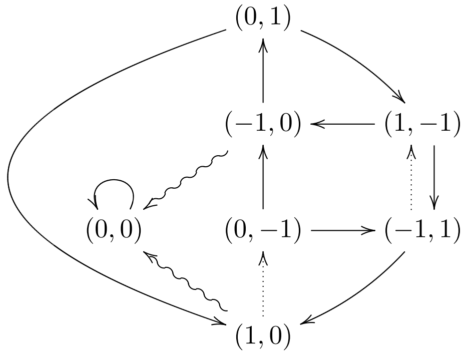

FIGURE 2. The graph $(\mathcal{V}, E)$ for Δ_1

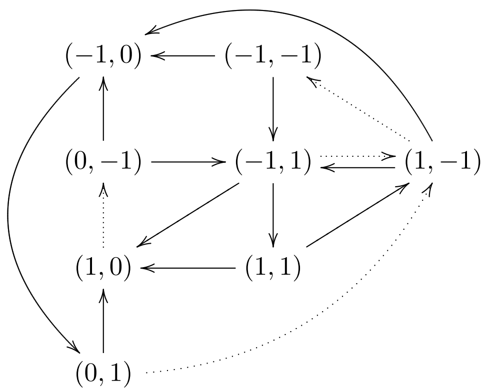

FIGURE 3. The graph $(\mathcal{V}, E)$ for Δ_2 without the trivial cycle $0 \to 0$

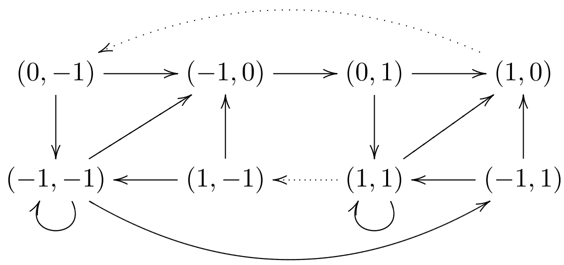

FIGURE 4. The graph $(\mathcal{V}, E)$ for Δ_3 without the trivial cycle $0 \to 0$

Removing the broken arrows only the primitive cycle $(0, 1); -1$ remains. This gives the cutout polygon $P((0, 1); -1) = \{(1, 1)\}$. Thus we proved that $\Delta_1 \setminus \{(0, 1), (1, 1)\} \subset \mathcal{D}_2^0$.

Hereafter we omit drawing the trivial cycle and its incoming edges. For Δ_2, the resulting graph is depicted in Figure 3. The broken arrows appear only for the points $(1, 0)$ or $(0, 1)$. Thus in $\Delta_2 \setminus \{(1, 0), (0, 1)\}$, the only non-trivial cycle is given by $(-1, 1); 1$. The cutout polygon $P((-1, 1); 1)$ is easily seen to be empty.

Finally for Δ_3, we have the graph depicted in Figure 4. The broken arrows appear only for the point $(1, 0)$. Therefore in $\Delta_3 \setminus \{(1, 0)\}$ there are only two relevant cycles $(1, 1)$; and $(-1, -1)$; which are the self loops $1 \to 1$ and $-1 \to -1$. Both of the associated cutout polygons are irrelevant.

□

In general it is easier to examine if a certain region does not belong to $\mathcal{D}_2^0$ than the opposite. The next result contains some regions of $\mathcal{D}_2 \setminus \mathcal{D}_2^0$.

Proposition 3.7. *Set*

$$\begin{aligned} E_1 &= \left\{(x,y) \;\middle|\; x<1,\ y<2x,\ \frac{2x}{3}+1\le y\right\}, \\ E_2 &= \left\{(x,y) \;\middle|\; x<1,\ \frac{x}{2}+1<y<2x,\ y<\frac{2x}{3}+1\right\}, \\ E_3 &= \left\{(x,y) \;\middle|\; x<1,\ -x+\frac{1}{2}\le y<2x-2,\ y<-\frac{x}{3}\right\}, \\ E_4 &= \left\{(x,y) \;\middle|\; x<1,-2x+1\le y<-\frac{1}{2}x\right\}. \end{aligned}$$

Then

$$E_1\cup E_2\cup E_3\cup E_4 \subseteq \mathcal{D}_2\setminus\mathcal{D}_2^0.$$

In fact these regions are the four dark cutout polygons depicted in Figure 1.

Proof. It follows from Theorem 2.1 that E_i is a subset of $\mathcal{D}_2$ for $i\in\{1,2,3,4\}$. Thus it remains to show that they have empty intersection with $\mathcal{D}_2^0$.

Each of the sets E_i $(1\le i\le 4)$ corresponds to a cutout polygon related to a certain period. Consider the period $\pi_4=(2,1);-1,-1,1$. From the definition of a cutout polygon we see that $P((2,1);-1,-1,1)$ is given by the set of all points (x,y) satisfying

$$\begin{aligned} 0 &\le 2x+y-1<1, \\ 0 &\le x-y-1<1, \\ 0 &\le -x-y+1<1, \\ 0 &\le -x+y+2<1, \\ 0 &\le x+2y+1<1. \end{aligned}$$

Simplifying this system of inequalities we get

$$P((2,1);-1,-1,1)=\left\{(x,y) \;\middle|\; x-2<y<-\frac{x}{2},\ y\ge -2x+1\right\}. \tag{3.3}$$

From this we see that $E_4=P((2,1);-1,-1,1)\cap\mathcal{D}_d$ and the proposition is proved for E_4. E_1, E_2 and E_3 correspond to longer cycles. With the same type of arguments we can show that

$$\begin{aligned} E_1 &= P((1,-2);3,-3,3,-2,1)\cap\mathcal{D}_d, \\ E_2 &= P((1,-2);3,-2,1)\cap\mathcal{D}_d, \\ E_3 &= P((2,-1);-2,1,3,1,-2,-1,2)\cap\mathcal{D}_d. \end{aligned}$$

This proves the result[8]. □

Remark 3.8. It is possible to show an analogue of Corollary 3.6 for $\mathcal{D}_3$ also. In fact, let

$$\begin{aligned} F_1 &= \{(x,y,z)\in\mathbb{R}^3 \mid 0\le x\le y\le z\le 1\}, \\ F_2 &= \{(x,y,z)\in\mathbb{R}^3 \mid x\ge 0, y\ge 0,\ 0\le x+y+z\le 1,\ z\ge x+y-1\} \quad \text{and} \\ F &= (F_1\cup F_2)\setminus\{(0,1,0),(0,1,1)\}\cup\{(x,x,1)\in\mathbb{R}^3\mid x\ge 0\}\cup\{(x,0,1-x)\in\mathbb{R}^3\mid x\ge 0\}. \end{aligned}$$

Then $F\subset\mathcal{D}_3^0$.

The proof of this result is much more involved than the proof of Corollary 3.6 and will appear elsewhere.

4. Subsets of $\mathcal{D}_2^0$ near to the boundary of $\mathcal{D}_2$

The characterization of $\mathcal{D}_2^0$ becomes more and more difficult the nearer we approach to $\partial\mathcal{D}_2$. In this section we show two characterization results of $\mathcal{D}_2^0$ near the boundary of $\mathcal{D}_2$.

[8] Explicit representations of these cutout polygons are given in Section 5.

4.1. **An SRS region near the upper boundary of $\mathcal{D}_2$.**

In this subsection we describe another idea to exhibit a region belonging to $\mathcal{D}_2^0$. Let R be the subset of $\mathcal{D}_2$ given by

$$R = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, 0 < x < 1,\ y > 0,\ y < x+1,\ x < \frac{y^2}{4}\right\}.$$

For $(x,y) \in R$ consider the characteristic polynomial of the matrix[9]

$$\begin{pmatrix} 0 & 1 \\ -x & -y \end{pmatrix}$$

given by $\chi(t) = t^2 + yt + x$. Denote by α and β the two roots of $\chi(t)$. As $(x,y) \in R$, α and β are real and have modulus less than 1. Clearly, we have $y = -(\alpha+\beta) > 0$ and $x = \alpha\beta$. As $\chi(0) = x > 0$, $\chi(-1) = 1 - y + x > 0$ and $\chi(-\frac{y}{2}) < 0$, we may assume $-1 < \alpha < \beta < 0$.

We denote by

$$\Pi(x,y) = \{\mathbf{a} \in \mathbb{Z}^2 \mid \tau_{(x,y)}^{\ell}(\mathbf{a}) = \mathbf{a} \text{ for some } \ell > 0\}$$

the purely periodic elements associated to $\tau_{(x,y)}$. For an element $\mathbf{a} \in \Pi(x,y)$ of period length L, i. e.

$$(a_1, a_2); a_3, \ldots, a_L$$

we let for convenience $\Xi_{\mathbf{a}} = \ldots a_{-2}a_{-1}a_0a_1a_2\ldots a_L\ldots$ be the bi-infinite periodic word generated by $\mathbf{a}$. If a_i is a letter in the word $\Xi_{\mathbf{a}}$ then we will write $a_i \in \Xi_{\mathbf{a}}$.

Proposition 4.1. *Let $(x,y) \in R$ and $\mathbf{a} \in \Pi(x,y)$. Then*

$$\frac{\beta}{1-\beta^2} \leq a_{i+1} - \alpha a_i \leq \frac{1}{1-\beta^2} \qquad \textit{and} \tag{4.1}$$

$$\frac{\alpha}{1-\alpha^2} \leq a_{i+1} - \beta a_i \leq \frac{1}{1-\alpha^2} \tag{4.2}$$

hold for all consecutive letters $a_i, a_{i+1} \in \Xi_{\mathbf{a}}$.

Proof. We only show (4.1); (4.2) is proved in a similar way. By the definition of $\tau_{(x,y)}$ we have for $i \in \mathbb{Z}$

$$0 \leq x a_{i+1} + y a_{i+2} + a_{i+3} < 1.$$

Rewrite this into

$$0 \leq (a_{i+3} - \alpha a_{i+2}) - \beta(a_{i+2} - \alpha a_{i+1}) < 1. \tag{4.3}$$

Multiplying (4.3) by $\beta < 0$ and shifting indices of (4.3), respectively, we get

$$\begin{aligned} \beta &< \beta(a_{i+3} - \alpha a_{i+2}) - \beta^2(a_{i+2} - \alpha a_{i+1}) \leq 0 \\ 0 &\leq (a_{i+4} - \alpha a_{i+3}) - \beta(a_{i+3} - \alpha a_{i+2}) < 1. \end{aligned}$$

Adding these two chains of inequalities, we see that

$$\beta < (a_{i+4} - \alpha a_{i+3}) - \beta^2(a_{i+2} - \alpha a_{i+1}) < 1.$$

Repeating this and shifting indices, we have

$$\cdots + \beta^5 + \beta^3 + \beta < (a_{i+2} - \alpha a_{i+1}) - \beta^{n+1}(a_{i-n+1} - \alpha a_{i-n}) < 1 + \beta^2 + \beta^4 + \cdots$$

for all $n \in \mathbb{N}$. As $\Xi_{\mathbf{a}}$ is a periodic word its letters are uniformly bounded. Thus, taking $n \to \infty$ we get the result. $\square$

Remark 4.2. As $\alpha \neq \beta$, Proposition 4.1 gives lattice points in a parallelogram. However, if α is near to -1 the above inequalities neither give a uniform bound nor a "uniform" algorithm[10] to determine whether (x,y) belongs to $\mathcal{D}_2^0$ or not.

[9] See [4, Section 4].

[10] Like the one in Lemma 3.2.

For $\kappa \in \mathbb{R}$ let
$$R_\kappa = \left\{(x,y) \in R \,|\, x < \kappa y - \kappa^2\right\}.$$
In the following we assume $0 < \kappa \le \gamma_q$ where $q > 0$ is an integer and γ_q is the positive root of the polynomial $qt^3 + qt^2 - qt - q + 1$; in particular we have $\gamma_1 = \frac{1}{\omega}$ with $\omega = \frac{1+\sqrt{5}}{2}$. Observe that for $(x,y) \in R_\kappa$, the following inequalities hold:
$$\alpha < -\kappa < \beta, \quad \frac{1}{(1-\alpha)(1-\beta^2)} < q.$$

Lemma 4.3. *Let $(x,y) \in R_\kappa$ and $(a,b) \in \Pi(x,y)$ with $\min\{|a|,|b|\} \ge q$. Then $ab \le 0$.*

Proof. Let us assume $ab > 0$. If $a,b > 0$ then by Proposition 4.1 we find $(1-\alpha)\min\{a,b\} \le b - \alpha a \le \frac{1}{1-\beta^2}$ yielding
$$q \le \min\{a,b\} \le \frac{1}{(1-\alpha)(1-\beta^2)} < q$$
which is impossible. Analogously, if $a,b < 0$ we have $\frac{\beta}{1-\beta^2} \le b - \alpha a \le (1-\alpha)\max\{a,b\}$ yielding the contradiction
$$-q \ge \max\{a,b\} \ge \frac{\beta}{(1-\alpha)(1-\beta^2)} > \beta q > -q.$$
□

In the sequel, the (finite) set
$$A_{\kappa,q} = \left\{(a,b) \in \mathbb{Z}^2 \,\middle|\, |a| < q,\ -\frac{\kappa}{1-\kappa^2} - q + 1 < b < \frac{1}{1-\kappa^2} + q - 1\right\}$$
will help us to decide whether a given element of R_κ belongs to $\mathcal{D}_2^0$.

Lemma 4.4. *Let $(x,y) \in R_\kappa$ and $(a,b) \in \Pi(x,y)$ with $|a| < q$. Then $(a,b) \in A_{\kappa,q}$.*

Proof. By Proposition 4.1 we find $\frac{\beta}{1-\beta^2} - (q-1) \le b \le \frac{1}{1-\beta^2} + q - 1$ from which we easily deduce our assertion. □

Lemma 4.5. *Let $(x,y) \in R_\kappa$ and suppose that for all $(a,b) \in A_{\kappa,q}$ there exists a $k \in \mathbb{N}$ such that $\tau_{(x,y)}^k(a,b) = 0$. Then $(x,y) \in \mathcal{D}_2^0$.*

Proof. Assume that $(x,y) \notin \mathcal{D}_2^0$. Let $\mathbf{a} \in \Pi(x,y)$ be a non-zero periodic point associated to $\tau_{(x,y)}$.

(i) We first observe that $|a_i| \ge q$ for all $a_i \in \Xi_{\mathbf{a}}$. Indeed, if $|a_i| < q$ for some $i \in \mathbb{Z}$ then $(a_i, a_{i+1}) \in A_{\kappa,q}$ by Lemma 4.4, hence the orbit of $\mathbf{a}$ tends to zero contrary to our hypothesis.

(ii) By the periodicity of $\Xi_{\mathbf{a}}$ there exists some index i with $|a_{i+2}| \le |a_{i+1}|$. Using (i) we see that the element (a_{i+1}, a_{i+2}) belongs to the set
$$E = \{(c,d) \in \Pi(x,y) \,|\, q \le |d| \le |c|\}.$$

(iii) We claim that the set E is invariant under $\tau_{(x,y)}$: Indeed, for $(c,d) \in E$ we clearly have $\tau_{(x,y)}(c,d) \in \Pi(x,y) \setminus \{0\}$. We distinguish two cases:

Case 1. Assume that $d < 0$. Then Lemma 4.3 implies $c > 0$. Since $|d| \le |c|$, we see
$$\lfloor cx + dy \rfloor > cx + dy - 1 \ge (y-x)d - 1.$$
Therefore $|\lfloor cx + dy \rfloor| < |d| + 1$ which shows $|\lfloor cx + dy \rfloor| \le |d|$. Thus we get $\tau_{(x,y)}(c,d) \in E$.

Case 2. Assume that $d > 0$. Similarly we have $c < 0$. Since $|d| \le |c|$,
$$\lfloor cx + dy \rfloor \le cx + dy \le dy - dx < |d|.$$
Thus we have $|\lfloor cx + dy \rfloor| < |d|$ and again $\tau_{(x,y)}(c,d) \in E$.

(iv) We have shown that $|a_{i+1}| \le |a_i|$ for each i. Because of the periodicity this is possible only if $|a_{i+1}| = |a_i|$. Since $a_i \ne 0$, Lemma 4.3 implies that $a_{2i-1} = g$ and $a_{2i} = -g$ for some $g \ne 0$. Going back to the definition of $\tau_{(x,y)}$, we have
$$\begin{aligned} 0 &\le xg - yg + g < 1 \\ 0 &\le -xg + yg - g < 1. \end{aligned}$$
As $1 - y + x > 0$, this is possible only for $g = 0$. This yields a contradiction. □

We are now in a position to state the first theorem of this subsection.

Theorem 4.6. *The set*

$$\left\{(x,y)\in R \;\middle|\; x<\frac{1}{\omega^2} \quad or \quad y>\omega x+\frac{1}{\omega}\right\}$$

is contained in $\mathcal{D}_2^0$*.*

Proof. As $x=\kappa y-\kappa^2$ is the tangent line of $x=y^2/4$ at $(\kappa^2,2\kappa)$, we have

$$\bigcup_{0<\kappa\le 1/\omega} R_\kappa=\left\{(x,y)\in R \;\middle|\; x<\frac{1}{\omega^2} \quad \text{or} \quad y>\omega x+\frac{1}{\omega}\right\}.$$

Therefore it suffices to show that if $\kappa\le\frac{1}{\omega}$, then $R_\kappa\subset\mathcal{D}_2^0$. Taking $q=1$ we get

$$A_{\kappa,1}=\left\{(0,b)\in\mathbb{Z}^2 \;\middle|\; -\frac{\kappa}{1-\kappa^2}<b<\frac{1}{1-\kappa^2}\right\}$$

by the definition of $A_{\kappa,q}$ and

$$-1\le-\frac{\kappa}{1-\kappa^2}<b<\frac{1}{1-\kappa^2}\le\omega<1.7.$$

Thus $b\in\{0,1\}$. As $R\subset\{(x,y)\mid 0<x<1,\ x<y<x+1\}$, we easily see by direct computation that $\tau_{(x,y)}^4(0,1)=0$ for $(x,y)\in R$. Thus Lemma 4.5 concludes the proof. $\square$

It is of course possible to apply Lemma 4.5 to $q\ge 2$, but the corresponding graphs become very large and beyond hand computation. We need the following lemma.

Lemma 4.7. *Let $H\subset\mathcal{D}_d$ be the convex hull of $\mathbf{r}_1,\dots,\mathbf{r}_k\in\mathcal{D}_d$ and let $A\subset\mathbb{Z}^d$ be a finite set. Let $G_A(H)=(V,E)$ be the smallest graph with the following properties.*

(1) $A\subset V$.
(2) *If $\mathbf{z}=(z_1,\dots,z_d)\in V$ then $(z_2,\dots,z_d,j)\in V$ and $(z_1,\dots,z_d)\to(z_2,\dots,z_d,j)\in E$ if and only if*

$$j\in\left[\min_{1\le i\le k}\{-\lfloor\mathbf{r}_i\mathbf{z}\rfloor\},\ \max_{1\le i\le k}\{-\lfloor\mathbf{r}_i\mathbf{z}\rfloor\}\right].$$

If each infinite walk in $G_A(H)$ ends up in the zero cycle $0\to 0$ then

$$\forall\mathbf{r}\in H\ \forall\mathbf{a}\in A\ \exists k\in\mathbb{N}:\tau_{\mathbf{r}}^k(\mathbf{a})=0.$$

Proof. This follows immediately from the definition of $\tau_{\mathbf{r}}$ and the fact that $\min_{1\le i\le k}\{-\lfloor\mathbf{r}_i\mathbf{z}\rfloor\}\le-\lfloor\mathbf{r}\mathbf{z}\rfloor\le\max_{1\le i\le k}\{-\lfloor\mathbf{r}_i\mathbf{z}\rfloor\}$ holds for all $\mathbf{r}\in H$ (cf. [4, Theorem 4.6]). $\square$

The graph $G_A(H)$ can be constructed in an analogous way as the graph described in Lemma 3.2.

Theorem 4.8. $R_\kappa\subset\mathcal{D}_2^0$ *for* $\kappa=\gamma_2,\gamma_3,\ \gamma_4,\ \gamma_5$ *and* $\gamma_6\simeq 0.956458072$*.*

Proof. Suppose that the theorem is already proved for $q-1$. Start with an initial set of vertices $V=A_{\gamma_q,q}$ with $q\ge 2$ and construct $G_V(H)$ with

$$\begin{gathered}H=\Delta((\gamma_{q-1},1+\gamma_{q-1}),(\gamma_q,1+\gamma_q),(\gamma_{q-1}+\gamma_q,\gamma_{q-1}+\gamma_q))=\\ \Delta((0,1),(0,\gamma_q),(\gamma_q,1+\gamma_q))\setminus(\mathrm{Int}\Delta((0,1),(0,\gamma_{q-1}),(\gamma_{q-1},1+\gamma_{q-1}))),\end{gathered}$$

according to Lemma 4.7 (note that we may assume that the interior of $\Delta((0,1),(0,\gamma_{q-1}),(\gamma_{q-1},1+\gamma_{q-1}))$ is a subset of $\mathcal{D}_2^0$).

Delete edges $(-a,a)\to(a,-a)$ for $a=1,2,\dots$ which obviously only correspond to the boundary of R_{γ_q}. Delete also the trivial cycle $(0,0)\to(0,0)$. We call the resulting graph $\mathcal{G}_q$. The number of vertices and edges of $\mathcal{G}_q$ are listed in Table 1. If $\mathcal{G}_q$ is acyclic, then $R_{\gamma_q}\subset\mathcal{D}_2^0$ by Lemma 4.5. In fact, one can confirm that $\mathcal{G}_q$ is acyclic for $q=2,3,4,5,6$. $\square$

q	vertices	edges
2	294	538
3	1398	2292
4	3991	6554
5	8732	14408
6	16258	26951

TABLE 1. Size of $\mathcal{G}_q$

4.2. **An SRS region near the point** $(1,-1)$**.**

Our first aim is to show that the set

$$S := \left\{ (1-T, -1+cT) \,\middle|\, 0 < T \le \frac{1}{30},\, 1 \le c < 2 \right\}$$

is contained in $\mathcal{D}_2^0$. In particular, this shows that $(1,-1)$ is not a critical point[11]. Despite this is a very small region its characterization is the crucial part in proving Theorem 5.6, which characterizes a very big SRS region.

4.2.1. *Basic definitions.*

For some results it is convenient to deal with the following region:

$$R := \left\{ (1-T, -1+cT) \,\middle|\, 0 < T \le \frac{1}{30},\, 1 \le c \le 2 \right\}.$$

Let $\mathbf{r} \in \mathcal{D}_2$ and $(u,v) \in \mathbb{Z}^2$. We will need the following abbreviations

$$\begin{aligned} \alpha &:= \lfloor cTv - Tu \rfloor, \\ \beta &:= \lfloor -cT\alpha + (c-1)Tv - cTu \rfloor, \\ \gamma &:= \lfloor -cT\beta - (c-1)T\alpha - Tv - (c-1)Tu \rfloor. \end{aligned}$$

Furthermore, in what follows, we will set

$$(u_2, v_2) := \tau_{\mathbf{r}}^3(u,v).$$

From the definition of $\tau_{\mathbf{r}}$ this implies that

$$\begin{aligned} u_2 &= -\alpha - \beta - u, \\ v_2 &= -\beta - \gamma - v. \end{aligned} \tag{4.4}$$

The proof of the above-mentioned characterization result relies on a certain "structural stability" of $\tau_{\mathbf{r}}$ in $\mathbf{r}$. In fact, if we look at the orbit of a point (x,y) of $\tau_{\mathbf{r}}$ with $\mathbf{r} \in R$ essentially only one shape can occur. If T^{-1} is small compared to the modulus of the coordinates of (x,y) then the orbit of (x,y) is of a shape similar to the orbit in Figure 5. However, if T^{-1} is large compared to the coordinates (x,y) the orbit looks similar to the one depicted in Figure 6. (Note that near the origin of Figure 5 the orbit is of a similar shape as the orbit in Figure 6.) Looking at several examples of orbits of $\tau_{\mathbf{r}}$ ($\mathbf{r} \in R$) we are lead to conjecture that the following facts are always true: Each of the orbits consists of six "branches" (see Figures 5 and 6). If we number consecutively these branches from 1 to 6 then the following holds: If (x,y) is part of the branch 6 then $\tau_{\mathbf{r}}^k(x,y)$ is part of the branch $k \bmod 6$. Moreover, for points $(x,y) \in \mathbb{Z}^2$ and $\mathbf{r} \in R$ we always observe that $\tau_{\mathbf{r}}^3(x,y)$ is "near" to the point $(-x,-y)$. Because of this fact the third iterate of $\tau_{\mathbf{r}}$ plays a big role in our proofs. (Since $\tau_{\mathbf{r}}^6(x,y)$ is "near" to (x,y) in the orbits under consideration it may look more natural to deal with $\tau_{\mathbf{r}}^6$ rather than $\tau_{\mathbf{r}}^3$. However, this would cause much more involved proofs.) Let $\mathbf{r} \in R$ and $(x,y) \in \mathbb{Z}^2$. Consider an arbitrary branch of the orbit of (x,y). If this branch enters the second or fourth quadrant, it is farther away from the origin than it is when it exits this quadrant. Making precise these observations we will construct a sequence of points of each orbit with decreasing distance from the origin in the following way.

[11]For the definition see [4, Definition 7.1].

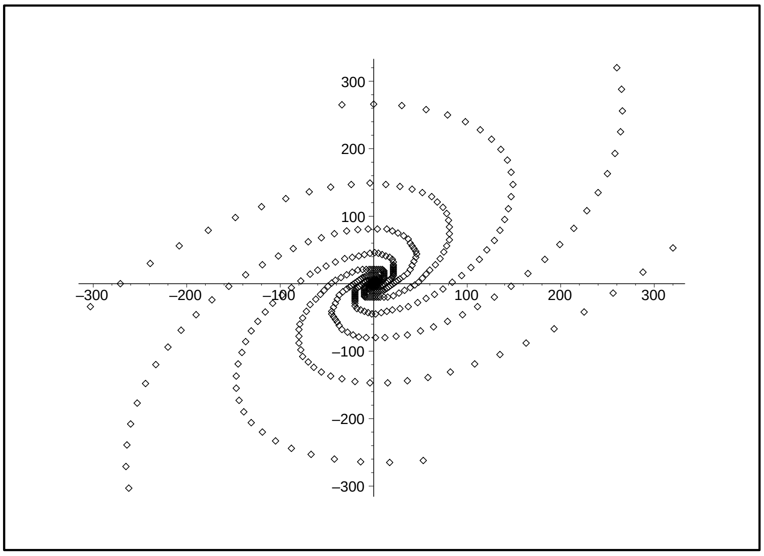

FIGURE 5. An example of an orbit

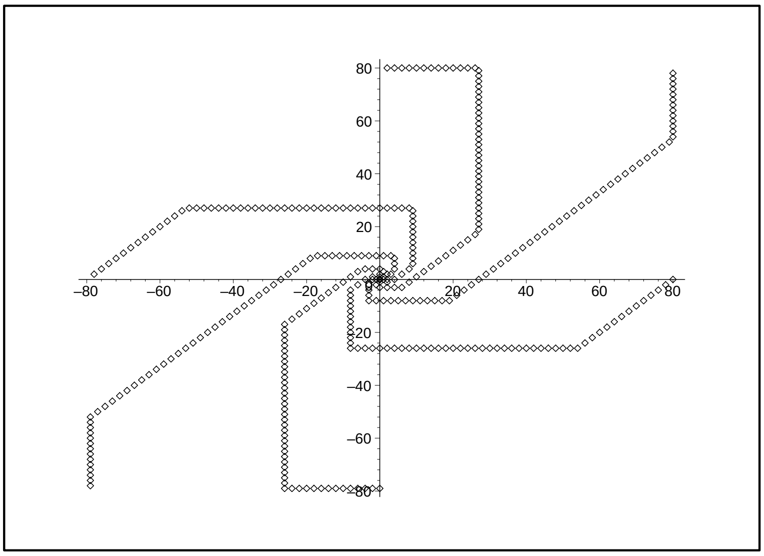

FIGURE 6. Another example of an orbit

Find an element (x_0, y_0) of the orbit of (x, y) contained in the second or fourth quadrant and follow (by iteration of $\tau_{\mathbf{r}}^3$) the branches of (x_0, y_0) and $\tau_{\mathbf{r}}^3(x_0, y_0)$ as long as they stay in the second and fourth quadrant, respectively. Denote the last element of this iteration process which stays in the second or fourth quadrant by (x_1, y_1). It turns out that $(x_2, y_2) = \tau_{\mathbf{r}}^2(x_1, y_1)$ is again contained in the second or fourth quadrant (but on another branch).

Now perform the following algorithm starting with $i = 1$ and $(x_0^{(1)}, y_0^{(1)}) := (x_2, y_2)$.

- Follow (by iteration of $\tau_{\mathbf{r}}^3$) the branches of $(x_0^{(i)}, y_0^{(i)})$ and $\tau_{\mathbf{r}}^3(x_0^{(i)}, y_0^{(i)})$ as long as they stay in the second and fourth quadrant, respectively. Denote the last element of this iteration process which stays in the second or fourth quadrant by $(x_1^{(i)}, y_1^{(i)})$.
- Set $(x_2^{(i)}, y_2^{(i)}) = \tau_{\mathbf{r}}^2(x_1^{(i)}, y_1^{(i)})$. This point is again contained in the second or fourth quadrant (but on another branch).
- If $\max\{|x_2^{(i)}|, |y_2^{(i)}|\} > 25$ then start again with $(x_0^{(i+1)}, y_0^{(i+1)}) := (x_2^{(i)}, y_2^{(i)})$.

We will show that either $\max\{|x_0^{(i+1)}|,|y_0^{(i+1)}|\} < \max\{|x_0^{(i)}|,|y_0^{(i)}|\}$ or $\max\{|x_0^{(i)}|,|y_0^{(i)}|\} \le 25$ holds. Thus the algorithm terminates after finitely many steps showing that each orbit contains a point (x',y') with $\max\{|x'|,|y'|\} \le 25$. Now in order to prove our result it remains to show that for each $\mathbf{r} \in S$ each (x',y') with $\max\{|x'|,|y'|\} \le 25$ has an orbit ending at $(0,0)$. This is done with computer aid.

4.2.2. *A series of lemmas.*

Before we can give our result, we need a series of technical lemmas. Some of these lemmas are valid even in larger domains than R. u_2 and v_2 are always defined as in (4.4).

Lemma 4.9. *Let $u \ge 0$ and $v \le 0$. Furthermore, suppose that $u \ge 2$ or $v \le -1$ holds. If $c \in [1,2]$ and $0 < T \le \frac{1}{5}$ then*

$$u + u_2 \ge v + v_2$$

holds.

Proof. By (4.4) the claim is equivalent to $\gamma \ge \alpha$. First observe that, since $(c-1)Tv - cTu \le 0$, we have

$$\begin{aligned} -cT\beta - (c-1)T\alpha &= -cT\lfloor -cT\alpha + (c-1)Tv - cTu \rfloor - (c-1)T\alpha \\ &\ge -cT\lfloor -cT\alpha \rfloor - (c-1)T\alpha \\ &\ge T(1-c+c^2T)\alpha. \end{aligned}$$

Inserting this in the definition of γ yields

$$\gamma \ge \left\lfloor T(1-c+c^2T)\alpha - Tv - (c-1)Tu \right\rfloor.$$

Suppose first that $1-c+c^2T \le 0$. Since $u \ge 0$ and $v \le 0$ we have $\alpha \le 0$ and thus

$$\gamma \ge \lfloor -Tv - (c-1)Tu \rfloor \ge \lfloor cTv - Tu \rfloor = \alpha.$$

Now suppose on the contrary that $1-c+c^2T > 0$. Since $T \le \frac{1}{5}$ this can happen only for $c < \frac{5}{2} - \frac{\sqrt{5}}{2} < \frac{7}{5}$. Now

$$\begin{aligned} \gamma &\ge \left\lfloor T(1-c+c^2T)\alpha - Tv - (c-1)Tu \right\rfloor \\ &\ge \left\lfloor T(1-c+c^2T)(cTv - Tu) - T(1-c+c^2T) - Tv - (c-1)Tu \right\rfloor \\ &= \left\lfloor c(1-c)T^2v + (c-1)T^2u + c^3T^3v - c^2T^3u - T(1-c+c^2T) - Tv - (c-1)Tu \right\rfloor. \end{aligned}$$

Since $1-c+c^2T < 1$, $c(1-c)T^2v \ge 0$ and $(c-1)T^2u \ge 0$ this implies that

$$\gamma \ge \left\lfloor c^3T^3v - c^2T^3u - T - Tv - (c-1)Tu \right\rfloor. \tag{4.5}$$

Now we have $u \ge 2$ or $v \le -1$. Suppose first that $u \ge 2$ holds. Then we have $-T \ge -\frac{1}{2}Tu$ and thus

$$\gamma \ge \left\lfloor \left(c^3T^2 - 1\right)Tv - \left(c^2T^2 + c - \frac{1}{2}\right)Tu \right\rfloor \ge \alpha.$$

The latter inequality follows because $c \le \frac{7}{5}$ and $T \le \frac{1}{5}$ imply that $c^3T^2 - 1 \le c$ and $c^2T^2 + c - \frac{1}{2} \le 1$.

If, on the other hand, $v \le -1$ we have $-T \ge Tv$ and thus

$$\gamma \ge \left\lfloor c^3T^3v - \left(c^2T^2 + c - 1\right)Tu \right\rfloor \ge \alpha.$$

The latter inequality follows because $c \le \frac{7}{5}$ and $T \le \frac{1}{5}$ imply that $c^3T^2 \le c$ and $c^2T^2 + c - 1 \le 1$. Thus the lemma is proved. $\square$

Lemma 4.10. *Let $u \le 0$, $v \ge 0$, $c \in [1,2]$ and $0 < T \le \frac{1}{5}$. Furthermore, suppose that $u \le -4$ or $v \ge 2$ holds. Then*

$$u + u_2 \le v + v_2.$$

Proof. It is easy to see that we have to prove $\gamma \le \alpha$. We first treat the case $v = 0$. Since $u \le 0$ we have $\alpha \ge 0$. Furthermore, $\beta = \lfloor -cT(u+\alpha)\rfloor$. Since

$$u + \alpha = u + \lfloor -Tu\rfloor \le u - Tu = (1-T)u \le 0$$

we also have $\beta \ge 0$. Thus

$$\gamma \le \lfloor -(c-1)Tu\rfloor \le \lfloor -Tu\rfloor = \alpha.$$

In what follows we may assume that $v \ge 1$. Observe that, because $(c-1)Tv - cTu \ge 0$, we have

$$\begin{aligned} -cT\beta - (c-1)T\alpha &= -cT\lfloor -cT\alpha + (c-1)Tv - cTu\rfloor - (c-1)T\alpha \\ &\le -cT\lfloor -cT\alpha\rfloor - (c-1)T\alpha \\ &\le T(1-c+c^2T)\alpha + cT. \end{aligned}$$

This implies that

$$\gamma \le \left\lfloor T(1-c+c^2T)\alpha + cT - Tv - (c-1)Tu\right\rfloor.$$

Suppose first that $1 - c + c^2T \le 0$. Then, since $\alpha \ge 0$ and $v \ge 1$ we have

$$\gamma \le \lfloor cT - Tv - (c-1)Tu\rfloor \le \lfloor (c-1)Tv - (c-1)Tu\rfloor \le \lfloor cTv - Tu\rfloor = \alpha.$$

Now suppose on the contrary that $1 - c + c^2T > 0$. Since $T \le \frac{1}{5}$ this can happen only for $c \le \frac{5}{2} - \frac{\sqrt{5}}{2} < \frac{7}{5}$. Since $1 - c + c^2T < 1$ we get

$$\begin{aligned} \gamma &\le \left\lfloor T(1-c+c^2T)(cTv - Tu) + cT - Tv - (c-1)Tu\right\rfloor \\ &\le \left\lfloor cT^2v - T^2u + cT - Tv - (c-1)Tu\right\rfloor. \end{aligned} \tag{4.6}$$

Now we have either $u \le -4$ or $v \ge 2$. Suppose first that $u \le -4$ holds. Since $c < \frac{7}{5}$ we have $Tc \le -\frac{2}{5}Tu$ and this yields

$$\gamma \le \left\lfloor (cT-1)\,Tv - \left(T + c - 1 + \frac{2}{5}\right)Tu\right\rfloor.$$

Since $c \le \frac{7}{5}$ and $T \le \frac{1}{5}$ this implies that $\gamma \le \alpha$. If, on the other hand, $v \ge 2$ holds, we have $cT \le Tv$,

$$\gamma \le \left\lfloor cT^2v - (T + c - 1)\,Tu\right\rfloor$$

and the result follows as well. □

Lemma 4.11. *Let $u \ge 0$ and $v \le -2$ and assume that*

$$-v \ge \frac{3}{2}u. \tag{4.7}$$

If $0 < T \le \frac{1}{10}$ and $c \in [1,2]$ then we have

$$u + u_2 \ge 2(v + v_2).$$

Proof. In view of (4.4) we have to show $2\gamma + \beta \ge \alpha$. As in Lemma 4.9 we derive

$$\gamma \ge \left\lfloor T(1-c+c^2T)\alpha - Tv - (c-1)Tu\right\rfloor.$$

Furthermore, since $\alpha \le 0$ we have

$$\beta = \lfloor -cT\alpha + (c-1)Tv - cTu\rfloor \ge \lfloor (c-1)Tv - cTu\rfloor.$$

We distinguish two cases. First suppose that $1 - c + c^2T \le 0$. Combining the above estimates for α, β and γ and using (4.7) we derive

$$\begin{aligned} 2\gamma + \beta &\ge 2\left\lfloor T(1-c+c^2T)\alpha - Tv - (c-1)Tu\right\rfloor + \lfloor (c-1)Tv - cTu\rfloor \\ &\ge 2\left\lfloor \left(1 - c + \frac{3}{2}\right)Tu\right\rfloor + \lfloor (c-1)Tv - cTu\rfloor \end{aligned}$$

Since $c \le 2$ and $u \ge 0$, the first term in the second line is non-negative. Thus using (4.7) again we get

$$2\gamma + \beta \ge \lfloor (c-1)Tv - cTu\rfloor \ge \left\lfloor cTv - \left(c - \frac{3}{2}\right)Tu\right\rfloor \ge \lfloor cTv - Tu\rfloor = \alpha$$

and we are done in this case.

Now suppose that $1 - c + c^2T > 0$. As above this implies that $c < \frac{7}{5}$. As in Lemma 4.9, inequality (4.5), we derive

$$\gamma \geq \left\lfloor c^3T^3v - c^2T^3u - T - Tv - (c-1)Tu \right\rfloor .$$

Since $v \leq -2$ we have $-T \geq \frac{1}{2}Tv$ and this implies

$$\gamma \geq \left\lfloor \left(c^3T^2 - \frac{1}{2}\right) Tv - \left(c^2T^2 + c - 1\right) Tu \right\rfloor .$$

Together with (4.7) this yields

$$\gamma \geq \left\lfloor -\left(c - \frac{7}{4} + T^2 \left(c^2 + \frac{3}{2}c^3\right)\right) Tu \right\rfloor .$$

Since $c - \frac{7}{4} + T^2 \left(c^2 + \frac{3}{2}c^3\right) \leq 0$ this implies that $\gamma \geq 0$. Thus

$$2\gamma + \beta \geq \beta \geq \lfloor (c-1)Tv - cTu \rfloor .$$

Using (4.7) again this yields

$$2\gamma + \beta \geq \lfloor (c-1)Tv - cTu \rfloor \geq \left\lfloor cTv - \left(c - \frac{3}{2}\right) Tu \right\rfloor \geq \lfloor cTv - Tu \rfloor = \alpha.$$

□

Lemma 4.12. *Let $u \leq 0$ and $v \geq 6$ and assume that*

$$v \geq -\frac{3}{2}u \tag{4.8}$$

holds. If $0 < T \leq \frac{1}{10}$ and $c \in [1, 2]$ then

$$u + u_2 \leq 2(v + v_2).$$

Proof. We have to show that $2\gamma + \beta \leq \alpha$. As in Lemma 4.10 we derive

$$\gamma \leq \left\lfloor T(1 - c + c^2T)\alpha + cT - Tv - (c-1)Tu \right\rfloor .$$

Since $\alpha \geq 0$ we have

$$\beta \leq \lfloor (c-1)Tv - cTu \rfloor .$$

Again we distinguish two cases. First assume that $1 - c + c^2T \leq 0$. Since $v \geq 6$ we get, using (4.8) in the form $-v \leq \frac{3}{2}u$,

$$\begin{aligned} \gamma &\leq \lfloor cT - Tv - (c-1)Tu \rfloor \leq \left\lfloor \left(\frac{c}{6} - 1\right) Tv - (c-1)Tu \right\rfloor \\ &\leq \left\lfloor -\frac{3}{2}\left(\frac{c}{6} - 1\right) Tu - (c-1)Tu \right\rfloor = \left\lfloor \left(1 - c + \frac{3}{2} - \frac{c}{4}\right) Tu \right\rfloor \leq 0. \end{aligned}$$

Note that the last inequality holds because $c \leq 2$. Now the desired estimate follows easily via

$$2\gamma + \beta \leq \beta \leq \lfloor (c-1)Tv - cTu \rfloor \leq \left\lfloor cTv - \left(c - \frac{3}{2}\right) Tu \right\rfloor \leq \lfloor cTv - Tu \rfloor = \alpha.$$

Now suppose that $1 - c + c^2T > 0$. Again this implies that $c \leq \frac{7}{5}$. As in Lemma 4.10, inequality (4.6) we derive

$$\gamma \leq \left\lfloor cT^2v - T^2u + cT - Tv - (c-1)Tu \right\rfloor .$$

Since $v \geq 6$ we have $cT < \frac{1}{3}Tv$. This implies that

$$\gamma \leq \left\lfloor \left(cT - \frac{2}{3}\right) Tv - (c - 1 + T) Tu \right\rfloor .$$

Using (4.8) this implies

$$\gamma \leq \left\lfloor \left(-\frac{3}{2}Tc - T - c + 2\right) Tu \right\rfloor \leq 0.$$

The last inequality is a consequence of $T \le \frac{1}{10}$ and $c \le \frac{7}{5}$. Summing up we get (arguing as in the first part of the proof)

$$2\gamma + \beta \le \beta \le \lfloor (c-1)Tv - cTu \rfloor \le \lfloor cTv - Tu \rfloor = \alpha.$$

$\square$

Lemma 4.13. *If $u \ge 0$, $v \le 0$, $(u,v) \ne (0,0)$, $c \in [1,2]$ and $0 < T \le \frac{1}{4}$ then $u + u_2 \ge 1$.*

Proof. We have to show that $\beta + \alpha \le -1$. Is is clear that $\alpha \le -1$. If $\alpha = -1$ then $\beta \le 0$ because $cT < 1$ and the result follows. If $\alpha \le -2$ then

$$\alpha + \beta \le (1-cT)\alpha + (c-1)Tv - cTu \le -1.$$

The latter inequality is true because $1 - cT \ge \frac{1}{2}$. $\square$

Lemma 4.14. *Let $u \le 0$, $v \ge 0$, $c \in [1,2]$ and $0 < T \le \frac{1}{2}$. Then $u + u_2 \le 0$.*

Proof. We have to show that $\alpha + \beta \ge 0$. Note that $\alpha \ge 0$. If $\alpha = 0$ then obviously $\beta \ge 0$ and we are done. If $\alpha \ge 1$ then

$$\alpha + \beta = \lfloor (1-cT)\alpha + (c-1)Tv - cTu \rfloor \ge \lfloor (1-cT) + (c-1)Tv - cTu \rfloor \ge 0$$

since $1 - cT \ge 0$. $\square$

Lemma 4.15. *Let $u \ge 0$, $v \le 0$, $c \in [1,2]$, $0 < T \le \frac{1}{2}$ and $u - v \le \ell$. Then $u + u_2 \le -\lfloor -3T\ell \rfloor + 1$.*

Proof. Since $u + u_2 = -\alpha - \beta$ we will establish the desired bound for $-\alpha - \beta$.

$$\begin{aligned}
-\alpha - \beta &= -\lfloor (1-cT)\alpha + (c-1)Tv - cTu \rfloor \\
&\le -\lfloor (1-cT)(cTv - Tu) + (c-1)Tv - cTu \rfloor + 1 \\
&\le -\lfloor cTv - Tu + (c-1)Tv - cTu \rfloor + 1 \\
&= -\lfloor (2c-1)Tv - (1+c)Tu \rfloor + 1.
\end{aligned}$$

Since $2c - 1 \le 1 + c$ and $u - v \le \ell$ this implies that

$$-\alpha - \beta \le -\lfloor -(1+c)T\ell \rfloor + 1 \le -\lfloor -3T\ell \rfloor + 1.$$

$\square$

Lemma 4.16. *Let $u \le 0$, $v \ge 0$, $c \in [1,2]$, $0 < T \le \frac{1}{2}$ and $-u + v \le \ell$. Then $u + u_2 \ge \lfloor -3T\ell \rfloor - 1$.*

Proof. It suffices to establish the desired lower bound for $-\alpha - \beta$.

$$\begin{aligned}
-\alpha - \beta &= -\lfloor (1-cT)\alpha + (c-1)Tv - cTu \rfloor \\
&\ge -\lfloor (1-cT)(cTv - Tu) + (c-1)Tv - cTu \rfloor \\
&\ge -\lfloor cTv - Tu + (c-1)Tv - cTu \rfloor \\
&= -\lfloor (2c-1)Tv - (1+c)Tu \rfloor .
\end{aligned}$$

Since $2c - 1 \le 1 + c$ and $-u + v \le \ell$ this implies

$$-\alpha - \beta \ge -\lfloor (1+c)T\ell \rfloor \ge -\lfloor 3T\ell \rfloor \ge \lfloor -3T\ell \rfloor - 1.$$

$\square$

Lemma 4.17. *Let $u \ge 0$, $v \le 0$, $c \in [1,2]$, $0 < T \le \frac{1}{10}$. If $u \ge 2$ or $v \le -1$ holds, then $v_2 \ge 0$.*

Proof. Note that the assertion is equivalent to $\gamma + \beta \le -v$. We have

$$\begin{aligned}
\gamma + \beta &\le (1-cT)\beta - (c-1)T\alpha - Tv - (c-1)Tu \\
&\le (1-cT)(-cT\alpha + (c-1)Tv - cTu) - (c-1)T\alpha - Tv - (c-1)Tu \\
&= (1 - 2c + c^2T)T\alpha + (c - 2 + cT - c^2T)Tv - (2c - 1 - c^2T)Tu \\
&\le (1 - 2c + c^2T)T(cTv - Tu) + (c - 2 + cT - c^2T)Tv - (2c - 1 - c^2T)Tu \\
&\quad + (2c - 1 - c^2T)T \\
&= (c - 2 + 2cT - 3c^2T + c^3T^2)Tv + (1 - 2c - T + 2cT + c^2T - c^2T^2)Tu \\
&\quad + (2c - 1 - c^2T)T.
\end{aligned}$$

Note that $u \geq 2$ or $v \leq -1$. Suppose first that $u \geq 2$. Then $(2c-1-c^2T)T \leq \frac{1}{2}(2c-1-c^2T)Tu$. Thus

$$\begin{aligned}\gamma+\beta &\leq (c-2+2cT-3c^2T+c^3T^2)Tv\\ &\quad+\left(1-2c-T+2cT+c^2T-c^2T^2+\frac{1}{2}(2c-1-c^2T)\right)Tu\\ &\leq (c-2+2cT-3c^2T+c^3T^2)Tv \leq -v.\end{aligned}$$

These inequalities follow since $T \leq \frac{1}{10}$. If, on the other hand, $v \leq -1$ holds, then

$$\gamma+\beta \leq \left(c-2+2cT-3c^2T+c^3T^2-(2c-1-c^2T)\right)Tv \leq -v.$$

□

Lemma 4.18. *Let $u \leq 0$, $v \geq 0$, $c \in [1,2]$, $0 < T \leq \frac{1}{10}$. If $\max\{-u,v\} \geq 3$ then $v_2 \leq 0$.*

Proof. We have to show that $\gamma+\beta \geq -v$.

$$\begin{aligned}\gamma+\beta &= \lfloor (1-cT)\beta-(c-1)T\alpha-Tv-(c-1)Tu\rfloor\\ &= \lfloor (1-cT)\lfloor -cT\alpha+(c-1)Tv-cTu\rfloor-(c-1)T\alpha-Tv-(c-1)Tu\rfloor\\ &\geq \lfloor (1-cT)\lfloor -cT(cTv-Tu)+(c-1)Tv-cTu\rfloor-(c-1)T\alpha-Tv-(c-1)Tu\rfloor.\end{aligned}$$

Suppose first that $-u=\max\{-u,v\}$, i.e. $-v \geq u$. Then

$$-cT(cTv-Tu)+(c-1)Tv-cTu \geq -c^2T^2v-c(1-T)Tu \geq -(c-cT-c^2T)Tu \geq 0.$$

Thus

$$\begin{aligned}\gamma+\beta &\geq \lfloor -(c-1)T\alpha-Tv-(c-1)Tu\rfloor\\ &\geq \lfloor -(c-1)T(cTv-Tu)-Tv-(c-1)Tu\rfloor\\ &= \lfloor (-1-(c-1)cT)Tv-(c-1)(1-T)Tu\rfloor.\end{aligned}\tag{4.9}$$

Since $c \leq 2$ and $T \leq \frac{1}{10}$ this yields

$$\gamma+\beta \geq \lfloor (-1-(c-1)cT)Tv-(c-1)(1-T)Tu\rfloor \geq \lfloor (-1-(c-1)cT)Tv\rfloor \geq -v.$$

Now suppose that $v=\max\{-u,v\}$. Then

$$-cT(cTv-Tu)+(c-1)Tv-cTu \geq -c^2T^2v$$

and thus as in (4.9) we see that

$$\gamma+\beta \geq \lfloor -c^2T^2v\rfloor+\lfloor (-1-(c-1)cT)Tv-(c-1)(1-T)Tu\rfloor.$$

Since $v \geq 3$, $c \leq 2$ and $T \leq \frac{1}{10}$,

$$\gamma+\beta \geq \lfloor -c^2T^2v\rfloor+\lfloor (-1-(c-1)cT)Tv\rfloor \geq (-1-(2c-1)cT)Tv-2 \geq -\frac{3}{10}v-2 \geq -v.$$

□

In what follows set $(u_1,v_1) := \tau_{\mathbf{r}}^2(u,v)$, this implies that

$$u_1=-\alpha+v-u \qquad \text{and} \qquad v_1=-\beta-\alpha-u.$$

Lemma 4.19. *Let $u > 0$, $v \leq 0$, $u_2 \geq 0$, $v_2 \geq 0$, $c \in [1,2]$ and $T \leq \frac{1}{10}$. If $\max\{u,-v\} \geq 4$ then we have $u_1 < 0$, $v_1 \geq 0$ and $u_1+v_1 \leq 0$.*

Proof. Since $v_1=u_2$ we trivially have $v_1 \geq 0$. In order to prove that $u_1 < 0$ note that

$$u_1=-\alpha+v-u \leq -\alpha+\min\{v,-u\}.$$

If $-v=\max\{u,-v\}$ then

$$u_1 \leq -\lfloor cTv-Tu\rfloor+v \leq -(cTv-Tu)+v+1 \leq -(c+1)Tv+v+1 < 0.$$

If $u = \max\{u, -v\}$ then

$$u_1 \leq -\lfloor cTv - Tu \rfloor - u \leq -(cTv - Tu) - u + 1 \leq (c+1)Tu - u + 1 < 0$$

and we are done. To prove $u_1 + v_1 \leq 0$ observe that $u_1 + v_1 = -2\alpha - \beta + v - 2u$. If $-v = \max\{u, -v\}$ we get (note $u \geq 1$ and $\lfloor (c+1)Tv \rfloor \leq \alpha \leq 0$)

$$\begin{aligned}
u_1 + v_1 &= -2\alpha - \lfloor -cT\alpha + (c-1)Tv - cTu \rfloor + v - 2u \\
&< -2\alpha + cT\alpha - (c-1)Tv + cTu + 1 + v - 2u \\
&\leq -2\alpha - (c-1)Tv + cTu + v - 2u + 1 \\
&\leq -2\lfloor (c+1)Tv \rfloor - (c-1)Tv - cTv + v - 2 + 1 \\
&< 2(-(c+1))Tv + 2 - (2c-1)Tv + v - 1 \\
&= -(2(c+1) + 2c - 1)Tv + v + 1 \\
&= (1 - (4c+1)T)v + 1 \leq 1.
\end{aligned}$$

The last inequality follows from the restriction on T. Since $u_1 + v_1$ is an integer $u_1 + v_1 < 1$ implies that $u_1 + v_1 \leq 0$. If $u = \max\{u, -v\}$ we derive

$$\begin{aligned}
u_1 + v_1 &\leq -2\alpha + cT\alpha - (c-1)Tv + cTu + v - 2u \\
&\leq -(2 - cT)\alpha + (c-1)Tu + cTu - 2u \\
&\leq -(2 - cT)\lfloor -(c+1)Tu \rfloor + (2c-1)Tu - 2u \\
&< (2 - cT)((c+1)Tu + 1) + (2c-1)Tu - 2u \\
&= (4c + 1 - cT(c+1))Tu + 2(1-u) - cT \\
&\leq ((4c+1)T - 2)u + 2 - cT < 0.
\end{aligned}$$

The last inequality follows from the restriction on T. ☐

Lemma 4.20. *Let $u < 0$, $v \geq 0$, $u_2 \leq 0$, $v_2 \leq 0$, $c \in [1, 2]$ and $T \leq \frac{1}{7}$. If $\max\{u, -v\} \geq 2$ then we have $u_1 > 0$, $v_1 \leq 0$ and $u_1 + v_1 \geq 0$.*

Proof. Again $v_1 \leq 0$ follows because $v_1 = u_2$. Furthermore, we have

$$u_1 \geq -cTv + Tu + v - u = (1 - cT)v - (1 - T)u \geq (1 - T)(-u) \geq 1 - T > 0.$$

Thus it remains to prove that $u_1 + v_1 \geq 0$. Since $u_1 + v_1 = -2\alpha - \beta + v - 2u$ we have (note $\alpha \geq 0$)

$$\begin{aligned}
u_1 + v_1 &= -2\alpha - \lfloor -cT\alpha + (c-1)Tv - cTu \rfloor + v - 2u \\
&\geq -2\alpha + cT\alpha - (c-1)Tv + cTu + v - 2u \\
&\geq -2\alpha - (c-1)Tv + cTu + v - 2u \\
&\geq -2(cTv - Tu) - (c-1)Tv + cTu + v - 2u \\
&= (T - 3cT)v + (c+2)Tu + v - 2u \\
&= (1 + T(1 - 3c))v + ((c+2)T - 2)u \\
&> (1 - 5T)v - 2u \geq -2u \geq 2.
\end{aligned}$$

☐

4.2.3. *The characterization result and its proof.*

First we shall prove the following result.

Theorem 4.21. *In order to characterize the SRS in the region R we need at most 51^2 cutout polygons. Thus the point $(1, -1)$ is not a critical point*[12].

[12] See [3, Definition 7.1] for the definition of "critical point".

For $\ell \in \mathbb{N}$ we need the the following sets.

$$\begin{aligned} M_\ell^{(1)} &:= \left\{(u,v) \,\middle|\, u>0,\, v\le 0,\, u-v\le \ell,\, u-2v \le \frac{8\ell}{5} - \lfloor -3T\ell \rfloor + 1 \right\}, \\ M_\ell^{(2)} &:= \left\{(u,v) \,\middle|\, u<0,\, v\ge 0,\, v-u\le \ell,\, 2v-u \le \frac{8\ell}{5} - \lfloor -3T\ell \rfloor + 1 \right\}, \\ M_\ell &:= M_\ell^{(1)} \cup M_\ell^{(2)}. \end{aligned}$$

Now we use the lemmas of the previous subsection to establish the following results. From now we always assume that $\mathbf{r} \in R$.

We want to show that the orbit of each element $(x,y) \in \mathbb{Z}^2$ contains an element of M_{25}. In a first step we show that we can confine ourselves to studying elements which are contained in M_ℓ for a certain $\ell \in \mathbb{N}$.

Lemma 4.22. *Let $\mathbf{r} \in R$ and $(x,y) \in \mathbb{Z}^2$ with $\max\{|x|,|y|\} \ge 20$. Then there exist $\ell, n \in \mathbb{N}$ such that $\tau_{\mathbf{r}}^n(x,y) \in M_\ell$.*

Proof. Using the definition of $\tau_{\mathbf{r}}$ it is easy to see that either $n=0$, $n=1$ or $n=2$ does the job for ℓ sufficiently large. □

Lemma 4.23. *Let $\mathbf{r} \in R$. If $(u,v) \in M_\ell^{(1)}$ with $\max\{u,-v\} \ge 20$ then $(u_2,v_2) \in M_\ell^{(2)}$ or $u_2, v_2 \ge 0$.*

If $(u,v) \in M_\ell^{(2)}$ with $\max\{-u,v\} \ge 20$ then $(u_2,v_2) \in M_\ell^{(1)}$ or $u_2,v_2 \le 0$.

Proof. Since $(u,v) \in M_\ell^{(1)}$ we have $u - v \le \ell$. Thus Lemma 4.9 implies that

$$-u_2 + v_2 \le \ell. \tag{4.10}$$

Next we want to show that

$$2v_2 - u_2 \le \frac{8\ell}{5} - \lfloor -3T\ell \rfloor + 1. \tag{4.11}$$

To this matter we distinguish two cases. Assume first that $v \ge -\frac{3\ell}{5}$. Then Lemma 4.9 yields

$$\begin{aligned} 2v_2 &= 2(v+v_2) - 2v \le 2(u+u_2) - 2v, \\ u_2 &= (u+u_2) - u. \end{aligned}$$

Since $u - v \le \ell$ and $-v \le \frac{3\ell}{5}$, Lemma 4.15 implies that

$$2v_2 - u_2 \le (u+u_2) - 2v + u \le \frac{8\ell}{5} - \lfloor -3T\ell \rfloor + 1.$$

If, on the contrary, $-v > \frac{3\ell}{5}$ then $u \le \frac{2\ell}{5}$ and thus $-v \ge \frac{3}{2}u$ holds. In this case Lemma 4.11 yields

$$2v_2 - u_2 \le -2v + u \le \frac{8\ell}{5} - \lfloor -3T\ell \rfloor + 1$$

and (4.11) is proved. Finally, note that Lemma 4.17 implies that

$$v_2 \ge 0. \tag{4.12}$$

Combining (4.10), (4.11) and (4.12) we get the first claim. The second claim is proved in an analogous way. Just use Lemmas 4.10, 4.12, 4.16 and 4.18 instead of Lemmas 4.9, 4.11, 4.15 and 4.17. □

Lemma 4.24. *Let $\mathbf{r} \in R$. Let $(x,y) \in M_\ell^{(1)}$, $(x',y') := \tau_{\mathbf{r}}^3(x,y) \in M_\ell^{(2)}$ and set $(x'',y'') := \tau_{\mathbf{r}}^6(x,y)$. Then $x'' < x$ or one of the pairs $(x,y),(x',y')$ has coordinate maximum less than 20 in modulus.*

Let $(x,y) \in M_\ell^{(2)}$, $(x',y') := \tau_{\mathbf{r}}^3(x,y) \in M_\ell^{(1)}$ and set $(x'',y'') := \tau_{\mathbf{r}}^6(x,y)$. Then $x'' > x$ or one of the pairs $(x,y),(x',y')$ has coordinate maximum less than 20 in modulus.

Proof. We only prove the first assertion, the second one is proved in the same way.

Applying Lemma 4.13 with $u = x$, $v = y$, $u_2 = x'$, $v_2 = y'$ we get

$$x + x' \geq 1.$$

Now we use Lemma 4.14 with $u = x'$, $v = y'$, $u_2 = x''$, $v_2 = y''$ in order to get

$$x' + x'' \leq 0.$$

Combining both inequalities yields the desired result. □

Lemma 4.25. *Let $\mathbf{r} \in R$. Let $(x, y) \in M_\ell$. Then there exists a least $n \in \mathbb{N}$ such that for*

$$\begin{aligned} (u, v) &:= \tau_{\mathbf{r}}^{3n}(x, y), \\ (u_2, v_2) &:= \tau_{\mathbf{r}}^{3n+3}(x, y) \end{aligned}$$

one of the following statements holds.

- $(u, v) \in M_\ell^{(1)}$ *and* $u_2, v_2 \geq 0$.
- $(u, v) \in M_\ell^{(2)}$ *and* $u_2, v_2 \leq 0$.
- $\max\{|u|, |v|\} \leq 20$.

Proof. This is an easy consequence of Lemmas 4.23 and 4.24. Note that Lemma 4.24 ensures that after finitely many iterations of $\tau_{\mathbf{r}}^3$ we must have $(u, v) \notin M_\ell$. □

Proposition 4.26. *Let $r \in R, l \in \mathbb{N}, l \geq 25$, and $(u, v) \in M_l$ with $max\{|u|, |v|\} > 20$. Then there exists $n \in \mathbb{N}$ such that $z := \tau_r^n(u, v)$ satisfies one of the following alternatives:*

(i) $|z|_\infty \leq 20$

(ii) $|z|_\infty > 20$ *and* $z \in M_{l_2}$ *for some* $l_2 \in \mathbb{N}$ *with* $l_2 < l$.

Proof. In view of Lemma 4.25 we can assume w.l.o.g. that (u, v) satisfies one of the three statements of that lemma.

Suppose that the first statement of Lemma 4.25 holds. Then Lemma 4.19 implies that for $(u_1, v_1) = \tau_{\mathbf{r}}^2(u, v)$ we have $u_1 < 0$, $v_1 \geq 0$ and $u_1 + v_1 \leq 0$. Recall that

$$\begin{aligned} u_1 &= -\alpha + v - u, \\ v_1 &= -\beta - \alpha - u. \end{aligned}$$

We claim that $(u_1, v_1) \in M_{\ell_2}$ for

$$\ell_2 := -v - \beta.$$

First we note that $v_1 - u_1 = -v - \beta = \ell_2$. Thus it remains to show that $2v_1 - u_1 \leq \frac{8\ell_2}{5} - \lfloor -3T\ell_2 \rfloor + 1$. Since $u_1 + v_1 \leq 0$ this follows by

$$2v_1 - u_1 \leq \frac{3}{2}(v_1 - u_1) \leq \frac{3}{2}(-v - \beta) \leq \frac{3}{2}\ell_2 \leq \frac{8\ell_2}{5} - \lfloor -3T\ell_2 \rfloor + 1.$$

Summing up we proved the claim. Now we need to show that $\ell_2 < \ell$. Since $(u, v) \in M_\ell^{(1)}$ we have

$$-v \leq \frac{4}{5}\ell + \frac{1}{2}\left(- \lfloor -3T\ell \rfloor + 1\right).$$

and $u - v \leq \ell$. Thus, since $\alpha < 0$

$$\begin{aligned} \ell_2 &\leq \frac{4}{5}\ell + \frac{1}{2}\left(- \lfloor -3T\ell \rfloor + 1\right) - \beta \\ &\leq \frac{4}{5}\ell + \frac{1}{2}\left(- \lfloor -3T\ell \rfloor + 1\right) + cT\alpha - (c-1)Tv + cTu + 1 \\ &\leq \frac{4}{5}\ell + \frac{1}{2}\left(- \lfloor -3T\ell \rfloor\right) + cT\ell + \frac{3}{2} \leq \left(\frac{4}{5} + \left(\frac{3}{2} + c\right) T\right)\ell + 2 \leq \frac{11}{12}\ell + 2 < \ell. \end{aligned}$$

If the second statement of Lemma 4.25 holds, a similar reasoning leads to the conclusion. If the third statement of Lemma 4.25 holds, there is nothing to prove. □

Theorem 4.21 now follows easily. Just start with Lemma 4.22 in order to get a point in the orbit which is contained in some M_ℓ. Then iterate Proposition 4.26 until you arrive at $(u,v) \in M_\ell$ for some $\ell \le 25$. It is easily seen that $(u,v) \in M_\ell$ with $\ell \le 25$ implies that $\max\{|u|,|v|\}) \le 25$. Thus each orbit contains a point (u,v) with $\max\{|u|,|v|\} \le 25$.

We now prove our main result.

Theorem 4.27. *Let* $\mathbf{r} \in S$. *Then* $\tau_{\mathbf{r}}$ *is an SRS.*

Proof. For $z \in \mathbb{R}$ set

$$\begin{aligned} R_1(z) &:= \left\{ (1-T, -1+cT) \;\middle|\; z \le T \le \frac{1}{30},\, 1 \le c \le 1.99 \right\}, \\ R_2(z) &:= \left\{ (1-T, -1+cT) \;\middle|\; z \le T \le \frac{1}{30},\, 1.99 \le c \le 2 \right\} \end{aligned}$$

and $Q_0 := \{(x,y) \,|\, \max\{|x|,|y|\} \le 25\}$. Furthermore we adopt the following notation. For a set $M \subset \mathbb{Z}^2$ we write

$$\tau_{\mathbf{r}} M := \{\tau_{\mathbf{r}}(x,y) \,|\, (x,y) \in M\}.$$

First we want to prove that $R_1(10^{-3})$ is a subset of $\mathcal{D}_2^0$. Define the sequence of sets

$$Q_{n+1} := \{\tau_{\mathbf{r}} Q_n \,|\, \mathbf{r} \in R_1(10^{-3})\}.$$

Note that just before we proved that for $\mathbf{r} \in R$ each orbit of $\tau_{\mathbf{r}}$ contains a point in Q_0. Thus what we have to show is that there exists an $n \in \mathbb{N}$ such that $Q_n = \{(0,0)\}$. For $z \in \mathbb{R}$ define the points

$$\begin{aligned} \mathbf{p}_1 &:= \left(1 - \frac{1}{30}, -1 + \frac{1}{30}\right), \quad & \mathbf{p}_2 &:= \left(1 - \frac{1}{30}, -1 + \frac{1.99}{30}\right), \\ \mathbf{p}_3 &:= (1-z, -1+z), & \mathbf{p}_4 &:= (1-z, -1+1.99z). \end{aligned}$$

Note that $R_1(10^{-3})$ is the convex hull of these points with $z = 10^{-3}$. By the definition of $\tau_{\mathbf{r}}$ we see that

$$Q_{n+1} \subset \{(y,j) \,|\, \min_i\{-\lfloor \mathbf{p}_i \cdot (x,y) \rfloor\} \le j \le \max_i\{-\lfloor \mathbf{p}_i \cdot (x,y) \rfloor\} \text{ for some } x \in \mathbb{R} \text{ with } (x,y) \in Q_n\}$$

(the dot "$\cdot$" denotes scalar multiplication). Thus we set $P_0 := Q_0$ and

$$P_{n+1} := \{(y,j) \,|\, \min_i\{-\lfloor \mathbf{p}_i \cdot (x,y) \rfloor\} \le j \le \max_i\{-\lfloor \mathbf{p}_i \cdot (x,y) \rfloor\} \text{ for some } x \in \mathbb{R} \text{ with } (x,y) \in P_n\}.$$

Since $Q_n \subset P_n$ what remains to prove is that for some n we have $P_n = \{(0,0)\}$. With help of an easy computer program we find that this is true for $n = 500$.

Performing the same procedure for $R_2(10^{-3})$ we get that for $n = 500$

$$Q_n \subset \{(-1,-1),(-1,1),(0,0),(1,-1),(1,0),(1,2),(2,1)\}.$$

If $c < 2$, i.e. $\mathbf{r} \in R_2(10^{-3}) \cap S$, we can easily see by direct calculation that each of these points (x,y) admits an $n \in \mathbb{N}$ such that $\tau_{\mathbf{r}}(x,y) = (0,0)$ for all $R_2(10^{-3}) \cap S$. Summing up we have shown that

$$\left\{ (1-T, -1+cT) \;\middle|\; 10^{-3} \le T \le \frac{1}{30},\, 1 \le c < 2 \right\}$$

is a subset of $\mathcal{D}_2^0$. Now we have to make the bound 10^{-3} smaller. First consider $R_1(z)$ for $0 < z \le 10^{-3}$. The sequence P_n only depends on the minimal and maximal values of $-\lfloor \mathbf{p}_i \cdot (x,y) \rfloor$ for $i \in \{1,2,3,4\}$. Since $\mathbf{p}_1$ and $\mathbf{p}_2$ do not depend on z we need to examine what happens with the functions

$$f_i(x,y;z) := -\lfloor \mathbf{p}_i \cdot (x,y)) \rfloor \qquad i = 3,4$$

for $0 \le z \le 10^{-3}$. First we note that all elements (x,y) occurring in the sets P_n have $\max\{|x|,|y|\} \le 100$ (this can easily be checked by the above mentioned computer program) and that

$$f_3(x,y;z) = -\lfloor (-x+y)z \rfloor - x + y.$$

Thus, since $\max\{|x|,|y|\} \le 100$ and $0 < z \le 10^{-3}$ the value of the function $f_3(x,y;z)$ only depends on x and y and *not* on z. The same follows for $f_4(x,y;z)$ by similar reasoning. Thus the sequence

$$\begin{tikzcd} (2,1) \arrow[r] & (1,-1) \arrow[r] & (-1,-1) \arrow[r] & (-1,1) \arrow[r] & (1,2) \arrow[llll, bend left] \end{tikzcd}$$

FIGURE 7. The essential subgraph for Lemma 5.2

of the P_n is not altered if we replace $R_1(10^{-3})$ by $R_1(z)$ for some $0 < z \leq 10^{-3}$. Summing up we have shown that

$$\left\{(1-T, -1+cT) \,\middle|\, 0 < T \leq \frac{1}{30},\, 1 \leq c \leq 1.99\right\}$$

is contained in $\mathcal{D}_2^0$. Performing the same considerations for $R_2(z)$ mutatis mutandis the result follows. □

5. Computational results

By using the algorithm in Lemma 3.2 for a given small closed convex polygon $H \subset \mathcal{E}_2$ we can describe $H \cap \mathcal{D}_2^0$ explicitly. In this subsection, we give several examples to illustrate the efficiency of this algorithm.

5.1. Complete characterization of $\mathcal{D}_2^0$ for $x \leq \frac{2}{3}$.

Lemma 5.1. *The triangle* $\Delta((\frac{1}{2}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3}))$ *is contained in* $\mathcal{D}_2^0$.

Proof. We apply the algorithm of Lemma 3.2. Start with $V_0 = \{(\pm 1, 0), (0, \pm 1)\}$ and add successively all possible vertices and edges according to (2) of Lemma 3.2. In the present case this leads to the graph $(\mathcal{V}, E)$ of 21 vertices and 30 edges as follows.

(1,0), (0,1), (-1,0), (0,-1), (-1,1), (0,0), (1,-1), (-1,-1), (1,1), (-1,2), (1,-2), (-2,0), (-2,1), (2,-1), (2,0), (0,-2), (0,2), (-2,2), (2,-2), (-2,-1), (2,1).

(-2, -1)→(-1, 2), (-2, 0)→(0, 1), (-2, 0)→(0, 2), (-2, 1)→(1, 1), (-2, 2)→(2, 0), (-2, 2)→(2, 1), (-1, -1)→(-1, 1), (-1, -1)→(-1, 2), (-1, 0)→(0, 1), (-1, 1)→(1, 0), (-1, 1)→(1, 1), (-1, 2)→(2, 0), (0, -2)→(-2, 1), (0, -2)→(-2, 2), (0, -1)→(-1, 1), (0, 0)→(0, 0), (0, 1)→(1, 0), (0, 2)→(2, -1), (0, 2)→(2, 0), (1, -2)→(-2, 0), (1, -2)→(-2, 1), (1, -1)→(-1, 0), (1, 0)→(0, 0), (1, 1)→(1, -1), (2, -2)→(-2, 0), (2, -1)→(-1, -1), (2, -1)→(-1, 0), (2, 0)→(0, -1), (2, 1)→(1, -2), (2, 1)→(1, -1).

As this graph has only one cycle $(0,0) \to (0,0)$, the lemma follows from Lemma 3.2. □

Lemma 5.2. *The triangle* $\Delta((\frac{1}{2}, -\frac{1}{2}), (\frac{2}{3}, -\frac{1}{3}), (\frac{2}{3}, -\frac{2}{3}))$ *is contained in* $\mathcal{D}_2^0$.

Proof. We proceed in a similar manner as in Lemma 5.1. The graph $(\mathcal{V}, E)$ is given by 21 vertices and 30 edges:

(1,0), (0,1), (-1,0), (0,-1), (-1,-1), (0,0), (1,1), (-1,1), (1,-1), (-1,-2), (1,2), (-2,-1), (-2,0), (2,0), (2,1), (0,-2), (0,2), (-2,-2), (2,2), (-2,1), (2,-1).

(-2,-2)→(-2,0), (-2,-2)→(-2,1), (-2,-1)→(-1,1), (-2,0)→(0,1), (-2,0)→(0,2), (-2,1)→(1,2), (-1,-2)→(-2,0), (-1,-1)→(-1,0), (-1,-1)→(-1,1), (-1,0)→(0,1), (-1,1)→(1,1), (-1,1)→(1,2), (0,-2)→(-2,-1), (0,-2)→(-2,0), (0,-1)→(-1,0), (0,0)→(0,0), (0,1)→(1,1), (0,2)→(2,1), (0,2)→(2,2), (1,-1)→(-1,-1), (1,0)→(0,0), (1,1)→(1,0), (1,2)→(2,0), (1,2)→(2,1), (2,-1)→(-1,-2), (2,-1)→(-1,-1), (2,0)→(0,-1), (2,1)→(1,-1), (2,1)→(1,0), (2,2)→(2,0).

In view of Lemma 3.2 we are only interested in the non-trivial cycles of this graph. Remove the trivial edge $(0,0) \to (0,0)$ and take the essential subgraph [13] by successive removal of stranded vertices. Then we get the graph drawn in Figure 7, which is just the cycle $(2,1); -1, -1, 1$ of length 5. The associated cutout polygon $P((2,1); -1, -1, 1)$ is given by (3.3). It is easy to see that $P((2,1); -1, -1, 1) \cap \Delta((\frac{1}{2}, -\frac{1}{2}), (\frac{2}{3}, -\frac{1}{3}), (\frac{2}{3}, -\frac{2}{3})) = \emptyset$. An application of Lemma 3.2 proves the

[13] The maximum subgraph with the property that each vertex has at least one incoming edge and at least one outgoing edge.

i	vertices	edges
1	123	267
2	27	45
3	27	39
4	39	53
5	135	267
6	407	1040

TABLE 2. Size of the graphs $(\mathcal{V}, E)$

$$\begin{tikzcd}
(-3,3) \arrow[r] & (3,-2) \arrow[r] & (-2,1) \arrow[r] & (1,1) \arrow[r] & (1,-2) \arrow[r] & (-2,3) \arrow[r] \arrow[llll, bend right] & (3,-3) \arrow[llllll, bend left]
\end{tikzcd}$$

FIGURE 8. The essential subgraph of Δ_4 in Lemma 5.3

lemma. Note that $(\frac{2}{3}, -\frac{1}{3})$ is on the boundary of $P((2,1);-1,-1,1)$ but not in $P((2,1);-1,-1,1)$. $\square$

In the following we again use the constants γ_q defined in Section 4.1.

Lemma 5.3. *The convex hull H given by the four points $(\gamma_1^3, 1), (\gamma_1\gamma_2, \gamma_1+\gamma_2), (\frac{2}{3}, 1), (\frac{2}{3}, \frac{2}{3\gamma_2}+\gamma_2)$ is contained in $\mathcal{D}_2^0$.*

Proof. The whole set H is too large; an application of Lemma 3.2 is not possible for the whole set, because the construction of the set $\mathcal{V}$ does not seem to converge. Thus we are forced to subdivide H into 6 triangles:

$$\begin{aligned}
\Delta_1 &= \Delta((\gamma_1^3, 1), (\gamma_1\gamma_2, \gamma_1 + \gamma_2), (\gamma_1\gamma_2, 5/4)),\\
\Delta_2 &= \Delta((\gamma_1^3, 1), (\gamma_1\gamma_2, 5/4), (2/3, 1)),\\
\Delta_3 &= \Delta((\gamma_1\gamma_2, 5/4), (2/3, 1), (2/3, 5/4)),\\
\Delta_4 &= \Delta((\gamma_1\gamma_2, 5/4), (2/3, \gamma_1 + \gamma_2), (2/3, 5/4)),\\
\Delta_5 &= \Delta((\gamma_1\gamma_2, 5/4), (2/3, \gamma_1 + \gamma_2), (\gamma_1\gamma_2, \gamma_1 + \gamma_2)),\\
\Delta_6 &= \Delta((2/3, 2/(3\gamma_2) + \gamma_2), (2/3, \gamma_1 + \gamma_2), (\gamma_1\gamma_2, \gamma_1 + \gamma_2)).
\end{aligned}$$

Now we can apply Lemma 3.2 to each of these triangles. Table 2 gives the number of vertices and edges of the graphs $(\mathcal{V}, E)$ related to Δ_i $(i = 1, 2, \ldots, 6)$ after removing the trivial cycle $0 \to 0$. Apart from Δ_4, the graphs are acyclic. As in the proof of Lemma 5.1 this shows that $\Delta_i \subset \mathcal{D}_2^0$ for $i \in \{1, 2, 3, 5, 6\}$. The essential subgraph of Δ_4 is given in Figure 8. It contains two primitive cycles: $(1,-2); 3, -3, 3, -2, 1$ and $(1,-2); 3, -2, 1$. The corresponding cutouts are

$$P((1,-2); 3, -3, 3, -2, 1) = \left\{ (x,y) \;\middle|\; y < 2x,\ \frac{2x}{3} + 1 \le y,\ x + \frac{2}{3} < y < -x + 3 \right\}$$

and

$$P((1,-2); 3, -2, 1) = \left\{ (x,y) \;\middle|\; \frac{x}{2} + 1 < y < 2x,\ \frac{3}{2}x < y < \frac{2x}{3} + 1 \right\}.$$

It is easy to see that $P((1,-2); 3, -3, 3, -2, 1) \cup P((1,-2); 3, -2, 1)$ has no intersection with Δ_4. Note that the point $(\frac{2}{3}, \frac{4}{3})$ is on the boundary of $P((1,-2); 3, -2, 1)$ but it is not contained in $P((1,-2); 3, -2, 1)$. $\square$

Summing up we have shown the following characterization result for $\mathcal{D}_2^0$.

Theorem 5.4. $\left\{(x,y) \;\middle|\; 0 \le x \le \frac{2}{3},\ y < x + 1,\ y \ge -x \right\}$ *is contained in $\mathcal{D}_2^0$.*

Proof. The assertion is the combination of Corollary3.6, Theorems 4.6 and 4.8 with $q = 2$, and Lemmas 5.1, 5.2 and 5.3. □

Note that the range for x can not go beyond $\frac{2}{3}$ since $(\frac{2}{3}, -\frac{1}{3})$ and $(\frac{2}{3}, \frac{4}{3})$ are on the boundary of a cutout polygon.

5.2. **Computational results near to the boundary of $\mathcal{D}_2$.**

The nearer we approach $\partial\mathcal{D}_2$ the more extensive calculations are necessary in order to perform the algorithm given in Lemma 3.2. Here are two examples.

Lemma 5.5. *The convex hull H of the four points $(\frac{2}{3}, -\frac{1}{3}), (\frac{2}{3}, -\frac{2}{3}), (\frac{29}{30}, -\frac{14}{15}), (\frac{29}{30}, -\frac{29}{30})$ is contained in $\mathcal{D}_2^0$ apart from the line connecting $(\frac{2}{3}, -\frac{1}{3})$ and $(\frac{29}{30}, -\frac{14}{15})$.*

Proof. Define two triangles $\Delta_{n,m} := \Delta((1-1/n, 2/n-1), (1-1/n, 1/n-1), (1-1/m, 2/m-1))$ and $\Delta'_{n,m} := \Delta((1-1/n, 1/n-1), (1-1/m, 2/m-1), (1-1/m, 1/m-1))$. Subdivide the convex hull into 12 triangles: $\Delta_{3,5}, \Delta'_{3,5}, \Delta_{5,10}, \Delta'_{5,10}, \Delta_{10,15}, \Delta'_{10,15}, \Delta_{15,20}, \Delta'_{15,20}, \Delta_{20,25}, \Delta'_{20,25}, \Delta_{25,30}, \Delta'_{25,30}$ for example[14]. Then for each invariant graph, apart from the trivial cycle there exists only the cycle $(2, 1); -1, -1, 1$. This cycle already appeared in the proof of Lemma 5.2 and $P((2, 1); -1, -1, 1)$ intersects the convex hull H only along the line connecting $(\frac{2}{3}, -\frac{1}{3})$ and $(\frac{29}{30}, -\frac{14}{15})$. □

Putting together Theorem 4.27, Lemma 5.5 Lemma 5.2, Theorem 3.3 and Theorem 3.4 we arrive at the following result.

Theorem 5.6. *We have*
$$\{(x, y) \mid x > 0, -x \le y < 1 - 2x\} \subset \mathcal{D}_2^0.$$

Lemma 5.7. *The convex hull H of the four points $(\frac{2}{3}, 1), (\frac{2}{3}, \frac{4}{3}), (\frac{29}{30}, 1), (\frac{29}{30}, \frac{31}{30})$ is contained in $\mathcal{D}_2^0$.*

Proof. In this case we subdivide H in the following way. Let $\Delta_n := ((1 - \frac{1}{n}, 1), (1 - \frac{1}{n}, 1 + \frac{1}{n}), (1 - \frac{1}{n+1}, 1 + \frac{1}{n+1}))$ and $\Delta'_n := ((1 - \frac{1}{n}, 1), (1 - \frac{1}{n+1}, 1 + \frac{1}{n+1}), (1 - \frac{1}{n+1}, 1))$. Then we subdivide H into the 54 triangles $\Delta_3, \dots, \Delta_{29}, \Delta'_3, \dots, \Delta'_{29}$. The corresponding invariant graphs are acyclic in most cases after removing the trivial cycle. However, the two non trivial cycles $(2, 0); -1, 2, -2, 1, 1, -2$ and $(2, 0); -1, 2, -2$ appear when we consider the triangle Δ_3. Both cycles give the same cutout point $(1, \frac{3}{2})$. □

5.3. **Complete characterization of $\mathcal{D}_2^0$ for $\frac{2}{3} \le x \le \frac{5}{6}$.**

In this subsection Lemma 3.2 is applied in order to characterize the set $\mathcal{D}_d^0$ completely in the strip $\frac{2}{3} \le x \le \frac{5}{6}$. We do not give all the details. Our aim is just to give the subdivision of this strip that is needed to apply Lemma 3.2 in all its subregions which have not yet been characterized in former results. Together with Theorem 5.4 this will lead to the following result.

Theorem 5.8. *Let E_1, E_2 and E_4 be given as in Proposition 3.7 and define*
$$L = \left\{ (x, y) \middle| 0 \le x \le \frac{5}{6},\ y < x + 1,\ y \ge -x \right\}.$$
Then
$$\mathcal{D}_2^0 \cap L = L \setminus (E_1 \cup E_2 \cup E_4).$$
This is a complete characterization of $\mathcal{D}_2^0$ for $x < \frac{5}{6}$.

The characterization of $L \cap \{(x, y) \mid 0 < x < \frac{2}{3}\}$ is already contained in Theorem 5.4. Thus we may confine ourselves to the characterization of
$$L' := L \cap \left\{ (x, y) \,\middle|\, \frac{2}{3} \le x \le \frac{5}{6} \right\}.$$
In what follows $\gamma_2 > \frac{5}{6}$ is defined as in Section 4.1. We already characterized certain subsets of L' in previous theorems. These results are given in Table 3. Thus in order to prove Theorem 5.8 it remains to characterize the regions which are treated in the following four lemmas.

[14] Finer subdivision would give smaller graphs.

Region	characterized in	contained in $\mathcal{D}_2^0$
$L' \cap \{(x,y) \mid \frac{x}{\gamma_2} + \gamma_2 < y < x+1\}$	Theorem 4.8 for $\kappa = \gamma_2$	yes
$L' \cap \{(x,y) \mid 1 + \frac{x}{2} < y < 2x\}$	Proposition 3.7 (E_1 and E_2)	no
$L' \cap \{(x,y) \mid x \le y \le 2-x\}$	Lemma 5.7 and Corollary 3.6	yes
$L' \cap \{(x,y) \mid -1+x \le y \le 1-x\}$	Corollary 3.6	yes
$L' \cap \{(x,y) \mid 1-2x \le y < -\frac{x}{2}\}$	Proposition 3.7 (E_4)	no
$L' \cap \{(x,y) \mid -x \le y < 1-2x\}$	Theorem 5.6	yes

Table 3. Results on the SRS in L' that have been proved already

Lemma 5.9. *Let*

$$A_1 := \left\{ (x,y) \,\middle|\, \frac{2}{3} \le x \le \frac{5}{6},\ 2x \le y \le \frac{x}{\gamma_2} + \gamma_2 \right\}.$$

Then $A_1 \subset \mathcal{D}_2^0$.

Proof. This lemma is proved with help of the algorithm in Lemma 3.2. To this matter we need to cover A_1 with small regions. These are the convex hulls of the following sets of points

$$\{(3/4, 2/(3\gamma_2)+\gamma_2), (2/3, 2/(3\gamma_2)+\gamma_2), (3/4, 8/5)\};$$
$$\{(3/4, 2/(3\gamma_2)+\gamma_2), (2/3, 2/(3\gamma_2)+\gamma_2), (3/4, 3/(4\gamma_2)+\gamma_2)\};$$
$$\{(4/5, 2/(3\gamma_2)+\gamma_2), (3/4, 2/(3\gamma_2)+\gamma_2), (4/5, 8/5)\};$$
$$\{(4/5, 2/(3\gamma_2)+\gamma_2), (3/4, 2/(3\gamma_2)+\gamma_2), (4/5, 3/(4\gamma_2)+\gamma_2)\};$$
$$\{(3/4, 3/(4\gamma_2)+\gamma_2), (3/4, 2/(3\gamma_2)+\gamma_2), (4/5, 3/(4\gamma_2)+\gamma_2)\};$$
$$\{(3/4, 3/(4\gamma_2)+\gamma_2), (4/5, 4/(5\gamma_2)+\gamma_2), (4/5, 3/(4\gamma_2)+\gamma_2)\};$$
$$\{(4/5, 8/5), (5/6, 5/3), (4/5, 5/3)\};$$
$$\{(4/5, 31/18), (5/6, 4/(5\gamma_2)+\gamma_2), (5/6, 31/18)\};$$
$$\{(4/5, 31/18), (4/5, 4/(5\gamma_2)+\gamma_2), (5/6, 4/(5\gamma_2)+\gamma_2)\};$$
$$\{(9/11, 4/(5\gamma_2)+\gamma_2), (4/5, 4/(5\gamma_2)+\gamma_2), (9/11, 9/(11\gamma_2)+\gamma_2)\};$$
$$\{(9/11, 4/(5\gamma_2)+\gamma_2), (5/6, 4/(5\gamma_2)+\gamma_2), (5/6, 9/(11\gamma_2)+\gamma_2)\};$$
$$\{(9/11, 4/(5\gamma_2)+\gamma_2), (5/6, 9/(11\gamma_2)+\gamma_2), (9/11, 9/(11\gamma_2)+\gamma_2)\};$$
$$\{(5/6, 5/(6\gamma_2)+\gamma_2), (5/6, 19/(23\gamma_2)+\gamma_2), (19/23, 19/(23\gamma_2)+\gamma_2)\};$$
$$\{(19/23, 19/(23\gamma_2)+\gamma_2), (5/6, 19/(23\gamma_2)+\gamma_2), (19/23, 9/(11\gamma_2)+\gamma_2)\};$$
$$\{(5/6, 19/(23\gamma_2)+\gamma_2), (5/6, 9/(11\gamma_2)+\gamma_2), (19/23, 9/(11\gamma_2)+\gamma_2)\};$$
$$\{(19/23, 19/(23\gamma_2)+\gamma_2), (9/11, 9/(11\gamma_2)+\gamma_2), (19/23, 9/(11\gamma_2)+\gamma_2)\};$$
$$\{(2/3, 3/2), (2/3, 4/3), (3/4, 3/2)\};$$
$$\{(2/3, 3/2), (2/3, 2/(3\gamma_2)+\gamma_2), (3/4, 3/2)\};$$
$$\{(3/4, 3/2), (2/3, 2/(3\gamma_2)+\gamma_2), (3/4, 8/5)\};$$
$$\{(3/4, 3/2), (3/4, 2/(3\gamma_2)+\gamma_2), (4/5, 8/5)\};$$
$$\{(4/5, 5/3), (4/5, 31/18), (5/6, 31/18), (5/6, 5/3)\}.$$

For each of these convex hulls the graph constructed with help of Lemma 3.2 is either acyclic or contains cycles corresponding to empty cutout polygons. This proves the lemma. □

Lemma 5.10. *Let*

$$A_2 := \left\{ (x,y) \,\middle|\, \frac{2}{3} \le x \le \frac{5}{6},\ 2-x \le y \le 1 + \frac{x}{2} \right\}.$$

Then $A_2 \subset \mathcal{D}_2^0$.

Proof. This lemma is proved with help of the algorithm in Lemma 3.2. To this matter we need to cover A_2 with small regions. These are the convex hulls of the following sets of points

$$\{(4/5, 7/5), (4/5, 6/5), (2/3, 4/3)\};$$
$$\{(4/5, 5/4), (4/5, 6/5), (5/6, 7/6), (5/6, 5/4)\};$$
$$\{(4/5, 7/5), (4/5, 5/4), (5/6, 5/4), (5/6, 17/12)\}.$$

For each of these convex hulls the graph constructed with help of Lemma 3.2 is either acyclic or contains cycles corresponding to empty cutout polygons. This proves the lemma. □

Lemma 5.11. *Let*

$$A_3 := \left\{(x,y) \,\middle|\, \frac{2}{3} \le x \le \frac{5}{6},\ 1-x \le y \le x\right\}.$$

Then $A_3 \subset \mathcal{D}_2^0$.

Proof. This lemma is proved with help of the algorithm in Lemma 3.2. To this matter we need to cover A_3 with small regions. These are the convex hulls of the following sets of points

$$\begin{gathered}
\{(2/3,2/3),(2/3,1/3),(4/5,1/2)\};\\
\{(2/3,2/3),(4/5,1/2),(4/5,4/5)\};\\
\{(4/5,4/5),(5/6,4/5),(5/6,5/6)\};\\
\{(4/5,4/5),(4/5,3/5),(5/6,3/5),(5/6,4/5)\};\\
\{(4/5,1/2),(4/5,3/5),(5/6,3/5),(5/6,1/2)\};\\
\{(4/5,1/5),(4/5,1/3),(5/6,1/3),(5/6,1/5)\};\\
\{(4/5,1/5),(5/6,1/6),(5/6,1/5)\};\\
\{(4/5,1/5),(2/3,1/3),(4/5,1/2)\};\\
\{(4/5,1/2),(4/5,1/3),(5/6,1/3),(5/6,1/2)\}.
\end{gathered}$$

For each of these convex hulls the graph constructed with help of Lemma 3.2 is either acyclic or contains cycles corresponding to empty cutout polygons. This proves the lemma. □

Lemma 5.12. *Let*

$$A_4 := \left\{(x,y) \,\middle|\, \frac{2}{3} \le x \le \frac{5}{6},\ -\frac{x}{2} \le y \le -1+x\right\}.$$

Then $A_4 \subset \mathcal{D}_2^0$.

Proof. This lemma is proved with help of the algorithm in Lemma 3.2. To this matter we need to cover A_4 with small regions. These are the convex hulls of the following sets of points

$$\begin{gathered}
\{(2/3,-1/3),(4/5,-2/5),(4/5,-1/5)\};\\
\{(4/5,-1/5),(4/5,-1/3),(5/6,-1/3),(5/6,-1/6)\};\\
\{(4/5,-1/3),(4/5,-2/5),(5/6,-5/12),(5/6,-1/3)\}.
\end{gathered}$$

For the first two convex hulls the graph constructed with help of Lemma 3.2 is either acyclic or contains cycles corresponding to empty cutout polygons. The last convex hull gives rise to a graph having a cycle which leads to the cutout polygon $P((2,-1);-2,1,3,1,-2,-1,2)$. This is the polygon causing the cutout E_3 of Proposition 3.7. Since $E_3 \cap A_4 = \emptyset$ this cutout is not relevant for the characterization of the SRS in A_4. This proves the lemma. □

Summing up we finish the proof of Theorem 5.8.

6. Some Conjectures

We finish this paper with the statement of some conjectures.

Conjecture 6.1. $\mathcal{D}_2$ coincides with the set D defined in Theorem 2.1. In particular, what remains to be proved in view of that theorem is

$$\{(1,y) \mid |y| < 2\} \subset \mathcal{D}_2.$$

In other words, let $|\lambda| < 2$ and let $(a_n)_{n=1}^{\infty}$ be a sequence which satisfies

$$0 \le a_n + \lambda a_{n+1} + a_{n+2} < 1 \qquad (n \in \mathbb{N}).$$

Then $(a_n)_{n=1}^{\infty}$ is periodic.

You find partial results concerning this conjecture in [5]. Especially we prove that it is true for $\lambda = \frac{1+\sqrt{5}}{2}$.

Conjecture 6.2. The interior of the region defined by the convex hull of the points

$$\left\{(1,1), \left(\frac{29}{30}, 1\right), \left(\frac{29}{30}, \frac{31}{30}\right)\right\}$$

is contained in $\mathcal{D}_2^0$. This is the region on the right hand side of the quadrangle characterized by Lemma 5.7 in Figure 1.

The interior of the triangle defined by the convex hull of

$$\left\{(1,2), \left(\frac{5}{6}, \frac{11}{6}\right), \left(\frac{5}{6}, \frac{10}{6}\right)\right\}$$

is contained in $\mathcal{D}_2^0$. This is the light grey region beyond E_1 in Figure 1.

Conjecture 6.3. The number of critical points[15] of $\mathcal{D}_d$ is finite. $\mathcal{D}_2$ has only two critical points. These are the points $(1,0)$ and $(1,1)$.

In the first part of this series of papers we showed that the set of weak critical points is compact.

In an earlier version of this paper we conjectured too: Let M be a positive integer and set

$$\begin{aligned} N(d,M) &= |\{(p_1,\dots,p_{d-1}) \in \mathbb{Z}^{d-1} \mid (M,p_1,\dots,p_{d-1}) \in \mathcal{C}_d\}|, \\ N^0(d,M) &= |\{(p_1,\dots,p_{d-1}) \in \mathbb{Z}^{d-1} \mid (M,p_1,\dots,p_{d-1}) \in \mathcal{C}_d^0\}|. \end{aligned}$$

Then

$$\lim_{M\to\infty} \frac{N(d+1,M)}{M^d} = \lambda_d(\mathcal{D}_d) \quad \text{and} \quad \lim_{M\to\infty} \frac{N^0(d+1,M)}{M^d} = \lambda_d\left(\mathcal{D}_d^0\right),$$

where λ_d denotes the d-dimensional Lebesgue measure (the Lebesgue measurability of $\mathcal{D}_d$ and $\mathcal{D}_d^0$ is proved in [4, Theorem 4.10]). In the meantime we proved both assertions and the result will appear in part III of this series of papers.

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