PERIODICITY OF CERTAIN PIECEWISE AFFINE PLANAR MAPS

SHIGEKI AKIYAMA, HORST BRUNOTTE, ATTILA PETHŐ, AND WOLFGANG STEINER

ABSTRACT. We determine periodic and aperiodic points of certain piecewise affine maps in the Euclidean plane. Using these maps, we prove that all integer sequences $(a_k)_{k\in\mathbb{Z}}$ satisfying $0 \le a_{k-1} + \lambda a_k + a_{k+1} < 1$ for some (fixed) $\lambda \in \{\frac{\pm 1 \pm \sqrt{5}}{2}, \pm\sqrt{2}\}$ are periodic.

1. INTRODUCTION

In the past few decades discontinuous piecewise affine maps have found considerable interest in the theory of dynamical systems. For an overview we refer the reader to [1, 7, 12, 13, 16, 17], for particular instances to [27, 15, 24] (polygonal dual billiards), [14] (polygonal exchange transformations), [10, 29, 11, 8] (digital filters) and [18, 20, 21] (propagation of round-off errors in linear systems). The present note deals with a presumably folklore conjecture on the periodicity of a certain kind of these maps which was recently stated twice in the literature in an explicit form. The first version appeared in [4]:

Conjecture 1.1. *For every real λ with $|\lambda| < 2$, all integer sequences $(a_k)_{k\in\mathbb{Z}}$ satisfying*

$$0 \le a_{k-1} + \lambda a_k + a_{k+1} < 1 \tag{1.1}$$

for all $k \in \mathbb{Z}$ are periodic.

Vivaldi [25] established an equivalent formulation: If $\lambda \in (-2, 2)$ then all orbits of the lattice map $\mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ given by

$$(x, y) \longmapsto (\lfloor \lambda x \rfloor - y, x)$$

are periodic. These two formulations are in fact equivalent: Assuming the truth of Conjecture (1.1) then the integer sequence

$$x_{k+1} = \lfloor \lambda x_k \rfloor - x_{k-1}$$

is periodic because it satisfies

$$0 \le (-x_{k-1}) + (-\lambda)(-x_k) + (-x_{k+1}) < 1.$$

The reverse implication is seen analogously.

The last mentioned formulation of Conjecture (1.1) has originated from a discretization process in a rounding-off scheme occurring in computer simulation of dynamical systems (we refer the reader to [18] and [25] and the literature quoted there). On the other hand, the interest in integer sequences satisfying (1.1) arose in the study of shift radix systems (see [4] and [2] for details).

Conjecture (1.1) is trivially true for $\lambda = -1, 0, 1$. A computer assisted proof for $\lambda = \frac{1-\sqrt{5}}{2}$ was given by Lowenstein, Hatjispyros and Vivaldi [18], where also the solution for the golden mean $\lambda = \gamma = \frac{1+\sqrt{5}}{2}$ was mentioned. A short proof (without use of computers) of the case $\lambda = \gamma$ was given by the authors [3].

The proof in [18] is based on a non-ergodic piecewise affine map on the unit square which is treated by Kouptsov, Lowenstein and Vivaldi [17] for all quadratic λ corresponding to rational rotations ($\lambda = \frac{\pm 1 \pm \sqrt{5}}{2}, \pm\sqrt{2}, \pm\sqrt{3}$), by heavy use of computers. Important related work is due to Adler, Kitchens and Tresser [1], Poggiaspalla [23], Vivaldi [25], Vivaldi and Lowenstein [26] and others.

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In this note we further develop the method described in [18]. Let $\lambda^2 = b\lambda + c$ with $b, c \in \mathbb{Z}$. Set $x = \{\lambda a_{k-1}\}$ and $y = \{\lambda a_k\}$, where $\{z\} = z - \lfloor z \rfloor$ denotes the fractional part of z. Then we have $a_{k+1} = -a_{k-1} - \lambda a_k + y$ and

$$\{\lambda a_{k+1}\} = \{-\lambda a_{k-1} - \lambda^2 a_k + \lambda y\} = \{-x + (\lambda - b)y\} = \{-x + cy/\lambda\} = \{-x - \lambda' y\},$$

where λ' is the algebraic conjugate of λ. Therefore we are interested in the discontinuous measure-preserving piecewise affine planar map $T : [0,1)^2 \to [0,1)^2$ given by $T(x,y) = (y, \{-x - \lambda' y\})$. Obviously, it suffices to study the periodicity of $(T^k(z))_{k\in\mathbb{Z}}$ for points $z = (x,y) \in (\mathbb{Z}[\lambda] \cap [0,1))^2$ in order to prove the conjecture. Note that people interested in dynamical systems usually neglect sets of measure zero like the discontinuity lines and its images, but we cannot neglect them since many points $z \in (\mathbb{Z}[\lambda] \cap [0,1))^2$ lie in these sets.

It turns out that investigating the dynamical system $([0,1)^2, T)$ allows proving the conjecture for all quadratic λ corresponding to rational rotations; furthermore, all possible period lengths are explicitly determined and aperiodic points in $(\mathbb{Q}(\lambda) \cap [0,1))^2$ are given. Note that the set of aperiodic points can be constructed similarly to a Cantor set, and that it is an open question of Mahler [22] whether there exist algebraic points in the triadic Cantor set.

The paper is organized as follows. In Section 2 we reprove the conjecture for the above mentioned simplest non-trivial case, i.e. where λ equals the golden mean. Here our main tool for the description of T is introduced, namely a related map S constructed from T by some scaling procedure. An exposition of our domain exchange method is given in Section 3 where the ideas of Section 2 are extended and the properties of S on suitable subsets of the unit square are exploited; here and in the subsequent four sections we prove the conjecture for the cases $\lambda = -\gamma, \pm 1/\gamma, \pm\sqrt{2}$ thereby making use of the fact that the scaling factor in the definition of S is the conjugate of a Pisot unit in the quadratic number field $\mathbb{Q}(\lambda)$. We conclude this note by an observation relating the famous Thue-Morse sequence to the trajectory of points for $\lambda = \pm\gamma, \pm 1/\gamma$. The proofs of the conjecture for $\lambda = \pm\sqrt{3}$ are much more involved and therefore postponed to future work. For cubic λ, the analog of the map T is defined on $[0,1)^4$, and the proof of the conjecture requires further efforts.

2. The case $\lambda = \gamma = \frac{1+\sqrt{5}}{2} = -2\cos\frac{4\pi}{5}$

We consider first the golden mean $\lambda = \gamma = \frac{1+\sqrt{5}}{2}$, $\lambda^2 = \lambda + 1$. Note that T is given by

$$T(x,y) = (x,y)A + (0, \lceil x - y/\gamma \rceil) \quad \text{with} \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 1/\gamma \end{pmatrix}. \tag{2.1}$$

Therefore, we have $T(x,y) = (x,y)A$ if $x \le y/\gamma$ and $T(z) = zA + (0,1)$ for the other points $z \in [0,1)^2$, see Figure 2.1. A particular role is played by the set

$$\mathcal{R} = \{(x,y) \in [0,1)^2 : x > y/\gamma,\ x + y > 1,\ y > x/\gamma\} \cup \{(0,0)\}.$$

If $z \in \mathcal{R}$, $z \ne (0,0)$, then we have $T^{k+1}(z) = T^k(z)A + (0,1)$ for all $k \in \{0,1,2,3,4\}$, hence

$$T^5(z) = zA^5 + (0,1)(A^4 + A^3 + A^2 + A^1 + A^0) = z + (0,1)(A^5 - A^0)(A - A^0)^{-1} = z$$

since $A^5 = A^0$. It can be easily verified that the minimal period length is 5 for all $z \in \mathcal{R}$ except $(\frac{\gamma^2}{\gamma^2+1}, \frac{\gamma^2}{\gamma^2+1})$ and $(0,0)$, which are fixed points of T. Therefore, it is sufficient to consider the domain $\mathcal{D} = D_0 \cup D_1$ with $D_0 = \{(x,y) \in [0,1)^2 : x \le y/\gamma\} \setminus \{(0,0)\}$, and $D_1 = [0,1)^2 \setminus (D_0 \cup \mathcal{R})$ in the following.

In Figure 2.2, we scale the polygons D_0 and D_1 by the factor $1/\gamma^2$ and follow their T-trajectory until the return to $\mathcal{D}/\gamma^2$. Let $\mathcal{P}$ be the set of (gray) points in $\mathcal{D}$ which are not eventually mapped to $\mathcal{D}/\gamma^2$, i.e.,

$$\mathcal{P} = D_\alpha \cup T(D_\alpha) \cup D_\beta \cup T(D_\beta) \cup T^2(D_\beta),$$

where D_α is the closed pentagon $\{(x,y) \in D_0 : y \ge 1/\gamma^2, x + y \le 1, y \le (1+x)/\gamma\}$ and D_β is the open pentagon $\mathcal{R}/\gamma^2 \setminus \{(0,0)\}$. (In Figure 2.2, D_α is split up into $\{T^k(D_{\tilde{\alpha}}) : k \in \{0,2,4,6,8\}\}$, and D_β is split up into $\{T^k(D_{\tilde{\beta}}) : k \in \{0,3,6,9,12\}\}$.) All points in $\mathcal{P}$ are periodic (with minimal

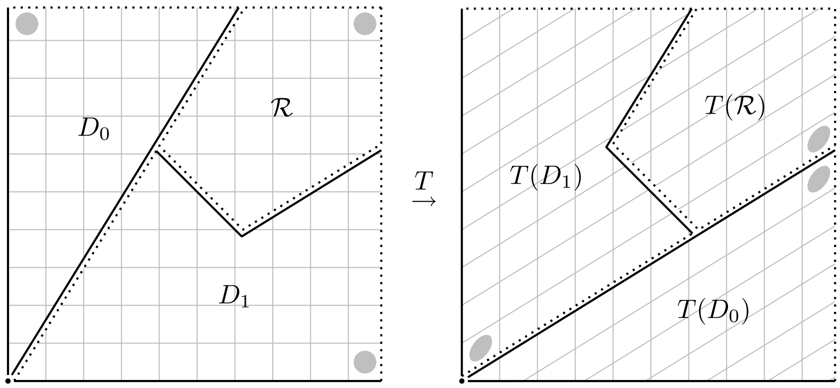

FIGURE 2.1. The piece-wise affine map T and the set $\mathcal{R}$, $\lambda = \gamma = \frac{1+\sqrt{5}}{2}$.

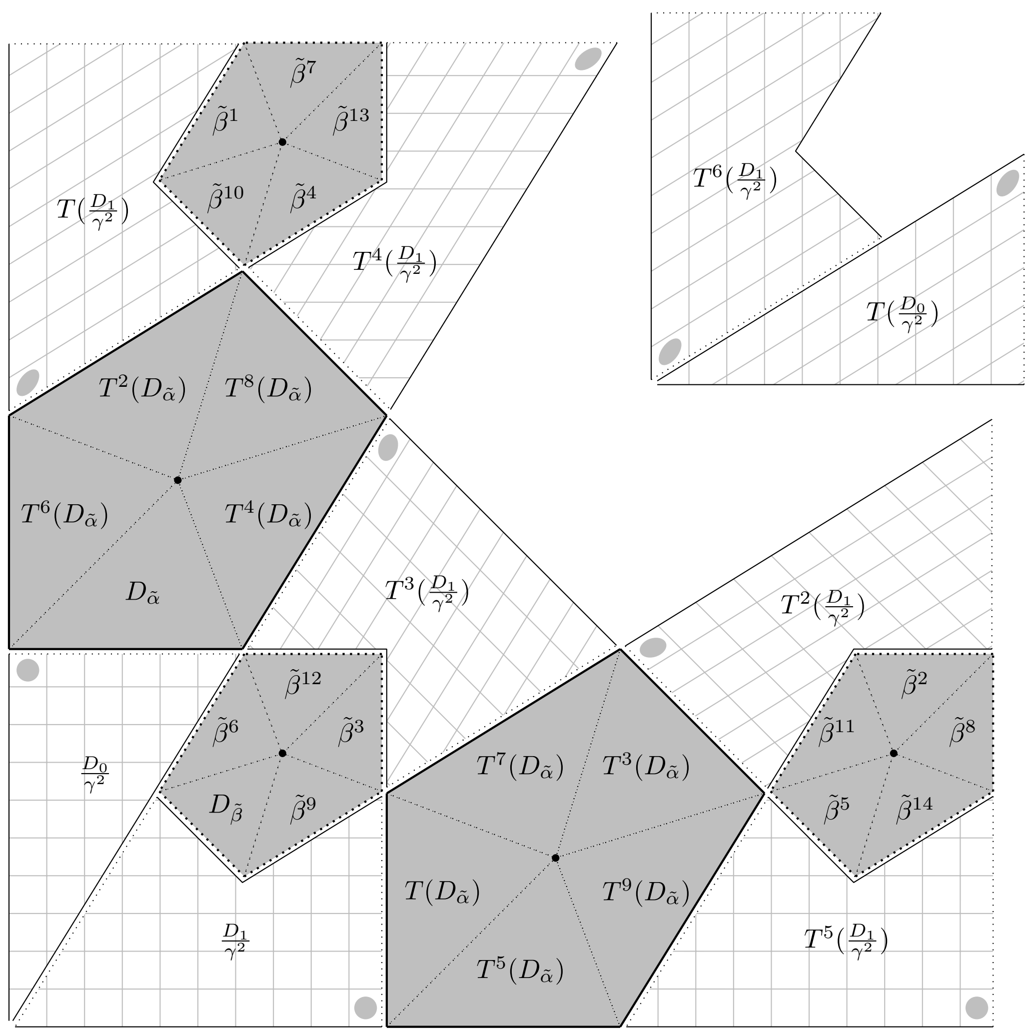

FIGURE 2.2. The trajectory of the scaled domains and the (gray) set $\mathcal{P}$, $\lambda = \gamma$. ($\tilde{\beta}^k$ stands for $T^k(D_{\tilde{\beta}})$.)

period length 1, 10 or 15). Figures 2.1 and 2.2 show that the action of the first return map on $\mathcal{D}/\gamma^2$ is similar to the action of T on $\mathcal{D}$, more precisely,

$$\frac{T(z)}{\gamma^2} = \begin{cases} T(z/\gamma^2) & \text{if } z \in D_0, \\ T^6(z/\gamma^2) & \text{if } z \in D_1. \end{cases} \tag{2.2}$$

For $z \in \mathcal{D} \setminus \mathcal{P}$, let $s(z) = \min\{m \geq 0 : T^m(z) \in \mathcal{D}/\gamma^2\}$. (Figure 2.2 shows $s(z) \leq 5$.) By the map

$$S: \mathcal{D} \setminus \mathcal{P} \to \mathcal{D}, \quad z \mapsto \gamma^2 T^{s(z)}(z),$$

we can completely characterize the periodic points. For $z \in [0,1)^2$, denote by $\pi(z)$ the minimal period length if $(T^k(z))_{k\in\mathbb{Z}}$ is periodic and set $\pi(z) = \infty$ if z is aperiodic.

Theorem 2.1. *$(T^k(z))_{k\in\mathbb{Z}}$ is periodic if and only if $z \in \mathcal{R}$ or $S^n(z) \in \mathcal{P}$ for some $n \geq 0$.*

We postpone the proof to Section 3, where the more general Proposition 3.3 and Theorem 3.4 are proved (with $U(z) = z/\gamma^2$, $R(z) = z$, $\hat{T}(z) = T(z)$, $\hat{\pi}(z) = \pi(z)$, and $z \in D_1$ or $T(z) \in D_1$ for all $z \in \mathcal{D}$, $|\sigma^n(1)| \to \infty$, see below).

(2.2) and Figure 2.2 suggest to define a substitution (or morphism) σ on the alphabet $\mathcal{A} = \{0,1\}$, i.e., a map $\sigma : \mathcal{A} \to \mathcal{A}^\star$ (where $\mathcal{A}^\star$ denotes the set of words with letters in $\mathcal{A}$), by

$$\sigma: \quad 0 \mapsto 0 \qquad 1 \mapsto 101101$$

in order to code the trajectory of the scaled domains: We have $T^{k-1}(D_\ell/\gamma^2) \subseteq D_{\sigma(\ell)[k]}$ and $T^{|\sigma(\ell)|}(z/\gamma^2) = T(z)/\gamma^2$ for all $z \in D_\ell$, where $w[k]$ denotes the k-th letter of the word w and $|w|$ denotes its length. Furthermore, we have $T^k(D_\ell/\gamma^2) \cap \mathcal{D}/\gamma^2 = \emptyset$ for all k, $1 \leq k < |\sigma(\ell)|$. Extend the definition of σ naturally to words in $\mathcal{A}^\star$ by setting $\sigma(vw) = \sigma(v)\sigma(w)$, where vw denotes the concatenation of v and w. Then we get the following lemma.

Lemma 2.2. *For every integer $n \geq 0$ and every $\ell \in \{0,1\}$, we have*

- $T^{|\sigma^n(\ell)|}(z/\gamma^{2n}) = T(z)/\gamma^{2n}$ *for all* $z \in D_\ell$,
- $T^{k-1}(D_\ell/\gamma^{2n}) \subseteq D_{\sigma^n(\ell)[k]}$ *for all* k, $1 \leq k \leq |\sigma^n(\ell)|$
- $T^k(D_\ell/\gamma^{2n}) \cap \mathcal{D}/\gamma^{2n} = \emptyset$ *for all* k, $1 \leq k < |\sigma^n(\ell)|$.

The proof is again postponed to Section 3, Lemma 3.1. This lemma allows to determine the minimal period lengths: If $z \in D_\alpha$, then

$$T^{|\sigma^n(0101010101)|}(z/\gamma^{2n}) = T^{|\sigma^n(101010101)|}(T(z)/\gamma^{2n}) = \cdots = T^{10}(z)/\gamma^{2n} = z/\gamma^{2n}$$

for all $n \geq 0$. The only points $T^k(z/\gamma^{2n})$ with $1 \leq k \leq 5|\sigma^n(01)|$ which lie in $\mathcal{D}/\gamma^{2n}$ are the points $T^m(z)/\gamma^{2n}$, $1 \leq m \leq 9$, which are all different from z/γ^{2n} if $\pi(z) = 10$. Therefore, we obtain $\pi(z/\gamma^{2n}) = 5|\sigma^n(01)|$ in this case. A point $\tilde{z}$ lies in the trajectory of z/γ^{2n} if and only if $S^n(\tilde{z}) = T^m(z)$ for some $m \in \mathbb{Z}$, see Lemma 3.2. This implies $\pi(\tilde{z}) = 5|\sigma^n(01)|$ for these $\tilde{z}$ as well. Similarly, we obtain $\pi(z) = 5|\sigma^n(101)|$ if $S^n(z) \in T^m(D_\beta)$ and $\pi(S^n(z)) = 15$. More precisely, the following theorem holds.

Theorem 2.3. *If $\lambda = \gamma$, then the minimal period lengths $\pi(z)$ of $(T^k(z))_{k\in\mathbb{Z}}$ are*

1	*if* $z = (0,0)$ *or* $z = (\frac{\gamma^2}{\gamma^2+1}, \frac{\gamma^2}{\gamma^2+1})$
5	*if* $z \in \mathcal{R} \setminus \{(0,0), (\frac{\gamma^2}{\gamma^2+1}, \frac{\gamma^2}{\gamma^2+1})\}$
$(5 \cdot 4^n + 1)/3$	*if* $S^n(z) = T^m(\frac{1/\gamma}{\gamma^2+1}, \frac{2}{\gamma^2+1})$ *for some* $n \geq 0$, $m \in \{0,1\}$
$5(5 \cdot 4^n + 1)/3$	*if* $S^n(z) \in T^m\big(D_\alpha \setminus \{(\frac{1/\gamma}{\gamma^2+1}, \frac{2}{\gamma^2+1})\}\big)$ *for some* $n \geq 0$, $m \in \{0,1\}$
$(10 \cdot 4^n - 1)/3$	*if* $S^n(z) = T^m(\frac{1}{\gamma^2+1}, \frac{1}{\gamma^2+1})$ *for some* $n \geq 0$, $m \in \{0,1,2\}$
$5(10 \cdot 4^n - 1)/3$	*if* $S^n(z) \in T^m\big(D_\beta \setminus \{(\frac{1}{\gamma^2+1}, \frac{1}{\gamma^2+1})\}\big)$ *for some* $n \geq 0$, $m \in \{0,1,2\}$
∞	*if* $S^n(z) \in \mathcal{D} \setminus \mathcal{P}$ *for all* $n \geq 0$

The minimal period length of $(a_k)_{k\in\mathbb{Z}}$ is $\pi(\{\gamma a_{k-1}\}, \{\gamma a_k\})$ (which does not depend on k).

Proof. By Theorem 2.1, Proposition 3.3 and the remarks preceding the theorem, it suffices to calculate $|\sigma^n(0)|$ and $|\sigma^n(1)|$. Clearly, we have $|\sigma^n(0)| = 1$ for all $n \geq 0$ and thus

$$|\sigma^n(1)| = |\sigma^{n-1}(101101)| = 4|\sigma^{n-1}(1)| + 2 = 4(5 \cdot 4^{n-1} - 2)/3 + 2 = (5 \cdot 4^n - 2)/3.$$

If $S^n(z) \in T^m(D_\alpha)$, then $\pi(z) = |\sigma^n(01)|$ and $\pi(z) = 5|\sigma^n(01)|$ respectively. If $S^n(z) \in T^m(D_\beta)$, then $\pi(z) = |\sigma^n(101)|$ and $\pi(z) = 5|\sigma^n(101)|$ respectively. □

Now consider aperiodic points $z \in [0,1)^2$, i.e., $S^n(z) \in \mathcal{D} \setminus \mathcal{P}$ for all $n \geq 0$. By (2.1) and Figure 2.2, we have

$$S(z) = \gamma^2 T^{s(z)}(z) = \gamma^2 \big(zA^{s(z)} + t(z)\big)$$

with

$$t(z) = \begin{cases} (0,0) & \text{if } s(z) = 0, \\ (0,1) & \text{if } s(z) \in \{1,2\}, \\ (0,1)A^2 + (0,1) = (1/\gamma, 1/\gamma^2) & \text{if } s(z) = 3, \\ (0,1)A^3 + (0,1)A^2 + (0,1) = (0, -1/\gamma) & \text{if } s(z) \in \{4,5\}. \end{cases}$$

Therefore we obtain inductively

$$S^n(z) = \gamma^{2n} z A^{s(z)+s(S(z))+\ldots+s(S^{n-1}(z))} + \sum_{k=0}^{n-1} \gamma^{2(n-k)} t(S^k(z)) A^{s(S^{k+1}(z))+\cdots+s(S^{n-1}(z))}$$

For $x \in \mathbb{Q}(\gamma)$, let x' be its algebraic conjugate, and $w' = (x', y')$ if $w = (x,y) \in \mathbb{Q}(\gamma)^2$. If $z \in \mathbb{Q}(\gamma)^2$, we have then

$$(S^n(z))' = \frac{\Big(zA^{s(z)+s(S(z))+\cdots+s(S^{n-1}(z))}\Big)'}{\gamma^{2n}} + \sum_{k=0}^{n-1} \frac{\Big(t(S^k(z))A^{s(S^{k+1}(z))+\cdots+s(S^{n-1}(z))}\Big)'}{\gamma^{2(n-k)}}$$

and

$$\|(S^n(z))'\|_\infty \leq \frac{\max_{h\in\mathbb{Z}} \|(zA^h)'\|_\infty}{\gamma^{2n}} + \sum_{k=0}^{n-1} \frac{\max_{h\in\mathbb{Z}, w\in\mathcal{D}\setminus\mathcal{P}} \|(t(w)A^h)'\|_\infty}{\gamma^{2n-k}}.$$

Since

$$\begin{aligned} t(w)A^h \in \big\{&(0,0),\ (0,1), (1,1/\gamma), (1/\gamma, -1/\gamma), (-1/\gamma, -1), (-1,0), \\ &(1/\gamma, 1/\gamma^2), (1/\gamma^2, -1/\gamma^2), (-1/\gamma^2, -1/\gamma), (-1/\gamma, 0), (0, 1/\gamma), \\ &(0,-1/\gamma), (-1/\gamma, -1/\gamma^2), (-1/\gamma^2, 1/\gamma^2), (1/\gamma^2, 1/\gamma), (1/\gamma, 0)\big\}, \end{aligned}$$

and zA^h takes only the values z, zA, zA^2, zA^3 and zA^4, we obtain

$$\|(S^n(z))'\|_\infty \leq \frac{\max_{h\in\mathbb{Z}} \|(zA^h)'\|_\infty}{\gamma^{2n}} + \sum_{k=0}^{n-1} \frac{\gamma^2}{\gamma^{2(n-k)}} < \frac{C(z)}{\gamma^{2n}} + \gamma$$

for some constant $C(z)$. If $z \in (\frac{1}{Q}\mathbb{Z}[\gamma])^2$ for some integer $Q \geq 1$, then $S^n(z) \in (\frac{1}{Q}\mathbb{Z}[\gamma])^2$. Since there exist only finitely many points $w \in (\frac{1}{Q}\mathbb{Z}[\gamma] \cap [0,1))^2$ with $\|w'\|_\infty < C(z) + \gamma$, we must have $\|(S^n(z))'\|_\infty \leq \gamma$ for some $n \geq 0$, which proves the following proposition.

Proposition 2.4. *Let $z \in (\frac{1}{Q}\mathbb{Z}[\gamma] \cap [0,1))^2$ be an aperiodic point. Then there exists an aperiodic point $\tilde{z} \in (\frac{1}{Q}\mathbb{Z}[\gamma])^2 \cap \mathcal{D}$ with $\|\tilde{z}'\|_\infty \leq \gamma$.*

For every denominator $Q \geq 1$, it is therefore sufficient to check the periodicity of the (finite set of) points $z \in (\frac{1}{Q}\mathbb{Z}[\gamma])^2 \cap \mathcal{D}$ with $\|z'\|_\infty \leq \gamma$ in order to determine if all points in $(\frac{1}{Q}\mathbb{Z}[\gamma] \cap [0,1))^2$ are periodic.

Clearly, $\mathbb{Z}[\gamma] \subset \frac{1}{2}\mathbb{Z}[\gamma]$. Therefore, we consider directly $Q = 2$. We have to take into account coordinates of the form $x = a + b\gamma \in [0,1)$ with $a, b \in \frac{1}{2}\mathbb{Z}$ and $|x'| = |a - b/\gamma| \leq \gamma$. For $b = 0$, we obtain $x = 0$ and $x = 1/2$; for $b = 1/2$, we obtain $x = \gamma/2$ and $x = -1/2 + \gamma/2 = 1/(2\gamma)$; for $b = 1$, we obtain $x = -1 + \gamma = 1/\gamma$; for other x with $b \geq 1$, we must have $a \leq -3/2$ and thus $x' \leq -3/2 - 1/\gamma < -\gamma$; for $b = -1/2$, we obtain $x = 1 - \gamma/2 = 1/(2\gamma^2)$; other values with $b \leq -1/2$ are again impossible since $a \geq 3/2$ implies $x' \geq 3/2 + \gamma/2 > \gamma$. Note that for $Q = 1$

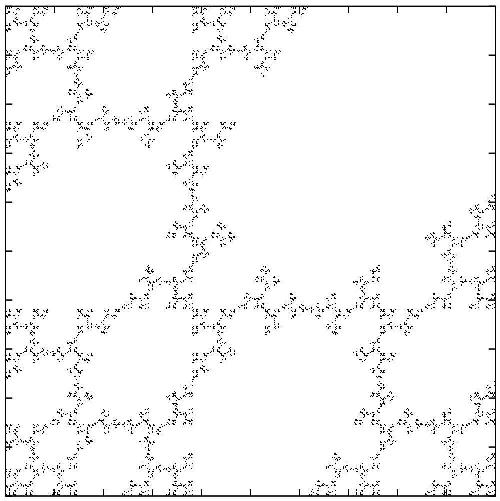

FIGURE 2.3. Aperiodic points, $\lambda = \gamma$.

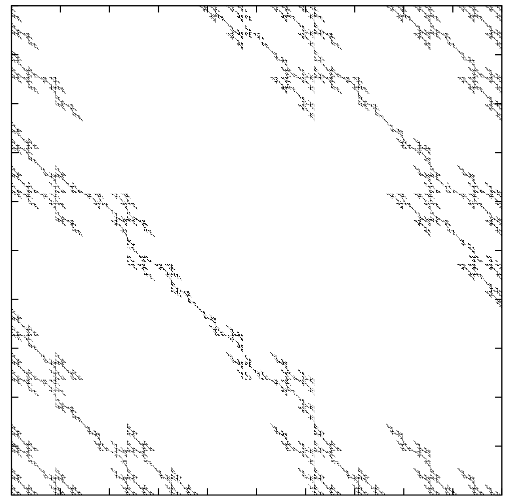

FIGURE 2.4. Aperiodic points, $\lambda = -1/\gamma$.

(which corresponds to the conjecture), only $x = 0$ and $x = 1/\gamma$ are possible. Now consider $S(x,y)$ for $(x,y) \in \{0, 1/(2\gamma^2), 1/(2\gamma), 1/2, 1/\gamma, \gamma/2\}^2$ (if $z \in \mathcal{D} \setminus \mathcal{P}$, otherwise we clearly have $\pi(z) < \infty$):

$x \setminus y$	0	$1/(2\gamma^2)$	$1/(2\gamma)$	$1/2$	$1/\gamma$	$\gamma/2$
0	$\in \mathcal{R}$	$(0,1/2)$	$(0,\gamma/2)$	$\in D_\alpha$	$\in D_\alpha$	$(0,1/2)$
$1/(2\gamma^2)$	$(1/2,0)$	$(1/2,1/2)$	$(1/2,\gamma/2)$	$\in D_\alpha$	$\in D_\alpha$	$(1/2,1/2)$
$1/(2\gamma)$	$(\gamma/2,0)$	$(\gamma/2,1/2)$	$(\gamma/2,\gamma/2)$	$\in D_\alpha$	$\in D_\alpha$	$(1/2,\gamma/2)$
$1/2$	$\in T(D_\alpha)$	$\in T(D_\alpha)$	$\in T(D_\alpha)$	$(0,1/2)$	$\in \mathcal{R}$	$(0,1/2)$
$1/\gamma$	$\in T(D_\alpha)$	$\in T(D_\alpha)$	$\in T(D_\alpha)$	$\in \mathcal{R}$	$\in \mathcal{R}$	$\in \mathcal{R}$
$\gamma/2$	$(0,1/2)$	$(1/2,\gamma/2)$	$(1/2,1/2)$	$(0,1/2)$	$\in \mathcal{R}$	$\in \mathcal{R}$

For every point $z \in (\frac{1}{2}\mathbb{Z}[\gamma])^2 \cap \mathcal{D}$ with $\|z'\|_\infty \le \gamma$, we have therefore some $n \in \{0,1,2\}$ such that $S^n(z) \in \mathcal{P}$.

If $Q = 3$, then the situation is completely different. We have

$$S(0,1/3) = (0,\gamma^2/3), \qquad S(0,\gamma^2/3) = \gamma^2\big((0,\gamma^2/3)A^5 + (0,-1/\gamma)\big) = (0,2/3),$$
$$S(0,2/3) = \gamma^2\big((0,2/3)A^5 + (0,-1/\gamma)\big) = \big(0,1/(3\gamma^2)\big), \quad S^4(0,1/3) = S\big(0,1/(3\gamma^2)\big) = (0,1/3).$$

Therefore, we have $S^n(0,1/3) \in \mathcal{D} \setminus \mathcal{P}$ for all $n \ge 0$ and $\pi(0,1/3) = \infty$ by Theorem 2.3.

Theorem 2.5. *$\pi(z)$ is finite for all points $z \in (\frac{1}{2}\mathbb{Z}[\gamma] \cap [0,1))^2$, but $(T^k(0,1/3))_{k\in\mathbb{Z}}$ is aperiodic.*

3. General description of the method

In this section, we generalize the method presented in Section 2 in order to make it applicable for $\lambda = -\gamma, \pm 1/\gamma, \pm\sqrt{2}$.

For the moment, we only need that $T : X \to X$ is a bijective map on a set X. Fix $\mathcal{D} \subseteq X$, let

$$\mathcal{R} = \{z \in X : \ T^m(z) \notin \mathcal{D} \text{ for all } m \ge 0\}$$

set $r(z) = \min\{m \ge 0 : T^m(z) \in \mathcal{D}\}$ for $z \in X \setminus \mathcal{R}$, and

$$R : \ X \setminus \mathcal{R} \to \mathcal{D}, \qquad R(z) = T^{r(z)}(z).$$

Let $\hat{T}$ be the first return map (of the iterates by T) on $\mathcal{D}$, i.e.,

$$\hat{T} : \ \mathcal{D} \to \mathcal{D}, \qquad \hat{T}(z) = RT(z) = T^{r(T(z))+1}(z),$$

in particular $\hat{T}(z) = T(z)$ if $T(z) \in \mathcal{D}$. Let $\mathcal{A}$ be a finite set, $\{D_\ell : \ell \in \mathcal{A}\}$ a partition of $\mathcal{D}$ and define a coding map $\iota : \mathcal{D} \to \mathcal{A}^{\mathbb{Z}}$ by $\iota(z) = (\iota_k(z))_{k\in\mathbb{Z}}$ such that $\hat{T}^k(z) \in D_{\iota_k(z)}$ for all $k \in \mathbb{Z}$. Let

$U : \mathcal{D} \to \mathcal{D}$, $\varepsilon \in \{-1, 1\}$ and σ a substitution on $\mathcal{A}$ such that, for every $\ell \in \mathcal{A}$ and $z \in D_\ell$,

$$U\hat{T}(z) = \hat{T}^{\varepsilon|\sigma(\ell)|}U(z),$$

$\hat{T}^{\varepsilon k}U(z) \notin U(\mathcal{D})$ for all k, $1 \le k < |\sigma(\ell)|$, and

$$\sigma(\ell) = \begin{cases} \iota_0(U(z))\,\iota_1(U(z)) \cdots \iota_{|\sigma(\ell)|-1}(U(z)) & \text{if } \varepsilon = 1, \\ \iota_{-|\sigma(\ell)|}(U(z)) \cdots \iota_{-2}(U(z))\,\iota_{-1}(U(z)) & \text{if } \varepsilon = -1. \end{cases}$$

Then the following lemma holds, which resembles Proposition 1 in [23].

Lemma 3.1. *For every integer $n \ge 0$, every $\ell \in \mathcal{A}$ and $z \in D_\ell$, we have*

$$U^n\hat{T}(z) = \hat{T}^{\varepsilon^n|\sigma^n(\ell)|}U^n(z),$$

$\hat{T}^{\varepsilon^n k}U^n(z) \notin U^n(\mathcal{D})$ for all k, $1 \le k < |\sigma^n(\ell)|$, and

$$\begin{aligned} \iota_0(U^n(z))\,\iota_1(U^n(z)) \cdots \iota_{|\sigma^n(\ell)|-1}(U^n(z)) &= \sigma^n(\ell) && \text{if } \varepsilon = 1, \\ \iota_0(U^n(z))\,\iota_1(U^n(z)) \cdots \iota_{|\sigma^n(\ell)|-1}(U^n(z)) &= (\sigma\bar{\sigma})^{n/2}(\ell) && \text{if } \varepsilon = -1, \varepsilon^n = 1, \\ \iota_{-|\sigma^n(\ell)|}(U^n(z)) \cdots \iota_{-2}(U^n(z))\,\iota_{-1}(U^n(z)) &= (\sigma\bar{\sigma})^{(n-1)/2}\sigma(\ell) && \text{if } \varepsilon = -1, \varepsilon^n = -1, \end{aligned}$$

where $\bar{\sigma}(\ell) = \ell_m \cdots \ell_2\ell_1$ if $\sigma(\ell) = \ell_1\ell_2 \cdots \ell_m$.

Proof. The lemma is trivially true for $n = 0$, and for $n = 1$ by the assumptions on σ. If we suppose inductively that it is true for $n-1$, then let $\sigma(\ell) = \ell_1\ell_2 \cdots \ell_m$ if $\varepsilon = 1$, $\sigma(\ell) = \ell_m \cdots \ell_2\ell_1$ if $\varepsilon = -1$, and we obtain (by another induction) for all j, $1 \le j \le m$,

$$\hat{T}^{\varepsilon^n|\sigma^{n-1}(\ell_1\cdots\ell_{j-1}\ell_j)|}U^n(z) = \hat{T}^{\varepsilon^n|\sigma^{n-1}(\ell_j)|}U^{n-1}\hat{T}^{\varepsilon(j-1)}U(z) = U^{n-1}\hat{T}^{\varepsilon j}U(z) \tag{3.1}$$

If $\varepsilon = 1$, then this follows immediately from the induction hypothesis; if $\varepsilon = -1$, then this follows by setting $k = |\sigma^{n-1}(\ell_j)|$ in

$$\hat{T}^{(-1)^n k}U^{n-1}\hat{T}\big(\hat{T}^{-j}U(z)\big) = \hat{T}^{(-1)^n(k-|\sigma^{n-1}(\ell_j)|)}U^{n-1}\hat{T}^{-j}U(z). \tag{3.2}$$

Therefore, we have

$$\hat{T}^{\varepsilon^n|\sigma^n(\ell)|}U^n(z) = \hat{T}^{\varepsilon^n|\sigma^{n-1}(\ell_1\cdots\ell_{m-1}\ell_m)|}U^n(z) = U^{n-1}\hat{T}^{\varepsilon m}U(z) = U^{n-1}\hat{T}^{\varepsilon|\sigma(\ell)|}U(z) = U^n\hat{T}(z).$$

If $\varepsilon = 1$, then (3.1) implies that

$$\begin{aligned} \iota_0(U^n(z)) \cdots \iota_{|\sigma^n(\ell)|-1}(U^n(z)) &= \big(\iota_0(U^{n-1}U(z)) \cdots \iota_{|\sigma^{n-1}(\ell_1)|-1}(U^{n-1}U(z))\big) \cdots \\ &\quad \big(\iota_0(U^{n-1}\hat{T}^{m-1}U(z)) \cdots \iota_{|\sigma^{n-1}(\ell_m)|-1}(U^{n-1}\hat{T}^{m-1}U(z))\big) = \sigma^{n-1}(\ell_1) \cdots \sigma^{n-1}(\ell_m) = \sigma^n(\ell); \end{aligned}$$

if $\varepsilon = -1$ and $\varepsilon^n = 1$, then (3.1) and (3.2) provide

$$\begin{aligned} \iota_0(U^n(z)) \cdots \iota_{|\sigma^n(\ell)|-1}(U^n(z)) &= \big(\iota_{-|\sigma^{n-1}(\ell_1)|}(U^{n-1}T^{-1}U(z)) \cdots \iota_{-1}(U^{n-1}T^{-1}U(z))\big) \\ &\quad \cdots \big(\iota_{-|\sigma^{n-1}(\ell_m)|}(U^{n-1}\hat{T}^{-m}U(z)) \cdots \iota_{-1}(U^{n-1}\hat{T}^{-m}U(z))\big) \\ &= (\sigma\bar{\sigma})^{(n-2)/2}\sigma(\ell_1) \cdots (\sigma\bar{\sigma})^{(n-2)/2}\sigma(\ell_m) = (\sigma\bar{\sigma})^{n/2}(\ell); \end{aligned}$$

if $\varepsilon = -1$ and $\varepsilon^n = -1$, then

$$\begin{aligned} \iota_{-|\sigma^n(\ell)|}(U^n(z)) \cdots \iota_{-1}(U^n(z)) &= \big(\iota_0(U^{n-1}T^{-m}U(z)) \cdots \iota_{|\sigma^{n-1}(\ell_m)|-1}(U^{n-1}T^{-m}U(z))\big) \\ &\quad \cdots \big(\iota_0(U^{n-1}\hat{T}^{-1}U(z)) \cdots \iota_{|\sigma^{n-1}(\ell_1)|}(U^{n-1}\hat{T}^{-1}U(z))\big) \\ &= (\sigma\bar{\sigma})^{(n-1)/2}(\ell_m) \cdots (\sigma\bar{\sigma})^{(n-1)/2}(\ell_1) = (\sigma\bar{\sigma})^{(n-1)/2}\sigma(\ell). \end{aligned}$$

By (3.1), (3.2) and the induction hypothesis, the only points in $(\hat{T}^{\varepsilon^n k}U^n(z))_{1 \le k < |\sigma^n(\ell)|}$ lying in $U^{n-1}(\mathcal{D})$ are $U^n\hat{T}^{\varepsilon j}(z)$, $1 \le j < |\sigma(\ell)|$. Since $\hat{T}^{\varepsilon j}(z) \notin U(\mathcal{D})$ for these j, the lemma is proved. □

Remark. If $\tilde{z} = \hat{T}^{-1}(z) \in D_\ell$, then $U^n\hat{T}(\tilde{z}) = \hat{T}^{\varepsilon^n|\sigma^n(\ell)|}U^n(\tilde{z})$, hence $U^n\hat{T}^{-1}(z) = T^{-\varepsilon^n|\sigma^n(\ell)|}U^n(z)$.

As in Section 2, a key role will be played by the map S. Assume that U is injective, let

$$\mathcal{P} = \{z \in \mathcal{D} :\ \hat{T}^m(z) \notin U(\mathcal{D}) \text{ for all } m \in \mathbb{Z}\},$$

fix $\hat{s}(z) = \min\{m \geq 0 : \hat{T}^m(z) \in U(\mathcal{D})\}$ or $\hat{s}(z) = \max\{m \leq 0 : \hat{T}^m(z) \in U(\mathcal{D})\}$ for every $z \in \mathcal{D} \setminus \mathcal{P}$, let $s(z) \in \mathbb{Z}$ be such that $\hat{T}^{\hat{s}(z)}(z) = T^{s(z)}(z)$, and define

$$S: \mathcal{D} \setminus \mathcal{P} \to \mathcal{D}, \qquad z \mapsto U^{-1}\hat{T}^{\hat{s}(z)}(z) = U^{-1}T^{s(z)}(z).$$

Remark. Allowing $s(z)$ and $\hat{s}(z)$ to be negative decreases the δ in Proposition 3.5 in most cases.

Lemma 3.2. *If $S^n R(z)$ exists, then we have some $m \geq 0$ such that $U^n S^n R(z) = T^m(z)$, and*

$$\tilde{z} = T^m(z) \text{ for some } m \in \mathbb{Z} \text{ if and only if } S^n R(\tilde{z}) = \hat{T}^k S^n R(z) \text{ for some } k \in \mathbb{Z}.$$

Proof. Suppose that $S^n R(z)$ exists. Then we have

$$U^n S^n R(z) = U^{n-1}\hat{T}^{\hat{s}(S^{n-1}R(z))}S^{n-1}R(z) = \hat{T}^{m_1}U^{n-1}S^{n-1}R(z) = \cdots = \hat{T}^{m_1+\cdots+m_n}R(z) = T^m(z)$$

for some $m_1, \ldots, m_n, m \geq 0$.

If $S^n R(\tilde{z}) = \hat{T}^k S^n R(z)$ for some $k \in \mathbb{Z}$, then let $m_1, m_2 \geq 0$ be such that $U^n S^n R(z) = T^{m_1}(z)$, $U^n S^n R(\tilde{z}) = T^{m_2}(\tilde{z})$, and we have

$$T^{m_2}(\tilde{z}) = U^n S^n R(\tilde{z}) = U^n \hat{T}^k S^n R(z) = \hat{T}^{k_1} U^n S^n R(z) = T^{k_2+m_1}(z)$$

for some $k_1, k_2 \in \mathbb{Z}$, hence $\tilde{z} = T^m(z)$ with $m = k_2 + m_1 - m_2$.

If $\tilde{z} = T^m(z)$ for some $m \in \mathbb{Z}$ and $n = 0$, then we have $S^n R(\tilde{z}) = \hat{T}^{k_n} S^n R(z)$ for some $k_n \in \mathbb{Z}$. If we suppose inductively that this is true for $n-1$, then

$$S^n R(\tilde{z}) = S\hat{T}^{k_{n-1}}S^{n-1}R(z) = S\hat{T}^{k_{n-1}-\hat{s}(S^{n-1}R(z))}US^n R(z) = SU\hat{T}^{k_n}S^n R(z) = \hat{T}^{k_n}S^n R(z)$$

for some $k_{n-1}, k_n \in \mathbb{Z}$, and the statement is proved. □

If rT is constant on every D_ℓ, $\ell \in \mathcal{A}$, then we can define $\tau : \mathcal{A} \to \mathbb{N}_{>0}$ by $\tau(\ell) = r(T(z)) + 1$ for $z \in D_\ell$ (cf. the definition of $\hat{T}$) and extend τ naturally to words $w \in \mathcal{A}^\star$ by $\tau(w) = \sum_{\ell \in \mathcal{A}} |w|_\ell \tau(\ell)$.

Let $\pi(z)$, $\hat{\pi}(z)$ be the minimal period lengths of $(T^k(z))_{k\in\mathbb{Z}}$ and $(\hat{T}^k(z))_{k\in\mathbb{Z}}$ respectively, with $\pi(z) = \infty$, $\hat{\pi}(z) = \infty$ if the sequences are aperiodic. Then the following proposition holds.

Proposition 3.3. *If $\hat{\pi}(S^n R(z)) = p$ and $\ell_1 \cdots \ell_p = \iota_0(S^n R(z)) \cdots \iota_{p-1}(S^n R(z))$, then we have*

$$\hat{\pi}(R(z)) = |\sigma^n(\ell_1\ell_2\cdots\ell_p)| \quad \text{and} \quad \pi(z) = \tau(\sigma^n(\ell_1\ell_2\cdots\ell_p)) \text{ (if } \tau \text{ is well defined).}$$

Proof. Since $U^n S^n R(z) = T^m(z) = \hat{T}^{\hat{m}}R(z)$ for some $m, \hat{m} \in \mathbb{Z}$, and

$$T^{\tau(\sigma^n(\ell_1\ell_2\cdots\ell_p))}U^n S^n R(z) = \hat{T}^{|\sigma^n(\ell_1\ell_2\cdots\ell_p)|}U^n S^n R(z) = U^n \hat{T}^p S^n R(z) = U^n S^n R(z),$$

we have $\hat{\pi}(R(z)) \leq |\sigma^n(\ell_1\cdots\ell_p)|$ and $\pi(z) \leq \tau(\sigma^n(\ell_1\cdots\ell_p))$ (if τ exists). Since p is minimal, we can show similarly to the proof of Lemma 3.1 that these period lengths are minimal. □

We obtain the following characterization of periodic points $z \notin \mathcal{R}$. Note that all points in $\mathcal{P} \cup \mathcal{R}$ are periodic in our cases, hence the characterization is complete.

Theorem 3.4. *Let $R, S, T, \mathcal{D}, \mathcal{P}, \mathcal{R}, \sigma$ be as in the preceding paragraphs of this section. Assume that $\hat{\pi}(z)$ is finite for all $z \in \mathcal{P}$, and that for every $z \in \mathcal{D} \setminus \mathcal{P}$ there exist $m \in \mathbb{Z}$, $\ell \in \mathcal{A}$, such that $\hat{T}^m(z) \in D_\ell$ and $|\sigma^n(\ell)| \to \infty$ for $n \to \infty$. Then we have for $z \notin \mathcal{R}$:*

$$(T^k(z))_{k\in\mathbb{Z}} \text{ is periodic if and only if } S^n R(z) \in \mathcal{P} \text{ for some } n \geq 0.$$

Proof. If $S^n R(z) \in \mathcal{P}$, then we have $\hat{\pi}(R(z)) = \hat{\pi}(S^n R(z)) < \infty$, which implies $\pi(z) < \infty$.

Suppose now that $S^n R(z) \in \mathcal{D} \setminus \mathcal{P}$ for all $n \geq 0$. Then we have $m_n \in \mathbb{Z}$ and $\ell_n \in \mathcal{A}$ such that $\hat{T}^{m_n} S^n R(z) \in D_{\ell_n}$ and $|\sigma^n(\ell_n)| \to \infty$ for $n \to \infty$ (because $\mathcal{A}$ is finite). We have $U^n \hat{T}^{m_n} S^n R(z) = \hat{T}^{\tilde{m}_n} U^n S^n R(z) \in U^n(D_{\ell_n})$ for some $\tilde{m}_n \in \mathbb{Z}$, hence $\hat{T}^{\tilde{m}_n+k}U^n S^n R(z) \notin U^n(\mathcal{D})$ for all k, $1 \leq k < |\sigma^n(\ell_n)|$, which implies $\pi(z) \geq \hat{\pi}(R(z)) = \hat{\pi}(U^n S^n R(z)) \geq |\sigma^n(\ell_n)|$ for all $n \geq 0$, thus $\pi(z) = \infty$. □

Assume now $\lambda \in \{\pm\sqrt{2}, \frac{\pm1\pm\sqrt{5}}{2}, \pm\sqrt{3}\}$, let λ' be its algebraic conjugate, $T: [0,1)^2 \to [0,1)^2$,

$$T(x,y) = (x,y)A + (0, \lceil x + \lambda' y \rceil) \quad \text{with} \quad A = \begin{pmatrix} 0 & -1 \\ 1 & -\lambda' \end{pmatrix}, \tag{3.3}$$

$$U(z) = V^{-1}(\kappa V(z))$$

with $0 < \kappa < 1$, $\kappa \in \mathbb{Z}[\lambda]$, $|\kappa\kappa'| = 1$, and $V(z) = z - v$ or $V(z) = v - z$ for some $v \in \mathbb{Z}[\lambda]^2$. Let

$$t(z) = V\big(T^{s(z)}(z)\big) - V(z)A^{s(z)}$$

for $z \in \mathcal{D} \setminus \mathcal{P}$. Since $U^{-1}(z) = V^{-1}(V(z)/\kappa)$, we have

$$S(z) = U^{-1}T^{s(z)}(z) = V^{-1}\left(\frac{V(z)A^{s(z)} + t(z)}{\kappa}\right),$$

Note that $A^h = A^0$ for some $h \in \{5, 8, 10, 12\}$,

$$T^{-1}(x,y) = (x,y)A^{-1} + (\lceil \lambda' x + y \rceil, 0) \quad \text{with} \quad A^{-1} = \begin{pmatrix} -\lambda' & 1 \\ -1 & 0 \end{pmatrix},$$

and $T^{-1}(x,y) = (\tilde{x}, \tilde{y})$ with $(\tilde{y}, \tilde{x}) = T(y,x)$. Since $|\hat{s}(z)| < \max_{\ell \in \mathcal{A}} |\sigma(\ell)|$, there exist only a finite number of values for $t(z)$, and we obtain the following proposition.

Proposition 3.5. *Let T, V, κ be as above and the assumptions of Theorem 3.4 be satisfied. Suppose that $\pi(z) = \infty$ for some $z \in (\frac{1}{Q}\mathbb{Z}[\lambda] \cap [0,1))^2 \setminus \mathcal{R}$, where Q is a positive integer. Then there exists an aperiodic point $\tilde{z} \in (\frac{1}{Q}\mathbb{Z}[\lambda])^2 \cap \mathcal{D}$ with*

$$\|V(\tilde{z})'\|_\infty \le \frac{\delta}{|\kappa'| - 1}, \quad \textit{where } \delta = \max\{\|(t(z)A^h)'\|_\infty : z \in \mathcal{D} \setminus \mathcal{P}, h \in \mathbb{Z}\}.$$

Proof. First note that δ exists since $t(z)$ and A^h take only finitely many values. If $\pi(z) = \infty$ for some $z \in (\frac{1}{Q}\mathbb{Z}[\lambda] \cap [0,1))^2 \setminus \mathcal{R}$, then $S^n R(z) \in \mathcal{D} \setminus \mathcal{P}$ for all $n \ge 0$ by Theorem 3.4. We use the abbreviations $s_n = s(S^n R(z))$ and $t_n = t(S^n R(z))$. Then we obtain inductively, for $n \ge 1$,

$$VS^n R(z) = \frac{VS^{n-1}R(z)A^{s_{n-1}} + t_{n-1}}{\kappa} = \frac{VR(z)A^{s_0+s_1+\cdots+s_{n-1}}}{\kappa^n} + \sum_{k=0}^{n-1} \frac{t_k A^{s_{k+1}+\cdots+s_{n-1}}}{\kappa^{n-k}}.$$

If we look at the algebraic conjugates, then note that $|\kappa'| > 1$, and we obtain

$$\|(VS^n R(z))'\|_\infty < \frac{\left\|\left(VR(z)A^{s_0+s_1+\cdots+s_{n-1}}\right)'\right\|_\infty}{|\kappa'|^n} + \frac{\delta}{|\kappa'| - 1},$$

thus $\|(VS^n R(z))'\|_\infty \le \frac{\delta}{|\kappa'|-1}$ for some $n \ge 0$ (as in Section 2), and we can choose $\tilde{z} = S^n R(z)$. □

Remarks.

- The last proof shows that, for every $z \in (\mathbb{Q}(\lambda) \cap [0,1))^2 \setminus \mathcal{R}$ with $\pi(z) = \infty$, there are only finitely many possibilities for $VS^n R(z)$, hence $(S^n R(z))_{n \ge 0}$ is eventually periodic.
- For every $z \in [0,1)^2 \setminus \mathcal{R}$ with $\pi(z) = \infty$, we have

$$VR(z) = \Big(VS^n R(z)\kappa^n - \sum_{k=0}^{n-1} t_k A^{s_{k+1}+\cdots+s_{n-1}}\kappa^k\Big)A^{-s_0-\cdots-s_{n-1}} = -\sum_{k=0}^{\infty} t_k A^{-\sum_{j=0}^{k} s(S^j R(z))}\kappa^k,$$

  which is a κ-expansion ($\kappa < 1$) of $VR(z)$ with (two-dimensional) "digits" $-t_k A^{-s_0-s_1-\cdots-s_k}$.
- As a consequence of Lemma 3.2 and the definition of U, for every aperiodic point $z \in [0,1)^2 \setminus \mathcal{R}$ and every $c > 0$, there exists some $m \in \mathbb{Z}$ such that $\|T^m(z) - v\|_\infty < c$.
- In all our cases, we have $\varepsilon = \kappa\kappa'$.

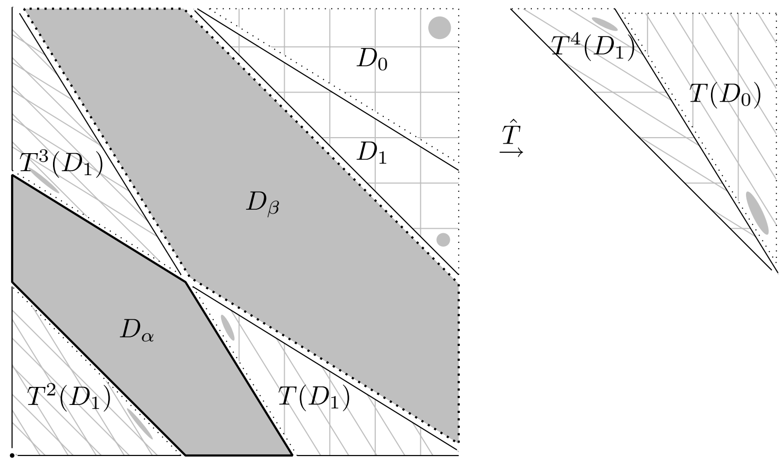

FIGURE 4.1. The map $\hat{T}$, $\hat{T}(D_0) = T(D_0)$, $\hat{T}(D_1) = T^4(D_1)$, and the (gray) set $\mathcal{R}$, $\lambda = -1/\gamma$.

4. The case $\lambda = -1/\gamma = \frac{1-\sqrt{5}}{2} = -2\cos\frac{2\pi}{5}$

Now we apply the method in Section 3 for $\lambda = -1/\gamma$, i.e., $\lambda' = \gamma$. To this end, set

$$\mathcal{D} = \{(x,y) \in [0,1)^2 : \ x + y \geq 3 - \gamma\} = D_0 \cup D_1$$

with $D_0 = \{(x,y) \in \mathcal{D} : x + \gamma y > 2\}$, $D_1 = \{(x,y) \in \mathcal{D} : x + \gamma y \leq 2\}$. Figure 4.1 shows that $\hat{T}$ is given by $\hat{T}(z) = T^{\tau(\ell)}(z)$ if $z \in D_\ell$, $\ell \in \mathcal{A} = \{0,1\}$, with $\tau(0) = 1$ and $\tau(1) = 4$, and $\mathcal{R} = \{(0,0)\} \cup D_\alpha \cup D_\beta$, with

$$D_\alpha = \{z \in [0,1)^2 : T^{k+1}(z) = T^k(z)A + (0,1) \text{ for all } k \geq 0\},$$
$$D_\beta = \{z \in [0,1)^2 : T^{k+1}(z) = T^k(z)A + (0,2) \text{ for all } k \geq 0\}.$$

As in Section 2, we have $T^5(z) = z$ for all $z \in \mathcal{R}$. If we set

$$U(z) = \frac{z}{\gamma^2} + \left(\frac{1}{\gamma}, \frac{1}{\gamma}\right) = (1,1) - \frac{(1,1) - z}{\gamma^2},$$

i.e., $V(z) = (1,1) - z$, $\kappa = 1/\gamma^2$, $\varepsilon = 1$ and

$$\sigma : \ 0 \mapsto 010 \qquad 1 \mapsto 01110,$$

then Figure 4.2 shows that σ satisfies the conditions in Section 3, and $\mathcal{P} = U(D_\alpha) \cup U(D_\beta)$. All points in $\mathcal{P}$ are periodic and $|\sigma^n(\ell)| \to \infty$ as $n \to \infty$ for all $\ell \in \mathcal{A}$. Therefore, all conditions of Proposition 3.3 and Theorem 3.4 are satisfied, and we obtain the following theorem.

Theorem 4.1. *If $\lambda = -1/\gamma$, then the period lengths $\pi(z)$ are*

1	*if* $z \in \{(0,0), (\frac{1}{\gamma^2+1}, \frac{1}{\gamma^2+1}), (\frac{2}{\gamma^2+1}, \frac{2}{\gamma^2+1})\}$
5	*for the other points of the pentagons* D_α, D_β
$2(5 \cdot 4^n + 1)/3$	*if* $S^n R(z) = (\frac{\gamma^2}{\gamma^2+1}, \frac{\gamma^2}{\gamma^2+1})$ *for some* $n \geq 0$
$10(5 \cdot 4^n + 1)/3$	*for the other points with* $S^n R(z) \in U(D_\alpha)$ *for some* $n \geq 0$
$(5 \cdot 4^n - 2)/3$	*if* $S^n R(z) = (\frac{3}{\gamma^2+1}, \frac{3}{\gamma^2+1})$ *for some* $n \geq 0$
$5(5 \cdot 4^n - 2)/3$	*for the other points with* $S^n R(z) \in U(D_\beta)$ *for some* $n \geq 0$
∞	*if* $S^n R(z) \in \mathcal{D} \setminus \mathcal{P}$ *for all* $n \geq 0$

Proof. We easily calculate

$$\begin{pmatrix} |\sigma^n(0)|_0 \\ |\sigma^n(0)|_1 \end{pmatrix} = 4^n \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}, \qquad \begin{pmatrix} |\sigma^n(1)|_0 \\ |\sigma^n(1)|_1 \end{pmatrix} = 4^n \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} + \begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix},$$

hence $\tau(\sigma^n(0)) = \frac{5}{3}4^n - \frac{2}{3}$, $\tau(\sigma^n(1)) = \frac{10}{3}4^n + \frac{2}{3}$. If $S^n R(z) \in U(D_\alpha)$, then $\pi(z) = \tau(\sigma^n(1))$ and $\pi(z) = \tau(\sigma^n(11111))$ respectively; if $S^n R(z) \in U(D_\beta)$, then $\pi(z) = \tau(\sigma^n(0))$ and $\pi(z) = 5\tau(\sigma^n(0))$ respectively. ∎

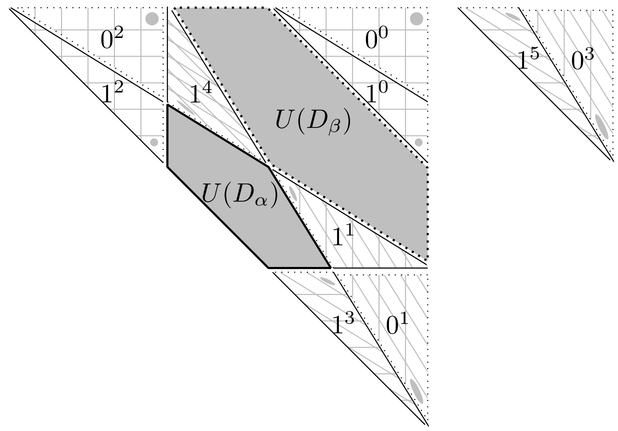

FIGURE 4.2. The trajectory of the scaled domains and $\mathcal{P}$, $\lambda = -1/\gamma$. (ℓ^k stands for $\hat{T}^k U(D_\ell)$.)

By Figure 4.2, we can choose $\hat{s}(z)$, $s(z)$ as follows and obtain the following $t(z)$:

$$\begin{aligned} z \in \hat{T}^2U(D_0) \cup \hat{T}^2U(D_1) &: \hat{s}(z) = -2, s(z) = -5,\ t(z) = V(\hat{T}^{-2}(z)) - V(z) = (-1/\gamma^2, 0) \\ z \in \hat{T}U(D_1) &: \hat{s}(z) = -1, s(z) = -1,\ t(z) = V(\hat{T}^{-1}(z)) - V(z)A^{-1} = (1/\gamma, 0) \\ z \in U(\mathcal{D}) &: \hat{s}(z) = 0, s(z) = 0,\ t(z) = (0,0) \\ z \in \hat{T}^4U(D_1) &: \hat{s}(z) = 1, s(z) = 1,\ t(z) = V(\hat{T}(z)) - V(z)A = (0, 1/\gamma) \\ z \in \hat{T}U(D_0) \cup \hat{T}^3U(D_1) &: \hat{s}(z) = 2, s(z) = 5,\ t(z) = V(\hat{T}^2(z)) - V(z) = (0, -1/\gamma^2) \end{aligned}$$

With

$$\begin{aligned} \{t(z)A^h : z \in \mathcal{D} \setminus \mathcal{P}, h \in \mathbb{Z}\} = \{(0,0),\ &\pm(0,1/\gamma), \pm(1/\gamma,-1), \pm(-1,1), \pm(1,1/\gamma), \pm(-1/\gamma,0), \\ &\pm(0,-1/\gamma^2), \pm(-1/\gamma^2,1/\gamma), \pm(1/\gamma,-1/\gamma), \pm(-1/\gamma,1/\gamma^2), \pm(1/\gamma^2,0)\}, \end{aligned}$$

we obtain $\delta = \frac{\gamma^2}{\gamma^2-1} = \gamma$ as in Section 2.

Theorem 4.2. *$\pi(z)$ is finite for all $z \in (\frac{1}{2}\mathbb{Z}[\gamma] \cap [0,1))^2$, but $\pi\big(1 - 1/(3\gamma), 1 - 2/(3\gamma)\big) = \infty$.*

Proof. By Proposition 3.5, it suffices to show that all points in $z \in (\frac{1}{2}\mathbb{Z}[\gamma])^2 \cap \mathcal{D}$ with $\|V(z)'\|_\infty \le 2$ are periodic. Since $V(\mathcal{D}) = \{(x,y) : x > 0, y > 0, x + y \le 1/\gamma\}$, we only have to take into account points $V(z)$ with coordinates in $\{1/(2\gamma^2), 1/(2\gamma), 1/2\}$ (cf. Section 2). In particular, we obtain immediately that the conjecture holds. Since $V\big(\frac{1}{2\gamma^2}, \frac{1}{2\gamma^2}\big)$ is in $U(D_\alpha)$, and $V\big(\frac{1}{2\gamma}, \frac{1}{2\gamma}\big)$, $V\big(\frac{1}{2\gamma}, \frac{1}{2\gamma^2}\big)$, $V\big(\frac{1}{2\gamma}, \frac{1}{2\gamma^2}\big)$ are in $U(D_\beta)$, we obtain $\pi(z) < \infty$ for all $z \in (\frac{1}{2}\mathbb{Z}[\gamma] \cap [0,1))^2$ as well.

Now let $V(z) = \big(1/(3\gamma), 2/(3\gamma)\big)$. Then

$$\begin{aligned} VS(z) &= \gamma^2\big(V(z)A^5 + (0, -1/\gamma^2)\big) = \big(\gamma/3, 1/(3\gamma^3)\big) \\ VS^2(z) &= \gamma^2\big(VS(z)A^{-5} + (-1/\gamma^2, 0)\big) = \big(2/(3\gamma), 1/(3\gamma)\big) \\ VS^3(z) &= \gamma^2\big(VS^2(z)A^{-5} + (0, -1/\gamma^2)\big) = \big(1/(3\gamma^3), \gamma/3)\big) \\ VS^4(z) &= \gamma^2\big(VS^3(z)A^5 + (0, -1/\gamma^2)\big) = \big(1/(3\gamma), 2/(3\gamma)\big) = V(z), \end{aligned}$$

hence $S^n(z) \in \mathcal{D} \setminus \mathcal{P}$ for all $n \ge 0$ and $\pi(z) = \infty$ by Theorem 4.1. $\square$

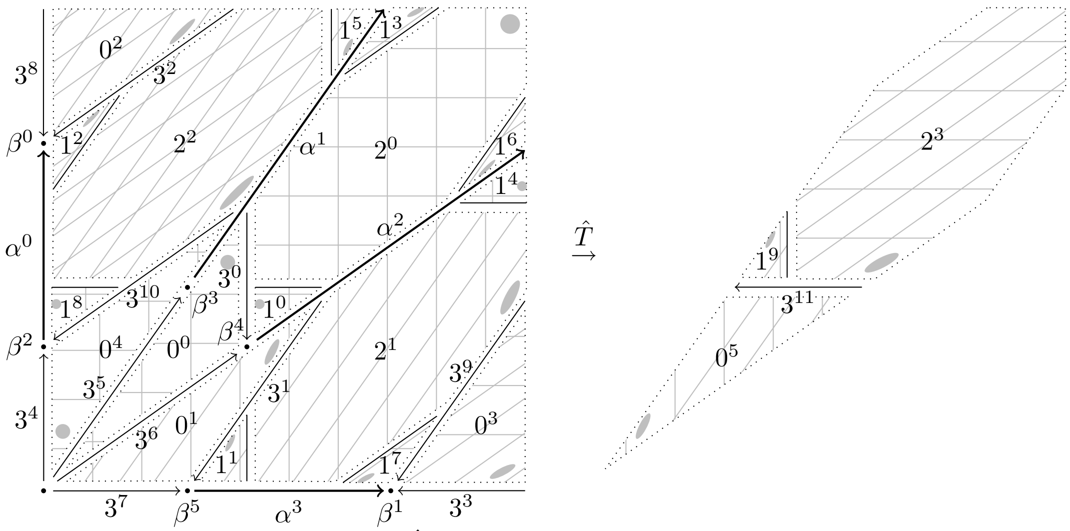

FIGURE 5.1. The map $\hat{T}$ and the set $\mathcal{R}$, $\lambda = \sqrt{2}$. (ℓ^k stands for $T^k(D_\ell)$.)

5. THE CASE $\lambda = \sqrt{2} = -2\cos\frac{3\pi}{4}$

Let $\lambda = \sqrt{2}$, i.e., $\lambda' = -\sqrt{2}$ and set

$$\mathcal{D} = \{(x,y) \in [0,1)^2 : \sqrt{2}-2 < x - \sqrt{2}y < 0,\ 0 < \sqrt{2}x - y < \sqrt{2}-2\} = \bigcup\nolimits_{\ell \in \mathcal{A} = \{0,1,2,3\}} D_\ell,$$

$$D_0 = \{(x,y) \in \mathcal{D} : x < \sqrt{2}-1\}, \quad D_1 = \{(x,y) \in \mathcal{D} : x > \sqrt{2}-1, y \le \sqrt{2}-1\},$$

$$D_2 = \{(x,y) \in \mathcal{D} : x > \sqrt{2}-1, y > \sqrt{2}-1\}, \quad D_3 = \{(x,y) \in \mathcal{D} : x = \sqrt{2}-1\}.$$

Figure 5.1 shows that $\hat{T}(z) = T^{\tau(\ell)}(z)$ if $z \in D_\ell$, with $\tau(0) = 5$, $\tau(1) = 9$, $\tau(2) = 3$, $\tau(3) = 11$, and $\mathcal{R} = \{(0,0)\} \cup \bigcup_{k=0}^{3} T^k(D_\alpha) \cup \bigcup_{k=0}^{5} T^k(D_\beta)$ with $D_\alpha = \{(0,y) : 1 - 1/\sqrt{2} < y < 1/\sqrt{2}\}$, $D_\beta = \{(0, 1/\sqrt{2})\}$. If we set $U(z) = (\sqrt{2}-1)z$, i.e., $V(z) = z$, $\kappa = \sqrt{2}-1$, $\varepsilon = -1$ and

$$\sigma :\ 0 \mapsto 010 \qquad 1 \mapsto 000 \qquad 2 \mapsto 0 \qquad 3 \mapsto 030,$$

then Figure 5.2 shows that σ satisfies the conditions in Section 3, and

$$\mathcal{P} = \{(x,y) \in \mathcal{D} : x, y \ge \sqrt{2}-1\} = D_2 \cup D_\delta \cup T(D_\zeta) \cup D_\eta$$

with $D_\zeta = \{(x, \sqrt{2}-1) : \sqrt{2}-1 < x < 2-\sqrt{2}\}$ and $D_\eta = \{(\sqrt{2}-1, \sqrt{2}-1)\}$. All points in $\mathcal{P}$ are periodic and $|\sigma^n(\ell)| \to \infty$ as $n \to \infty$ for all $\ell \in \mathcal{A}$. Therefore, all conditions of Proposition 3.3 and Theorem 3.4 are satisfied, and we obtain the following theorem.

Theorem 5.1. *If $\lambda = \sqrt{2}$, then the minimal period length $\pi(z)$ is*

1	*if $z = (0,0)$*
4	*if $z = T^m(0,1/2)$, $0 \le m \le 3$*
8	*for the other points of $T^m(D_\alpha)$, $0 \le m \le 3$*
6	*if $z = T^m(0,1/\sqrt{2})$, $0 \le m \le 5$*
$2 \cdot 3^n + (-1)^n$	*if $S^nR(z) = (1/\sqrt{2}, 1/\sqrt{2})$*
$8(2 \cdot 3^n + (-1)^n)$	*for the other points with $S^nR(z) \in D_2$*
$4(3^{n+1} + 1 + (-1)^n)$	*if $S^nR(z) \in \{(1/2, \sqrt{2}-1), (\sqrt{2}-1, 1/2)\}$*
$8(3^{n+1} + 1 + (-1)^n)$	*for the other points with $S^nR(z) \in D_\zeta \cup \hat{T}(D_\zeta)$*
$2 \cdot 3^{n+1} + 4 + (-1)^n$	*if $S^nR(z) = (\sqrt{2}-1, \sqrt{2}-1)$*
∞	*if $S^nR(z) \in \mathcal{D} \setminus \mathcal{P}$ for all $n \ge 0$*

Proof. We easily calculate

$$\begin{pmatrix} |\sigma^n(0)|_0 \\ |\sigma^n(0)|_1 \end{pmatrix} = 3^n \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} + (-1)^n \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix}, \qquad \begin{pmatrix} |\sigma^n(1)|_0 \\ |\sigma^n(1)|_1 \end{pmatrix} = 3^n \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} + (-1)^n \begin{pmatrix} -3/4 \\ 3/4 \end{pmatrix}$$

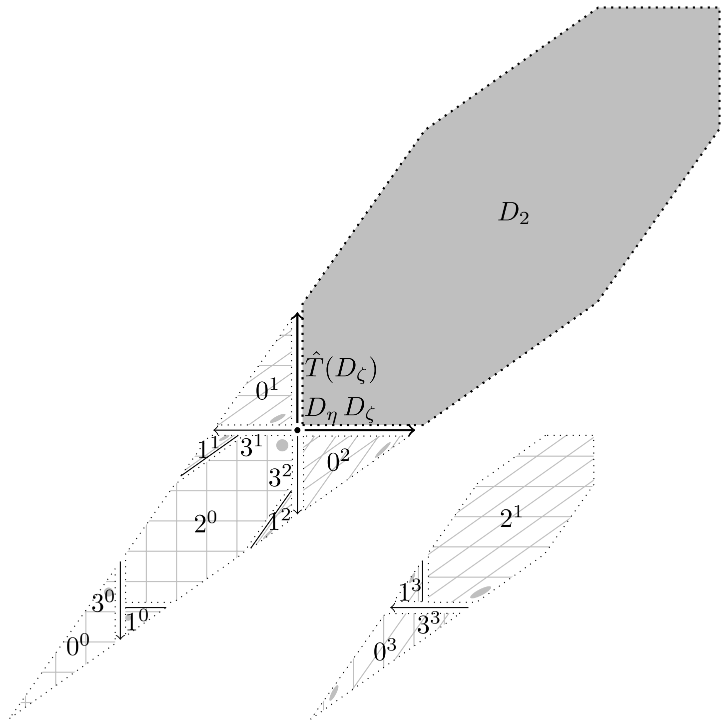

FIGURE 5.2. The trajectory of the scaled domains and $\mathcal{P}$, $\lambda = \sqrt{2}$. (ℓ^k stands for $\hat{T}^{-k}U(D_\ell)$.)

and obtain $\tau(\sigma^n(0)) = 2 \cdot 3^{n+1} - (-1)^n$, $\tau(\sigma^n(3)) = \tau(\sigma^{n-1}(030)) = 2 \cdot 3^{n+1} + 4 + (-1)^n$. If $S^n R(z) \in D_2$ and $n \geq 1$, then $\pi(z) = \tau(\sigma^n(2)) = \tau(\sigma^{n-1}(0))$ and $\pi(z) = 8\tau(\sigma^{n-1}(0))$ respectively; if $S^n R(z) \in D_\delta$, then $\pi(z) = \tau(\sigma^n(13)) = \tau(\sigma^{n-1}(000030))$ and $\pi(z) = 2\tau(\sigma^{n-1}(000030))$ respectively; if $S^n R(z) = (\sqrt{2}-1, \sqrt{2}-1)$, then $\pi(z) = \tau(\sigma^n(3))$. The given $\pi(z)$ hold for $n = 0$ as well. $\square$

By Figure 5.2, we can choose $\hat{s}(z)$, $s(z)$ as follows and obtain the following $t(z)$:

$$z \in \hat{T}^{-2}U(D_0 \cup D_1 \cup D_3) : \hat{s}(z) = -1, s(z) = -5,\ t(z) = \hat{T}^{-1}(z) - zA^{-5} = (\sqrt{2}-1, 2-\sqrt{2})$$
$$z \in U(\mathcal{D}) : \hat{s}(z) = 0, s(z) = 0,\ t(z) = (0,0)$$
$$z \in \hat{T}^{-1}U(D_0 \cup D_1 \cup D_3) : \hat{s}(z) = 1, s(z) = 5,\ t(z) = \hat{T}(z) - zA^5 = (2-\sqrt{2}, \sqrt{2}-1)$$

Since $A^4 = -A^0$, we obtain

$$\{t(z)A^h : z \in \mathcal{D}\backslash\mathcal{P}, h \in \mathbb{Z}\} = \{(0,0),\ \pm(2-\sqrt{2}, \sqrt{2}-1), \pm(\sqrt{2}-1, 0), \pm(0, 1-\sqrt{2}), \pm(1-\sqrt{2}, \sqrt{2}-2)\}$$

and $\delta = (2+\sqrt{2})/\sqrt{2} = \sqrt{2}+1$.

Theorem 5.2. *If $\lambda = \sqrt{2}$, then $\pi(z) < \infty$ for all $z \in (\frac{1}{2}\mathbb{Z}[\sqrt{2}] \cap [0,1))^2$ and $z \in (\frac{1}{3}\mathbb{Z}[\sqrt{2}] \cap [0,1))^2$, but $(T^k(\frac{3-\sqrt{2}}{4}, \frac{2\sqrt{2}-1}{4}))_{k\in\mathbb{Z}}$ is aperiodic.*

Proof. Consider first $z \in (\frac{1}{2}\mathbb{Z}[\sqrt{2}])^2 \cap \mathcal{D} \setminus \mathcal{P}$ with $\|z'\|_\infty \leq \sqrt{2}+1$. The coordinates of z satisfy $x, y \in (0, 2-\sqrt{2}) \cap \frac{1}{2}\mathbb{Z}[\sqrt{2}]$, $|x'|, |y'| \leq \sqrt{2}+1$, hence $x, y \in \{(\sqrt{2}-1)/2, (2-\sqrt{2})/2, \sqrt{2}-1, 1/2\}$ and $x < \sqrt{2}y$, $y < \sqrt{2}x$. Therefore it is easy to see that $z \in \mathcal{P}$ or $S(z) = (\sqrt{2}+1)z \in \mathcal{P}$ for all these points. In particular, the conjecture follows.

If $z \in (\frac{1}{3}\mathbb{Z}[\sqrt{2}])^2 \cap \mathcal{D} \setminus \mathcal{P}$ and $\|z'\|_\infty \leq \sqrt{2}+1$, then

$$z \in \Big\{1 - \frac{2\sqrt{2}}{3}, \frac{\sqrt{2}-1}{3}, \frac{2-\sqrt{2}}{3}, \frac{2\sqrt{2}-2}{3}, \frac{1}{3}, \frac{4-2\sqrt{2}}{3}, \sqrt{2}-1, \frac{\sqrt{2}}{3}, 1 - \frac{\sqrt{2}}{3}\Big\}^2.$$

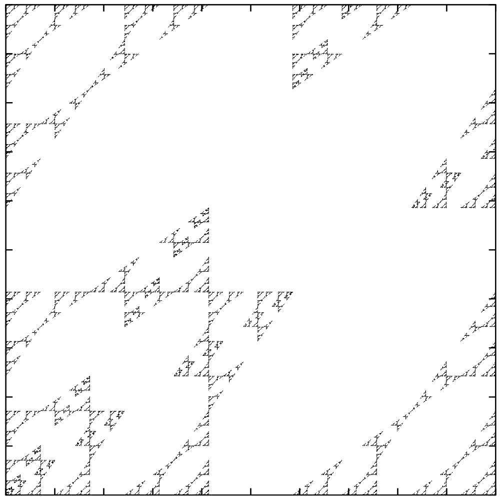

FIGURE 5.3. Aperiodic points, $\lambda = \sqrt{2}$.

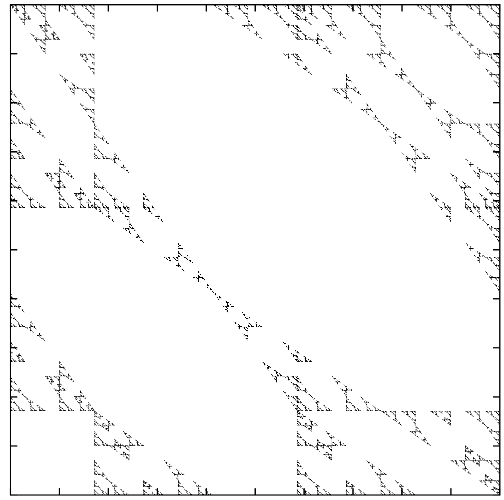

FIGURE 5.4. Aperiodic points, $\lambda = -\sqrt{2}$.

Since all points $(x,x) \in [0,1)^2$ are periodic, $S\big((2\sqrt{2}-2)/3, 1/3\big) = \big(2/3, (\sqrt{2}+1)/3\big) \in \mathcal{P}$, $S\big(1/3, (4-2\sqrt{2})/3\big) = \big((\sqrt{2}+1)/3, 2\sqrt{2}/3\big) \in \mathcal{P}$, $S\big(1/3, \sqrt{2}-1\big) = \big(\sqrt{2}-1, (4-2\sqrt{2})/3\big)$, $S\big((4-2\sqrt{2})/3, \sqrt{2}-1\big) = \big(\sqrt{2}-1, 1-\sqrt{2}/3\big) \in \mathcal{P}$, and $\pi(x,y) < \infty$ if and only if $\pi(y,x) < \infty$, all these points are periodic.

For $z = (\frac{3-\sqrt{2}}{4}, \frac{2\sqrt{2}-1}{4})$, we have

$$S(z) = \big(zA^5 + (2-\sqrt{2}, \sqrt{2}-1)\big)/\kappa = (\sqrt{2}+1)\big(\frac{9-6\sqrt{2}}{4}, \sqrt{2}-\frac{5}{4}\big) = \big(\frac{3\sqrt{2}-3}{4}, \frac{3-\sqrt{2}}{4}\big),$$

$$S^2(z) = \big(S(z)A^5 + (2-\sqrt{2}, \sqrt{2}-1)\big)/\kappa = (\sqrt{2}+1)\big(\frac{5-3\sqrt{2}}{4}, \sqrt{2}-\frac{5}{4}\big) = \big(\frac{2\sqrt{2}-1}{4}, \frac{3-\sqrt{2}}{4}\big),$$

$S^3(z) = \big(S^2(z)A^{-5} + (\sqrt{2}-1, 2-\sqrt{2})\big)/\kappa = (\frac{3-\sqrt{2}}{4}, \frac{3\sqrt{2}-3}{4})$ and $S^4(z) = (\frac{3-\sqrt{2}}{4}, \frac{2\sqrt{2}-1}{4}) = z$. □

6. THE CASE $\lambda = -\sqrt{2} = -2\cos\frac{\pi}{4}$

Let $\lambda = -\sqrt{2}$, i.e., $\lambda' = \sqrt{2}$, and set

$$\mathcal{D} = \{(x,y) \in [0,1)^2 : \sqrt{2}x + y > 2 \text{ or } x + \sqrt{2}y > 2\} = \bigcup\nolimits_{\ell \in \mathcal{A} = \{0,1,2\}} D_\ell,$$

with $D_0 = \{(x,y) \in \mathcal{D} : x + \sqrt{2}y > 2\}$, $D_1 = \{(x,y) \in \mathcal{D} : x + \sqrt{2}y < 2\}$, $D_2 = \{(x,y) \in \mathcal{D} : x + \sqrt{2}y = 2\}$. Figure 6.1 shows that $\hat{T}(z) = T^{\tau(\ell)}(z)$ if $z \in D_\ell$, with $\tau(0) = 1$, $\tau(1) = 21$, $\tau(2) = 31$, and

$$\mathcal{R} = \{(0,0)\} \cup \bigcup\nolimits_{k=0}^{3} T^k(D_\alpha) \cup D_\beta \cup D_\zeta \cup \bigcup\nolimits_{k=0}^{9} T^k(D_\eta)$$

with $D_\alpha = \{(x,y) : 0 \le x, y \le 3-2\sqrt{2}\} \setminus \{(0,0), (3-2\sqrt{2}, 3-2\sqrt{2})\}$, $D_\beta = \{z \in [0,1)^2 : T^{k+1}(z) = T^k(z)A + (0,1) \text{ for all } k \in \mathbb{Z}\}$, $D_\zeta = \{z \in [0,1)^2 : T^{k+1}(z) = T^k(z)A + (0,2) \text{ for all } k \in \mathbb{Z}\}$, $D_\eta = \{(1/\sqrt{2}, 0)\}$. If we set $U(z) = (\sqrt{2}-1)z + (2-\sqrt{2}, 2-\sqrt{2}) = (1,1) - (\sqrt{2}-1)\big((1,1)-z\big)$, i.e., $V(z) = (1,1) - z$, $\kappa = \sqrt{2}-1$, $\varepsilon = -1$ and

$$\sigma: \ 0 \mapsto 010 \qquad 1 \mapsto 000 \qquad 2 \mapsto 020,$$

then Figure 6.2 shows that σ satisfies the conditions in Section 3, and

$$\mathcal{P} = D_\vartheta \cup \bigcup\nolimits_{k=0}^{5} \hat{T}^k(D_\mu) \cup \bigcup\nolimits_{k=0}^{2} \hat{T}^k(D_\nu)$$

with $D_\vartheta = \{z \in [0,1)^2 : T^{k+1}(z) = T^k(z)A + (0,3) \text{ for all } k \in \mathbb{Z}\}$, $D_\mu = \{(x, 5-3\sqrt{2}) : 8-5\sqrt{2} \le x < 1\}$ and $D_\nu = \{(8-5\sqrt{2}, 8-5\sqrt{2})\}$. All points in $\mathcal{P}$ are periodic and $|\sigma^n(\ell)| \to \infty$ as $n \to \infty$

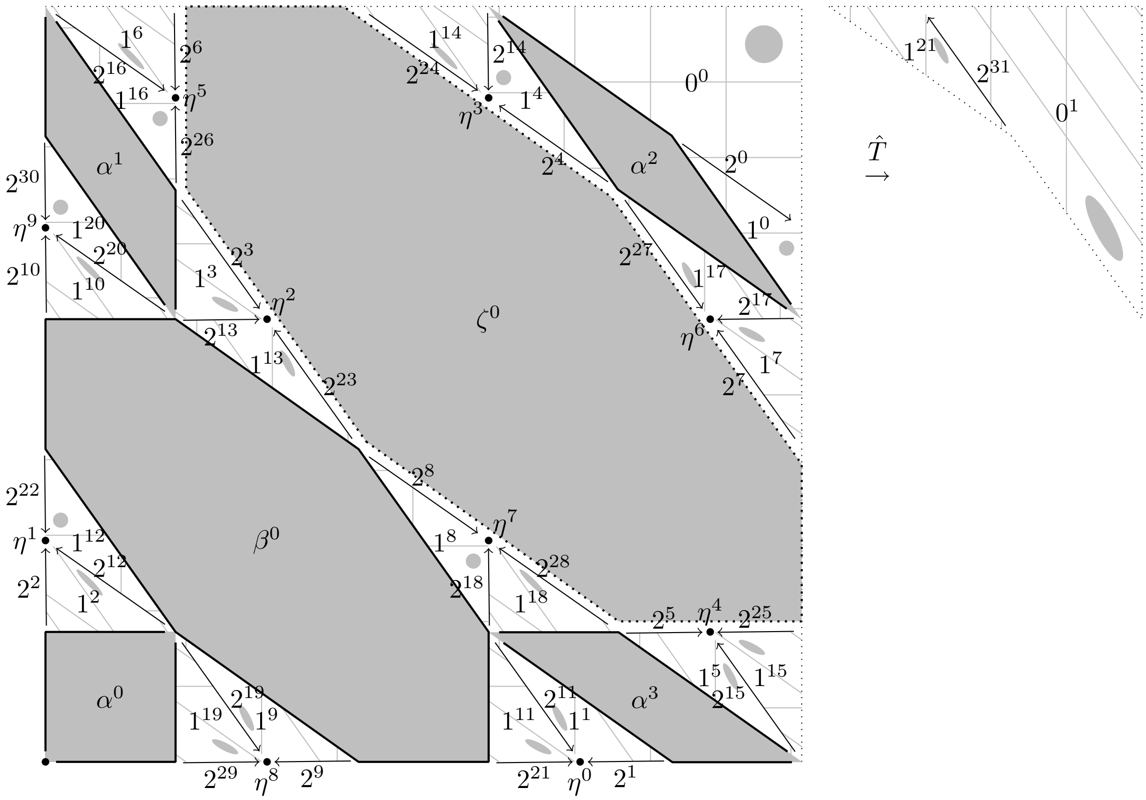

FIGURE 6.1. The map $\hat{T}$ and the set $\mathcal{R}$, $\lambda = -\sqrt{2}$. (ℓ^k stands for $T^k(D_\ell)$.)

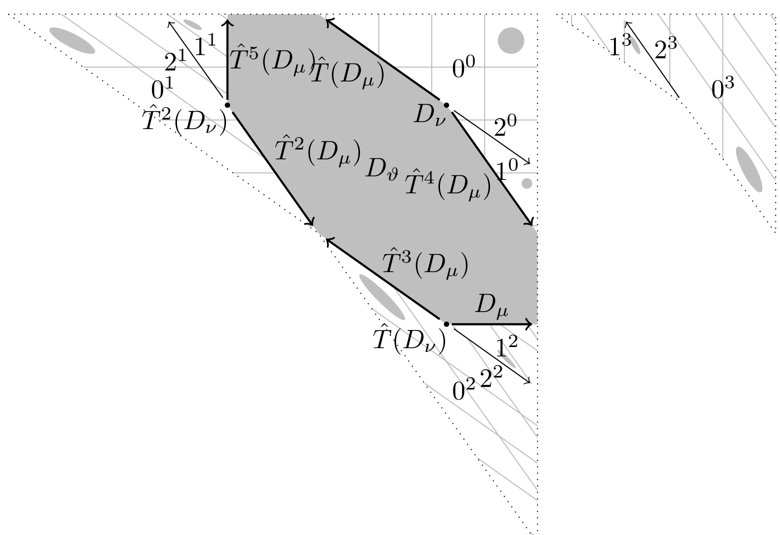

FIGURE 6.2. The trajectory of the scaled domains and $\mathcal{P}$, $\lambda = -\sqrt{2}$. (ℓ^k stands for $\hat{T}^{-k}U(D_\ell)$.)

for all $\ell \in \mathcal{A}$. Therefore, all conditions of Proposition 3.3 and Theorem 3.4 are satisfied, and we obtain the following theorem.

Theorem 6.1. *If $\lambda = -\sqrt{2}$, then the minimal period length $\pi(z)$ is*

1	*if* $z \in \{(0,0), (1/\sqrt{2}, 1/\sqrt{2}), (2-\sqrt{2}, 2-\sqrt{2})\}$
4	*if* $z = T^m(3/2-\sqrt{2}, 3/2-\sqrt{2})$ *for some* $m \in \{0,1,2,3\}$
10	*if* $z = T^m(1/\sqrt{2}, 0)$ *for some* $m \in \{0,1,\ldots,9\}$
8	*for the other points in* $\mathcal{R}$
$2\cdot 3^{n+1} - 5(-1)^n$	*if* $S^nR(z) = (3-3/\sqrt{2}, 3-3/\sqrt{2})$ *for some* $n \ge 0$
$8(2\cdot 3^{n+1} - 5(-1)^n)$	*for the other points with* $S^nR(z) \in D_\vartheta$
$4(3^{n+2} + 5 - 5(-1)^n)$	*if* $S^nR(z) = \hat{T}^m\big((9-5\sqrt{2})/2, 5-3\sqrt{2}\big)$ *for some* $m \in \{0,\ldots,5\}$, $n \ge 0$
$8(3^{n+2} + 5 - 5(-1)^n)$	*for the other points with* $S^nR(z) \in \hat{T}^m(D_\mu)$
$2\cdot 3^{n+2} + 20 - 5(-1)^n$	*if* $S^nR(z) = \hat{T}^m(8-5\sqrt{2}, 8-5\sqrt{2})$ *for some* $m \in \{0,1,2\}$, $n \ge 0$
∞	*if* $S^nR(z) \in \mathcal{D}\setminus\mathcal{P}$ *for all* $n \ge 0$.

Proof. As for $\lambda = \sqrt{2}$, we have

$$\begin{pmatrix}|\sigma^n(0)|_0\\|\sigma^n(0)|_1\end{pmatrix} = 3^n\begin{pmatrix}3/4\\1/4\end{pmatrix} + (-1)^n\begin{pmatrix}1/4\\-1/4\end{pmatrix}, \qquad \begin{pmatrix}|\sigma^n(1)|_0\\|\sigma^n(1)|_1\end{pmatrix} = 3^n\begin{pmatrix}3/4\\1/4\end{pmatrix} + (-1)^n\begin{pmatrix}-3/4\\3/4\end{pmatrix},$$

hence $\tau(\sigma^n(0)) = 2\cdot 3^{n+1} - 5(-1)^n$ and $\tau(\sigma^n(2)) = \tau(\sigma^{n-1}(020)) = 2\cdot 3^{n+1} + 20 + 5(-1)^n$. For $S^nR(z) \in D_\vartheta$, we have $\pi(z) = \tau(\sigma^n(0))$ and $\pi(z) = 8\tau(\sigma^n(0))$ respectively; if $S^nR(z) \in T^m(D_\mu)$, then $\pi(z) = \tau(\sigma^n(002000))$ and $\pi(z) = 2\tau(\sigma^n(002000))$ respectively; if $S^nR(z) = \hat{T}^m(D_\nu)$, then $\pi(z) = \tau(\sigma^n(020))$. □

By Figure 6.2, we can choose $\hat{s}(z)$, $s(z)$ as follows and obtain the following $t(z)$:

$$z \in \hat{T}^{-2}U(D_0\cup D_1\cup D_2) : \hat{s}(z) = -1, s(z) = -1,\ t(z) = V(\hat{T}^{-1}(z)) - V(z)A^{-1} = (\sqrt{2}-1, 0)$$
$$z \in U(\mathcal{D}) : \hat{s}(z) = 0, s(z) = 0,\ t(z) = (0,0)$$
$$z \in \hat{T}^{-1}U(D_0\cup D_1\cup D_2) : \hat{s}(z) = 1, s(z) = 1,\ t(z) = V(\hat{T}(z)) - V(z)A = (0, \sqrt{2}-1)$$

Since $A^4 = -A^0$, we obtain

$$\{(\sqrt{2}-1)t(z)A^h : z \in \mathcal{D}\setminus\mathcal{P}, h \in \mathbb{Z}\} = \{(0,0),\ \pm(1,0), \pm(0,-1), \pm(-1,\sqrt{2}), \pm(\sqrt{2},-1)\}$$

and $\delta = (\sqrt{2}+1)\sqrt{2}/\sqrt{2} = \sqrt{2}+1$.

Theorem 6.2. *If $\lambda = -\sqrt{2}$, then $\pi(z) < \infty$ for all $z \in (\frac{1}{2}\mathbb{Z}[\sqrt{2}]\cap[0,1))^2$ and $z \in (\frac{1}{3}\mathbb{Z}[\sqrt{2}]\cap[0,1))^2$, but $(T^k(\frac{3}{4}, \frac{5-\sqrt{2}}{4}))_{k\in\mathbb{Z}}$ is aperiodic.*

Proof. Note that $z \in \mathcal{D}$ means that $(\sqrt{2}+1)V(z) \in \{(x,y) \in [0,1)^2 : x+\sqrt{2}y < 1 \text{ or } \sqrt{2}x+y < 1\}$. If $z \in (\frac{1}{2}\mathbb{Z}[\sqrt{2}])^2 \cap \mathcal{D}\setminus\mathcal{P}$ and $\|(V(z))'\|_\infty \le \sqrt{2}+1$, we have thus $\|((\sqrt{2}+1)V(z))'\|_\infty \le 1$, hence $(\sqrt{2}+1)V(z) \in \{1/2, 1/\sqrt{2}\}^2$, which is impossible for $z \in \mathcal{D}\setminus\mathcal{P}$.

If $z \in (\frac{1}{3}\mathbb{Z}[\sqrt{2}])^2 \cap \mathcal{D}\setminus\mathcal{P}$ and $\|((\sqrt{2}+1)V(z))'\|_\infty \le 1$, then we have $(\sqrt{2}+1)V(z) \in \{(\sqrt{2}-1)/3, 1/3, 2/3, (1+\sqrt{2})/3\}^2$. Since $(\sqrt{2}+1)V\big(1/3, (\sqrt{2}-1)/3\big) \in (\sqrt{2}+1)V(\mathcal{P})$ and $(\sqrt{2}+1)VS\big(2/3, (\sqrt{2}-1)/3\big) = (1/3, 1/3) \in (\sqrt{2}+1)V(\mathcal{P})$, all these points are periodic.

For $z = (\frac{3}{4}, \frac{5-\sqrt{2}}{4})$, we have $(\sqrt{2}+1)V(z) = (\frac{\sqrt{2}+1}{4}, \frac{1}{4})$ and

$$(\sqrt{2}+1)VS(z) = \big((\sqrt{2}+1)V(z)A + (0,1)\big)/\kappa = (\sqrt{2}+1)\big(\frac{1}{4}, \frac{3-2\sqrt{2}}{4}\big) = \big(\frac{\sqrt{2}+1}{4}, \frac{\sqrt{2}-1}{4}\big),$$
$$(\sqrt{2}+1)VS^2(z) = \big((\sqrt{2}+1)VS(z)A + (0,1)\big)/\kappa = (\sqrt{2}+1)\big(\frac{1-\sqrt{2}}{4}, \frac{1}{4}\big) = \big(\frac{1}{4}, \frac{\sqrt{2}+1}{4}\big),$$

$(\sqrt{2}+1)VS^3(z) = (\frac{\sqrt{2}-1}{4}, \frac{\sqrt{2}+1}{4})$ and $(\sqrt{2}+1)VS^4(z) = (\frac{\sqrt{2}+1}{4}, \frac{1}{4}) = (\sqrt{2}+1)V(z)$. □

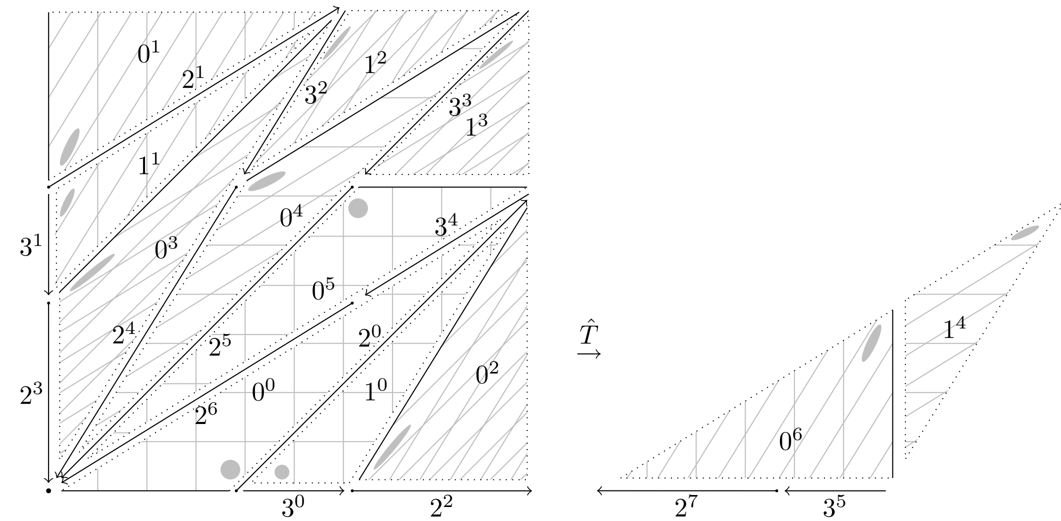

FIGURE 7.1. The map $\hat{T}$, $\lambda = 1/\gamma$. (ℓ^k stands for $T^k(D_\ell)$.)

7. THE CASE $\lambda = 1/\gamma = -2\cos\frac{3\pi}{5}$

Let $\lambda = 1/\gamma$, i.e., $\lambda' = -\gamma$, and set

$$\mathcal{D} = \{(x,y) \in [0,1)^2 : \gamma x - 1 < y < x/\gamma\} = \bigcup\nolimits_{\ell \in \mathcal{A} = \{0,1,2,3\}} D_\ell,$$

with $D_0 = \{(x,y) \in \mathcal{D} : y > x - 1/\gamma^2\}$, $D_1 = \{(x,y) \in \mathcal{D} : 0 < y < x - 1/\gamma^2\}$, $D_2 = \{(x,y) \in \mathcal{D} : y = x - 1/\gamma^2\}$, $D_3 = \{(x,0) : 1/\gamma^2 < x < 1/\gamma\}$. Figure 7.1 shows that $\hat{T}(z) = T^{\tau(\ell)}(z)$ if $z \in D_\ell$, with $\tau(0) = 6$, $\tau(1) = 4$, $\tau(2) = 7$, $\tau(3) = 5$, and $\mathcal{R} = \{(0,0)\}$. If we set $U(z) = z/\gamma^2$, i.e., $V(z) = z$, $\kappa = 1/\gamma^2$, $\varepsilon = 1$, and

$$\sigma:\ 0 \mapsto 010 \qquad 1 \mapsto 01110 \qquad 2 \mapsto 012 \qquad 3 \mapsto 01112,$$

then Figure 7.2 shows that σ satisfies the conditions in Section 3, and

$$\mathcal{P} = D_\alpha \cup D_\beta \cup D_3 \cup \bigcup\nolimits_{k=0}^{3} \hat{T}^k(D_\zeta) \cup D_\eta \cup \bigcup\nolimits_{k=0}^{1} \hat{T}^k(D_\vartheta)$$

with $D_\alpha = \{z \in \mathcal{D} : \hat{T}^k(z) \in D_0 \text{ for all } k \in \mathbb{Z}\}$, $D_\beta = \{z \in \mathcal{D} : \hat{T}^k(z) \in D_1 \text{ for all } k \in \mathbb{Z}\}$, $D_\zeta = \{(x,0) : 1/\gamma^3 < x < 1/\gamma^2\}$, $D_\eta = \{(1/\gamma^2, 0)\}$ and $D_\vartheta = \{(1/\gamma^3, 0)\}$. All points in $\mathcal{P}$ are periodic and $|\sigma^n(\ell)| \to \infty$ as $n \to \infty$ for all $\ell \in \mathcal{A}$. Therefore, all conditions of Proposition 3.3 and Theorem 3.4 are satisfied, and we obtain the following theorem.

Theorem 7.1. *If $\lambda = 1/\gamma$, then the minimal period length $\pi(z)$ is*

1	*if* $z = (0,0)$
$2(5\cdot 4^n + 4)/3$	*if* $S^n R(z) = \big(\frac{\gamma}{\gamma^2+1}, \frac{1/\gamma}{\gamma^2+1}\big)$ *for some* $n \ge 0$
$10(5\cdot 4^n + 4)/3$	*for the other points with* $S^n R(z) \in D_\alpha$
$4(5\cdot 4^n - 2)/3$	*if* $S^n R(z) = \big(\frac{\gamma^2}{\gamma^2+1}, \frac{1}{\gamma^2+1}\big)$ *for some* $n \ge 0$
$20(5\cdot 4^n - 2)/3$	*for the other points with* $S^n R(z) \in D_\beta$
$5(4^{n+1} - 1)/3$	*if* $S^n R(z) = (0, 1/2)$ *for some* $n \ge 0$
$10(4^{n+1} - 1)/3$	*for the other points with* $S^n R(z) \in D_3$
$5(2\cdot 4^{n+1} + 7)/3$	*if* $S^n R(z) = \hat{T}^m(1/(2\gamma), 0)$ *for some* $m \in \{0,1,2,3\}$ *and* $n \ge 0$
$10(2\cdot 4^{n+1} + 7)/3$	*for the other points with* $S^n R(z) \in \hat{T}^m(D_\zeta)$
$(10\cdot 4^n + 11)/3$	*if* $S^n R(z) = (1/\gamma^2, 0)$ *for some* $n \ge 0$
$(5\cdot 4^{n+1} + 19)/3$	*if* $S^n R(z) = \hat{T}^m(1/\gamma^3, 0)$ *for some* $m \in \{0,1\}$ *and* $n \ge 0$
∞	*if* $S^n R(z) \in \mathcal{D} \setminus \mathcal{P}$ *for all* $n \ge 0$.

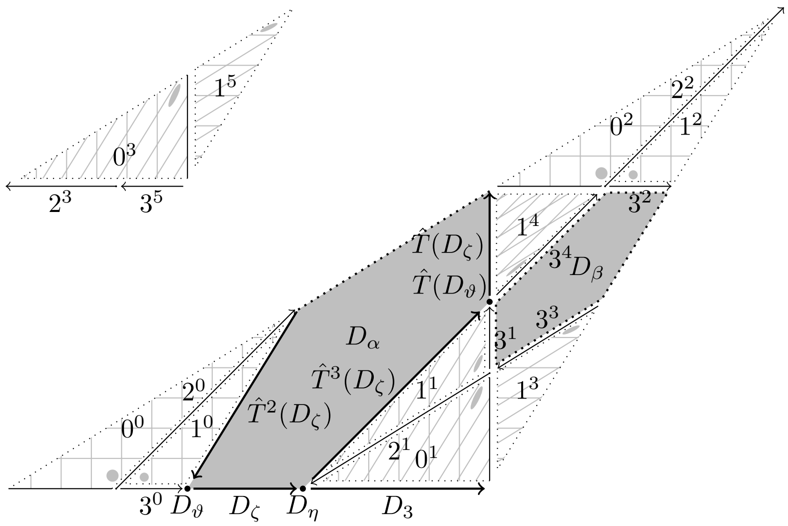

FIGURE 7.2. The trajectory of the scaled domains and $\mathcal{P}$, $\lambda = 1/\gamma$. (ℓ^k stands for $\hat{T}^k U(D_\ell)$.)

Proof. As for $\lambda = -1/\gamma$, we have

$$\begin{pmatrix} |\sigma^n(0)|_0 \\ |\sigma^n(0)|_1 \end{pmatrix} = 4^n \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}, \qquad \begin{pmatrix} |\sigma^n(1)|_0 \\ |\sigma^n(1)|_1 \end{pmatrix} = 4^n \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} + \begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix},$$

hence $\tau(\sigma^n(0)) = \frac{10}{3}4^n + \frac{8}{3}$, $\tau(\sigma^n(1)) = \frac{20}{3}4^n - \frac{8}{3}$, $\tau(\sigma^n(2)) = \frac{10}{3}4^n + \frac{11}{3}$, $\tau(\sigma^n(3)) = \frac{20}{3}4^n - \frac{5}{3}$. For $S^n R(z) \in D_\alpha$, we have $\pi(z) = \tau(\sigma^n(0))$ and $\pi(z) = 5\tau(\sigma^n(0))$ respectively; if $S^n R(z) \in D_\beta$, then $\pi(z) = \tau(\sigma^n(1))$ and $\pi(z) = 5\tau(\sigma^n(1))$ respectively; if $S^n R(z) \in D_3$, then $\pi(z) = \tau(\sigma^n(3))$ and $\pi(z) = 2\tau(\sigma^n(3))$ respectively; if $S^n R(z) \in D_\zeta$, then $\pi(z) = \tau(\sigma^n(0002))$ and $\pi(z) = 2\tau(\sigma^n(0002))$ respectively; if $S^n R(z) = (1/\gamma^2, 0)$ (D_η), then $\pi(z) = \tau(\sigma^n(2))$; if $S^n R(z) = \hat{T}^m(1/\gamma^3, 0)$ (D_ϑ), then $\pi(z) = \tau(\sigma^n(02))$. □

By Figure 7.2, we can choose $\hat{s}(z)$, $s(z)$ as follows and obtain the following $t(z)$:

$$\begin{aligned}
z \in \hat{T}^2 U(\mathcal{D}) &: \hat{s}(z) = -2, s(z) = -10,\ t(z) = \hat{T}^{-2}(z) - z = (-1/\gamma, -1/\gamma^2) \\
z \in \hat{T} U(D_1 \cup D_2 \cup D_3) &: \hat{s}(z) = -1, s(z) = -6,\ t(z) = \hat{T}^{-1}(z) + zA^{-1} = (1, 1/\gamma) \\
z \in U(\mathcal{D}) &: \hat{s}(z) = 0, s(z) = 0,\ t(z) = (0,0) \\
z \in \hat{T}^4 U(D_1) &: \hat{s}(z) = 1, s(z) = 6,\ t(z) = \hat{T}(z) + zA = (1/\gamma, 0) \\
z \in \hat{T}^4 U(D_3) &: \hat{s}(z) = 1, s(z) = 7,\ t(z) = \hat{T}(z) + zA^2 = (0, -1/\gamma) \\
z \in \hat{T} U(D_0) \cup \hat{T}^3 U(D_1) &: \hat{s}(z) = 2, s(z) = 10,\ t(z) = \hat{T}^2(z) - z = (-1/\gamma^2, 0) \\
z \in \hat{T}^3 U(D_3) &: \hat{s}(z) = 2, s(z) = 11,\ t(z) = \hat{T}^2(z) - zA = (0, 1/\gamma^2)
\end{aligned}$$

With

$$\begin{aligned}
\{t(z)A^h : z \in \mathcal{D} \setminus \mathcal{P}, h \in \mathbb{Z}\} = \{&(0,0),\ \pm(1,1/\gamma), \pm(1/\gamma,0), \pm(0,1/\gamma), \pm(1/\gamma,1), \pm(1,1), \\
&\pm(-1/\gamma,-1/\gamma^2), \pm(-1/\gamma^2,0), \pm(0,1/\gamma^2), \pm(1/\gamma^2,1/\gamma), \pm(1/\gamma,1/\gamma)\}
\end{aligned}$$

we obtain $\delta = \frac{\gamma^2}{\gamma^2-1} = \gamma$ as in Section 2.

Theorem 7.2. *$\pi(z)$ is finite for all $z \in (\frac{1}{2}\mathbb{Z}[\gamma] \cap [0,1))^2 \cup (\frac{1}{3}\mathbb{Z}[\gamma] \cap [0,1))^2$, but $\pi\big(1/4, 1/(4\gamma^3)\big) = \infty$.*

Proof. For $z \in (\frac{1}{2}\mathbb{Z}[\gamma] \cap [0,1))^2$, we have to show that all points $z \in \{0, 1/(2\gamma^2), 1/(2\gamma), 1/2, 1/\gamma, \gamma/2\}^2$ with $z \in \mathcal{D}$ are periodic, similarly to Section 2. All such points lie either in $\mathcal{P}$ or on a discontinuity line. Note that all points on the line $y = 0$ are periodic, hence all points on the discontinuity lines are periodic as well.

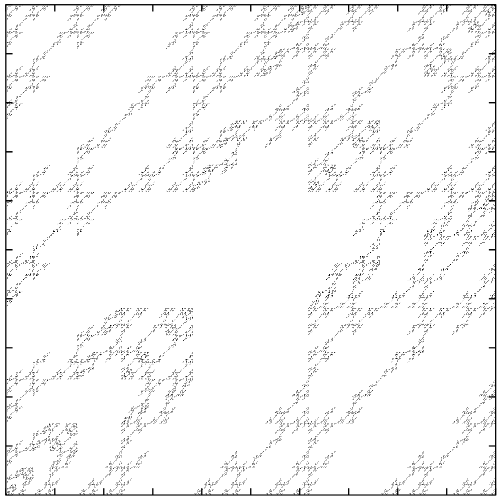

FIGURE 7.3. Aperiodic points, $\lambda = 1/\gamma$.

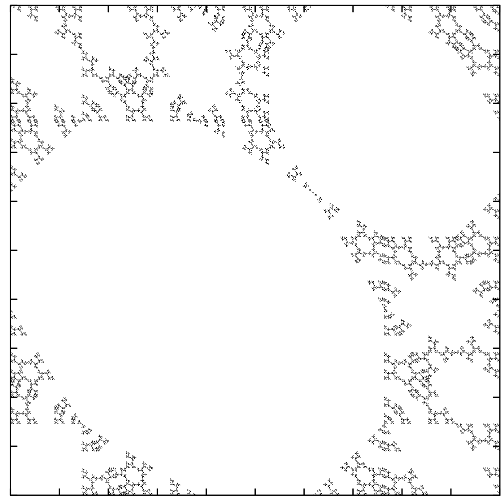

FIGURE 7.4. Aperiodic points, $\lambda = -\gamma$.

For $z \in (\frac{1}{3}\mathbb{Z}[\gamma] \cap [0,1))^2$, we have to consider

$$z \in \Big\{0, \frac{1}{3\gamma^3}, \frac{1}{3\gamma^2}, \frac{1}{3\gamma}, \frac{1}{3}, \frac{2}{3\gamma}, \frac{3-\gamma}{3}, \frac{\gamma}{3}, \frac{1}{\gamma}, \frac{2}{3}, \frac{2\gamma-1}{3}, \frac{4-\gamma}{3}, \frac{\gamma^2}{3}, \frac{3\gamma-2}{3}\Big\}^2.$$

All points on the discontinuity lines $y = 0$ and $x = 1/\gamma$ are periodic. For the other points $z \in \mathcal{D} \setminus \mathcal{P}$, $S(z)$ is given by $S(\frac{1}{3\gamma}, \frac{1}{3\gamma^3}) = (\frac{1}{3\gamma}, \frac{1}{3})$, $S(\frac{\gamma}{3}, \frac{1}{3\gamma^3}) = (\frac{2}{3\gamma}, \frac{1}{3\gamma})$, $S(\frac{\gamma}{3}, \frac{1}{3\gamma^2}) = (\frac{2}{3}, \frac{1}{3\gamma})$, $S(\frac{2}{3}, \frac{1}{3\gamma^2}) = (\frac{2\gamma-1}{3}, \frac{1}{3})$, $S(\frac{2}{3}, \frac{1}{3}) = (\frac{2\gamma-1}{3}, \frac{1}{3})$, $S(\frac{2\gamma-1}{3}, \frac{2}{3\gamma}) = (\frac{1}{3}, \frac{1}{3\gamma^3})$, $S(\frac{4-\gamma}{3}, \frac{2}{3\gamma}) = (\frac{3-\gamma}{3}, \frac{1}{3\gamma^3})$, $S(\frac{4-\gamma}{3}, \frac{3-\gamma}{3}) = (\frac{3-\gamma}{3}, \frac{1}{3\gamma})$ and $S(\frac{\gamma^2}{3}, \frac{3-\gamma}{3}) = (\frac{2}{3}, \frac{1}{3\gamma})$, hence $S(z) \in \mathcal{P}$.

If $z = \big(1/4, 1/(4\gamma^3)\big)$, then we have $S(z) = \big(\gamma^2/4, 1/(4\gamma)\big)$, $S^2(z) = \gamma^2\big(S(z) - (1/\gamma^2, 0)\big) = \big((3\gamma-2)/4, \gamma/4\big)$, and $S^3(z) = \gamma^2\big(S^2(z) - (1/\gamma, 1/\gamma^2)\big) = \big((1/4, 1/(4\gamma^3)\big) = z$. □

8. THE CASE $\lambda = -\gamma = -2\cos\frac{\pi}{5}$

Let $\lambda = -\gamma$, i.e., $\lambda' = 1/\gamma$, and set

$$\mathcal{D} = \{(x,y) \in [0,1)^2 : x < y, \gamma(1-x) + (1-y) \le 1/\gamma^3\} = D_0 \cup D_1$$

with $D_0 = \{(x,y) \in \mathcal{D} : 1 - x < 1/\gamma^5\}$, $D_1 = \{(x,y) \in \mathcal{D} : 1 - x \ge 1/\gamma^5\}$. Figure 8.1 shows that $\hat{T}(z) = T^{\tau(\ell)}(z)$ if $z \in D_\ell$, with $\tau(0) = 42$, $\tau(1) = 28$, and

$$\mathcal{R} = \{(0,0)\} \cup D_\alpha \cup D_\beta \cup \bigcup_{k=0}^{4} T^k(D_\zeta) \cup \bigcup_{k=0}^{1} T^k(D_\eta) \cup \bigcup_{k=0}^{24} T^k(D_\vartheta) \cup \bigcup_{k=0}^{10} T^k(D_\mu)$$

with $D_\alpha = \{z \in [0,1)^2 : T^{k+1}(z) = T^k(z)A + (0,1) \text{ for all } k \in \mathbb{Z}\}$, $D_\beta = \{z \in [0,1)^2 : T^{k+1}(z) = T^k(z)A + (0,2) \text{ for all } k \in \mathbb{Z}\}$, $D_\zeta = \{(x,y) \in [0,1)^2 : T^{2k+1}(z) = T^{2k}(z)A + (0,2), T^{2k}(z) = T^{2k-1}(z)A + (0,1) \text{ for all } k \in \mathbb{Z}\}$, $D_\eta = \{(x,y) : 0 \le x, y \le 1/\gamma^4\} \setminus \{(0,0), (1/\gamma^4, 1/\gamma^4)\}$, $D_\vartheta = \{(x,x) : 1 - 1/\gamma^5 < x < 1\}$, $D_\mu = \{(1 - 1/\gamma^5, 1 - 1/\gamma^5)\}$. If we set $U(z) = z/\gamma^2 + (1/\gamma, 1/\gamma) = (1,1) - \big((1,1) - z\big)/\gamma^2$, i.e., $V(z) = z$, $\kappa = 1/\gamma^2$, $\varepsilon = 1$, and

$$\sigma : \ 0 \mapsto 010 \qquad 1 \mapsto 01110,$$

then Figure 8.2 shows that σ satisfies the conditions in Section 3, and $\mathcal{P} = D_\rho \cup D_\nu$ with $D_\nu = \{z \in \mathcal{D} : T^k(z) \in D_0 \text{ for all } k \in \mathbb{Z}\}$, $D_\rho = \{z \in \mathcal{D} : T^k(z) \in D_1 \text{ for all } k \in \mathbb{Z}\}$. All points in $\mathcal{P}$ are periodic and $|\sigma^n(\ell)| \to \infty$ as $n \to \infty$ for all $\ell \in \mathcal{A}$. Therefore, all conditions of Proposition 3.3 and Theorem 3.4 are satisfied, and we obtain the following theorem.

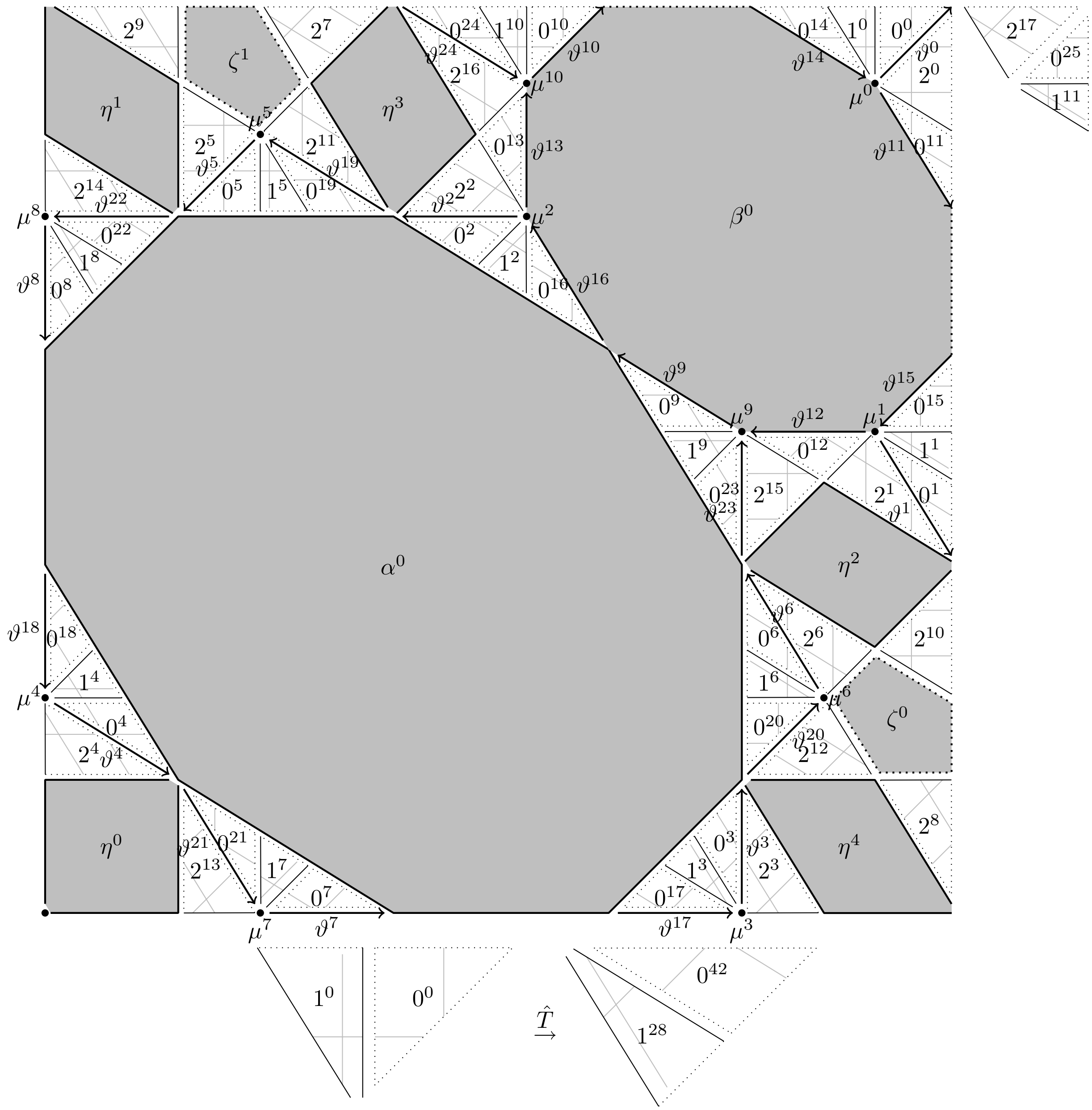

FIGURE 8.1. The map $\hat{T}$ and the set $\mathcal{R}$, $\lambda = -\gamma$. (ℓ^k stands for $T^k(D_\ell)$.)

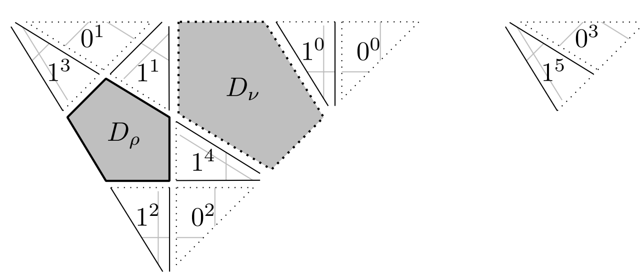

FIGURE 8.2. The trajectory of the scaled domains and $\mathcal{P}$, $\lambda = -\gamma$. (ℓ^k stands for $\hat{T}^k U(D_\ell)$.)

Theorem 8.1. *If $\lambda = -\gamma$, then the minimal period length $\pi(z)$ is*

1	*if* $z \in \{(0,0), (1/\gamma^2, 1/\gamma^2), (2/\gamma^2, 2/\gamma^2)\}$
2	*if* $z \in \{(\frac{5-\gamma}{\gamma^2+1}, \frac{2/\gamma^2}{\gamma^2+1}), (\frac{2/\gamma^2}{\gamma^2+1}, \frac{5-\gamma}{\gamma^2+1})\}$
5	*if* $z = T^m(1/(2\gamma^4), 1/(2\gamma^4))$ *for some* $m \in \{0,1,2,3,4\}$
10	*for the other points of* D_α, D_β, D_ζ, $T(D_\zeta)$, $T^m(D_\eta)$
11	*if* $z = T^m(1-1/\gamma^5, 1-1/\gamma^5)$ *for some* $m \in \{0,1,\ldots,10\}$
25	*if* $z = T^m(1-1/(2\gamma^5), 1-1/(2\gamma^5))$ *for some* $m \in \{0,1,\ldots,24\}$
50	*for the other points of* $T^m(D_\vartheta)$
$2(35\cdot 4^n+28)/3$	*if* $S^nR(z)$ *is the center of* D_ν
$10(35\cdot 4^n+28)/3$	*for the other points of* D_ν
$4(35\cdot 4^n-14)/3$	*if* $S^nR(z)$ *is the center of* D_ρ
$20(35\cdot 4^n-14)/3$	*for the other points of* D_ρ
∞	*if* $S^nR(z) \in \mathcal{D}\setminus\mathcal{P}$ *for all* $n \geq 0$.

Proof. As for $\lambda = -1/\gamma$ and $\lambda = 1/\gamma$, we have

$$\begin{pmatrix}|\sigma^n(0)|_0\\ |\sigma^n(0)|_1\end{pmatrix} = 4^n\begin{pmatrix}1/3\\1/3\end{pmatrix} + \begin{pmatrix}2/3\\-1/3\end{pmatrix}, \qquad \begin{pmatrix}|\sigma^n(1)|_0\\ |\sigma^n(1)|_1\end{pmatrix} = 4^n\begin{pmatrix}2/3\\2/3\end{pmatrix} + \begin{pmatrix}-2/3\\1/3\end{pmatrix},$$

hence $\tau(\sigma^n(0)) = (70\cdot 4^n + 56)/3$, $\tau(\sigma^n(1)) = (140\cdot 4^n - 56)/3$. For $S^nR(z) \in D_\nu$, we have $\pi(z) = \tau(\sigma^n(0))$ and $5\tau(\sigma^n(0))$ respectively; if $S^nR(z) \in D_\rho$, then $\pi(z) = \tau(\sigma^n(1))$ and $5\tau(\sigma^n(1))$ respectively. □

By Figure 8.2, we can choose $\hat{s}(z)$, $s(z)$ as follows and obtain the following $t(z)$:

$$\begin{aligned}
z \in \hat{T}^2U(D_0\cup D_1) &: \hat{s}(z) = -2, s(z) = -70,\ t(z) = V(\hat{T}^{-2}(z)) - V(z) = (-1/\gamma^6, -1/\gamma^6)\\
z \in \hat{T}U(D_1) &: \hat{s}(z) = -1, s(z) = -42,\ t(z) = V(\hat{T}^{-1}(z)) - V(z)A^{-2} = (1/\gamma^5, 1/\gamma^5)\\
z \in U(\mathcal{D}) &: \hat{s}(z) = 0, s(z) = 0,\ t(z) = (0,0)\\
z \in \hat{T}^4U(D_1) &: \hat{s}(z) = 1, s(z) = 42,\ t(z) = V(\hat{T}(z)) - V(z)A^2 = (1/\gamma^4, 0)\\
z \in \hat{T}U(D_0)\cup\hat{T}^3U(D_1) &: \hat{s}(z) = 2, s(z) = 70,\ t(z) = V(\hat{T}^2(z)) - V(z) = (-1/\gamma^5, 0)
\end{aligned}$$

With

$$\begin{aligned}
\{t(z)A^h/\gamma^4 : z \in \mathcal{D}\setminus\mathcal{P}, h \in \mathbb{Z}\} = \{&(0,0),\ \pm(1/\gamma,1/\gamma), \pm(1/\gamma,-1), \pm(-1,0), \pm(0,1), \pm(1,-1/\gamma),\\
&\pm(-1/\gamma^2,-1/\gamma^2), \pm(-1/\gamma^2,1/\gamma), \pm(1/\gamma,0), \pm(0,-1/\gamma), \pm(-1/\gamma,1/\gamma^2)\},
\end{aligned}$$

we obtain $\delta = \frac{\gamma^6}{\gamma^2-1} = \gamma^5$.

Theorem 8.2. *$\pi(z)$ is finite for all $z \in (\frac{1}{2}\mathbb{Z}[\gamma]\cap[0,1))^2$, but $\pi\big(1-1/(3\gamma^2), 1-1/(3\gamma^5)\big) = \infty$.*

Proof. We have $V(\mathcal{D}) = \{(x,y) \in (0,1/\gamma^4)^2 : x > y, \gamma x + y \leq 1/\gamma^3\}$. Hence we have to show that all points $z \in \mathcal{D}$ with $\gamma^4V(z) \in \{1/(2\gamma^2), 1/(2\gamma), 1/2, 1/\gamma, \gamma/2\}^2$ are periodic. This is true since $V\big(\frac{1}{2\gamma^5}, \frac{1}{2\gamma^6}\big) \in D_\nu$, $V\big(\frac{1}{2\gamma^4}, \frac{1}{2\gamma^6}\big) \in D_\nu$, $S^2V\big(\frac{1}{\gamma^5}, \frac{1}{2\gamma^6}\big) = SV\big(\frac{3}{2\gamma^5}, \frac{1}{2\gamma^7}\big) = SV\big(\frac{1}{2\gamma^4}, \frac{1}{2\gamma^5}\big) = V\big(\frac{1}{2\gamma^3}, \frac{1}{2\gamma^5}\big) \in D_\rho$, $V\big(\frac{1}{2\gamma^3}, \frac{1}{2\gamma^6}\big) \in D_\rho$ and $SV\big(\frac{1}{\gamma^5}, \frac{1}{2\gamma^4}\big) = V\big(\frac{1}{\gamma^5}, \frac{1}{2\gamma^5}\big) \in D_\rho$.

If $V(z) = \big(1/(3\gamma^2), 1/(3\gamma^5)\big)$, then we have

$$VS(z) = \gamma^2\Big(V(z) - (\frac{1}{\gamma^5}, 0)\Big) = (\frac{2}{3\gamma^4}, \frac{1}{3\gamma^3}),\ VS^2(z) = \gamma^2\Big(VS(z) - (\frac{1}{\gamma^6}, \frac{1}{\gamma^6})\Big) = (\frac{\gamma^2+1}{3\gamma^5}, \frac{2}{3\gamma^5}),$$

$$VS^3(z) = \gamma^2\Big(VS^2(z) - (\frac{1}{\gamma^6}, \frac{1}{\gamma^6})\Big) = (\frac{3\gamma-2}{3\gamma^4}, \frac{1}{3\gamma^7}),\ VS^4(z) = \gamma^2\Big(VS^3(z) - (\frac{1}{\gamma^5}, 0)\Big) = V(z).$$

□

9. The Thue-Morse sequence and the golden mean

We conclude by exhibiting a relation between the Thue-Morse sequence and substitutions we used in golden mean cases (see [6] for a survey on links between fractal objects and automatic sequences). The Thue-Morse sequence is a fixed point of the substitution $0 \mapsto 01$, $1 \mapsto 10$:

$$0\,1\,10\,1001\,10010110\,1001011001101001\,10010110011010010110100110010110\cdots$$

It can be written as

$$0^1 1^2 0^1 1^1 0^2 1^2 0^2 1^1 0^1 1^2 0^1 1^1 0^2 1^1 0^1 1^2 0^2 1^2 0^1 1^1 0^2 1^2 0^2 1^1 0^1 1^2 0^2 1^2 0^1 1^1 0^2 1^1 0^1 1^2 0^1 1^1 0^2 1^2 0^2 1^1 0^1 1^2 0^1 \cdots$$

By subtracting 1 from each term of the sequence of exponents (the run-lenghts of 0's and 1's) we obtain the sequence

$$0\,10\,01110010\,01001110011100111001001001110010\cdots$$

which is easily shown to be the fixed point of the substitution $0 \mapsto 010$, $1 \mapsto 01110$ (see [5]), which is equal to σ in the cases $\lambda = -1/\gamma$, $\lambda = 1/\gamma$, $\lambda = -\gamma$. In case $\lambda = \gamma$, we have that $\sigma^\infty(1)$ is the image of this word by the morphism $0 \mapsto 10$, $1 \mapsto 110$ since $\sigma(10) = (10)(110)(10)$ and $\sigma(110) = (10)(110)(110)(110)(10)$.

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Department of Mathematics, Faculty of Science Niigata University, Ikarashi 2-8050, Niigata 950-2181, Japan
E-mail address: akiyama@math.sc.niigata-u.ac.jp

Haus-Endt-Strasse 88, D-40593 Düsseldorf, Germany
E-mail address: brunoth@web.de

Department of Computer Science, University of Debrecen, P.O. Box 12, H-4010 Debrecen, Hungary
E-mail address: pethoe@inf.unideb.hu

LIAFA, CNRS, Université Paris Diderot, Case 7014, 2 place Jussieu, 75251 Paris Cedex 05, France
E-mail address: steiner@liafa.jussieu.fr