On a periodic expansion of algebraic numbers

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To the memory of B. Kovács and I. Környei

1 Introduction

Let $\mathbb{K}$ be an algebraic number field of degree n with ring of integers $\mathbb{Z}_{\mathbb{K}}$. Let $\mathbb{K}^{(i)},\ i = 1, \cdots, n$ be denote the conjugates of $\mathbb{K}$ in the field of the complex numbers $\mathbf{C}$. Similarly $\alpha^{(i)}$ will denote the i-th conjugate of $\alpha \in \mathbb{K},\ i = 1, \ldots, n$. For $\alpha \in \mathbb{Z}_{\mathbb{K}}$ and $\mathcal{N} \subseteq \mathbb{Z}$ the pair $\{\alpha, \mathcal{N}\}$ is called a number system $\mathcal{NS}$, if there exist uniquely for every $0 \neq \beta \in \mathbb{Z}[\alpha]$ a non-negative integer $L(\beta)$ and $b_0, \ldots, b_{L(\beta)} \in \mathcal{N}$ such that $b_{L(\beta)} \neq 0$ and

$$\beta = \sum_{i=0}^{L(\beta)} b_i \alpha^i. \tag{1}$$

After partial results Kovács and Pethő [8] gave a complete characterization of number systems in algebraic number fields. In [9] they gave asymptotic estimate for $L(\beta)$, which you find here as Lemma 1.

Having a number system it is natural to ask which elements of $\mathbb{R}(\alpha)$ have an infinite power series expansion of α with "digits" from $\mathcal{N}$. Remark that the field $\mathbb{R}(\alpha)$ is $\mathbb{R}$, the field of the real numbers and $\mathbf{C}$, according as α is real or non-real. Another natural question is whether the well-known rationality criterion of the ordinary q-ary representation of real numbers may be generalized for the new situation.

To be more precise, let $\{\alpha, \mathcal{N}\}$ be a $\mathcal{NS}$ such that $\mathbb{K} = \mathbf{Q}(\alpha)$. We shall denote by $\mathcal{S}(\alpha)$ the set of those complex numbers γ for which either $\gamma = 0$ or

$$\gamma = \sum_{i=L(\gamma)}^{\infty} a_{-i} \alpha^{-i} \tag{2}$$

with some $a_i \in \mathcal{N}, i = L(\gamma), L(\gamma) - 1, \ldots$ and $a_{-L(\gamma)} \neq 0$. We shall call (2) the $\alpha\mathcal{N}$-expansion of γ. This concept was introduced by Kátai and Szabó [3]. They proved that if α is a Gaussian integer and $\{\alpha, \mathcal{N}\}$ is a $\mathcal{NS}$ in $\mathbb{Z}[i]$ then

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any complex number has an $\alpha\mathcal{N}$-expansion. Later Kovács [5] characterized those $\{\alpha, \mathcal{N}\}$ number systems for which any $\gamma \in \mathbb{R}(\alpha)$ have $\alpha\mathcal{N}$-expansions. Using this result and properties of intevall filling sequences Kovács and Maksa [7] proved far reaching generalization of the theorem of Kátai and Szabó. They proved for example that if α is real $\mathcal{N} = \{0, 1, \ldots, |Norm(\alpha)| - 1\}$ then any real γ have $\alpha\mathcal{N}$-expansions. $Norm(\alpha)$ denotes the norm of α.

This problem was completely solved, even in more general setting, by Kátai and Környei [2]. By their result any $\gamma \in \mathbb{R}(\alpha)$ has an $\alpha\mathcal{N}$-expansion. Using this theorem Kovács and Környei [6] proved that the $\alpha\mathcal{N}$-expansion of a $\gamma \in \mathbb{R}(\alpha)$ is periodic if and only if $\gamma \in \mathbf{Q}(\alpha)$. Unfortunately, the method of Kovács and Környei is not algorithmic, one can hardly compute the periodic expansion of a given $\gamma \in \mathbf{Q}(\alpha)$.

The aim of this paper is to prove that at least one periodic $\alpha\mathcal{N}$- expansion of any $\gamma \in \mathbf{Q}(\alpha)$ can be found by using the arithmetic of $\mathbf{Z_K}$. Our method is independent from the above mentioned theorem of Kátai and Környei, it is essentially the same as the method which computes the periodic q-ary expansion of rational numbers. We are stating now our result.

Theorem 1 *Let $\{\alpha, \mathcal{N}\}$ be a $\mathcal{NS}$ such that $K = \mathbf{Q}(\alpha)$. Then there exists an algorithm, which computes a periodic $\alpha\mathcal{N}$-expansion for any $\gamma \in \mathbb{K}$.*

It is obvious from the theorem of Kátai and Környei, that any $\gamma \in \mathbb{R}(\alpha)$ may have many different $\alpha\mathcal{N}$-expansions. On the other hand it is not clear how many periodic $\alpha\mathcal{N}$-expansions of the elements of $\mathbb{K}$ may have. We shall point out in section 3, that even the periodic $\alpha\mathcal{N}$-expansion of the elements of $\mathbb{K}$ is not unique. In all of the examples, I studied the different expansions were closely related, more precisely they were different only in finitely many digits. It remains an open question whether there exist essentially different periodic expansions.

2 Proof of Theorem 1

An important tool in the proof of Theorem 1 is the following theorem of Kovács and Pethő [9].

Lemma 1 *Let $\{\alpha, \mathcal{N}\}$ be a $\mathcal{NS}$, α being of degree n and $0 \neq \gamma \in \mathbb{Z}[\alpha]$. Then there exist constants $c_1(\alpha, \mathcal{N}), c_2(\alpha, \mathcal{N})$ such that*

$$\max_{1 \leq i \leq n} \frac{\log |\gamma^{(i)}|}{\log |\alpha^{(i)}|} + c_1(\alpha, \mathcal{N}) \leq L(\gamma) \leq \max_{1 \leq i \leq n} \frac{\log |\gamma^{(i)}|}{\log |\alpha^{(i)}|} + c_2(\alpha, \mathcal{N}).$$

Remark that the constants c_1 and c_2 do not depend on γ.

In the proof of Theorem 1. we follow essentially the proof of the analogous statement for the q-ary representation of real numbers as given in Bundschuh [1]. Let $0 \neq \gamma \in \mathbb{K} = \mathbf{Q}(\alpha)$. Write $\gamma = \frac{\beta}{\delta}$ with $\beta, \delta \in \mathbb{Z}[\alpha]$.

Assume first, that $(\delta) + (\alpha) = (1)$, where and in the sequel (δ) denotes the principal ideal generated by δ in $\mathbb{Z}_{\mathbb{K}}$. There exists an integer $d > 0$ such that $\alpha^d \equiv 1 \pmod{\delta}$ holds in $\mathbb{Z}[\alpha]$. Let fix d and put $\alpha^d - 1 = \delta\kappa_1$. Then for $m \geq 1$ integer δ divides obviously $\alpha^{dm} - 1$. Put $\alpha^{dm} - 1 = \delta\kappa_m$. Then we have $\kappa_m \in \mathbb{Z}[\alpha]$ and

$$\gamma = \frac{\beta\kappa_1}{\alpha^d - 1} = \frac{\beta\kappa_m}{\alpha^{dm} - 1} = \frac{\beta\kappa_1(\alpha^{d(m-1)} + \ldots + 1)}{\alpha^{dm} - 1} = \frac{\varepsilon_m}{\alpha^{dm} - 1}. \tag{3}$$

As $0 \neq \varepsilon_m \in \mathbb{Z}[\alpha]$, thus ε_m can be represented in $\{\alpha, \mathcal{N}\}$ and

$$L(\varepsilon_m) \leq \max_{1 \leq i \leq n} \frac{\log |\varepsilon_m^{(i)}|}{\log |\alpha^{(i)}|} + c_2 = \max_{1 \leq i \leq n} \frac{\log |(\beta\kappa_1)^{(i)}| + \log |\frac{\alpha^{(i)dm} - 1}{\alpha^{(i)} - 1}|}{\log |\alpha^{(i)}|} + c_2$$

$$< \max_{1 \leq i \leq n} \frac{\log |(\beta\kappa_1)^{(i)}| - \log |\alpha^{(i)} - 1|}{\log |\alpha^{(i)}|} + c_2 + dm.$$

Notice that the first two summands are independent from m, hence

$$|L(\varepsilon_m) - dm| \leq c_3,$$

with c_3 independent from m, and we can write

$$\varepsilon_m = \sum_{i=0}^{dm-1} a_{mi}\alpha^i + \sum_{i=dm}^{L(\varepsilon_m)} a_{mi}\alpha^i = \omega_m + \alpha^{dm}\tau_m,$$

with $a_{mi} \in \mathcal{N},\ i = 0, \dots, L(\varepsilon_m)$. As the length of τ_m is bounded by a constant, which does not depend on m and the "digits" $a_{mi},\ i = dm, \dots, L(\varepsilon_m)$ belong to a finite set, there are only finitely many possibilites for $\tau_m,\ m = 1, 2, \dots$. Thus there exist $0 < \ell < k$ integers such that $\tau_{2^\ell} = \tau_{2^k} = \tau$. Let fix ℓ and k for the sequel. We have

$$\begin{aligned}
\varepsilon_{2^k} &= \beta\kappa_1 \frac{\alpha^{d\cdot 2^k} - 1}{\alpha^d - 1} = \beta\kappa_1 \frac{\alpha^{d\cdot 2^\ell} - 1}{\alpha^d - 1} \cdot \frac{\alpha^{d\cdot 2^k} - 1}{\alpha^{d\cdot 2^\ell} - 1} \\
&= \varepsilon_{2^\ell}(1 + \alpha^{d\cdot 2^\ell} + \cdots + \alpha^{d(2^k - 2^\ell)}) \\
&= \omega_{2^\ell} + (\omega_{2^\ell} + \tau)\alpha^{d\cdot 2^\ell}(1 + \alpha^{d\cdot 2^\ell} + \cdots + \alpha^{d(2^k - 2^{\ell+1})}) + \alpha^{d\cdot 2^k}\tau.
\end{aligned}$$

On the other hand

$$\varepsilon_{2^k} = \omega_{2^k} + \alpha^{d\cdot 2^k}\tau.$$

Both ω_{2^ℓ} and τ are assumed already represented in $\{\alpha, \mathcal{N}\}$. If we write $\omega_{2^\ell} + \tau$ in $\{\alpha, \mathcal{N}\}$ then it may happen that the length of $\omega_{2^\ell} + \tau$ is longer then $d \cdot 2^\ell - 1$. Let

$$\eta_k = (\omega_{2^\ell} + \tau) \sum_{i=0}^{2^{k-\ell}-2} \alpha^{d\cdot 2^\ell i}.$$

We get in this notation

$$\varepsilon_{2^k} = \omega_{2^\ell} + \alpha^{d\cdot 2^\ell}\eta_k + \alpha^{d\cdot 2^k}\tau.$$

We have on the other hand

$$\varepsilon_{2^k} = \omega_{2^k} + \alpha^{d\cdot 2^k}\tau,$$

where both ω_{2^k} and τ are written in $\{\alpha, \mathcal{N}\}$. Thus $\omega_{2^k} = \omega_{2^\ell} + \alpha^{d\cdot 2^\ell}\eta_k$ and we have

$$L(\eta_k) \le d(2^k - 2^\ell) - 1. \tag{4}$$

for the $\{\alpha, \mathcal{N}\}$ expansion of η_k.

Let now $t \geq 0$ an integer and consider $\varepsilon_{2^k+t(2^k-2^\ell)}$. We have similarly as above

$$\begin{aligned}
\varepsilon_{2^k+t(2^k-2^\ell)} &= \varepsilon_{2^\ell} \sum_{i=0}^{(t+1)(2^{k-\ell}-1)} \alpha^{d\cdot 2^\ell i} \\
&= \omega_{2^\ell} + (\omega_{2^\ell}+\tau) \sum_{i=1}^{(t+1)(2^{k-\ell}-1)} \alpha^{d\cdot 2^\ell i} + \tau \cdot \alpha^{d\cdot 2^\ell[(t+1)(2^{k-\ell}-1)+1]} \\
&= \omega_{2^\ell} + \sum_{j=0}^{t} \alpha^{d\cdot 2^\ell (j(2^{k-\ell}-1)+1)} (\omega_{2^\ell}+\tau) \sum_{i=0}^{2^{k-\ell}-2} \alpha^{d\cdot i 2^\ell} \\
&\quad + \tau \cdot \alpha^{d\cdot 2^\ell[(t+1)(2^{k-\ell}-1)+1]} \\
&= \omega_{2^\ell} + \eta_k \alpha^{d\cdot 2^\ell} \sum_{j=0}^{t} \alpha^{j\cdot d(2^k-2^\ell)} + \tau \alpha^{d[(t+1)(2^k-2^\ell)+2^\ell]}.
\end{aligned}$$

This is by (4) already the $\{\alpha, \mathcal{N}\}$ expansion of $\varepsilon_{2^k+t(2^k-2^\ell)}$ if we insert the $\{\alpha, \mathcal{N}\}$ expansions of ω_{2^ℓ}, η_k and τ.

Let $A = \max\{|b|,\ b \in \mathcal{N}\}$ then we have

$$|\beta| \leq A \cdot |\alpha|^{L(\beta)} \frac{|\alpha|}{|\alpha|-1} \tag{5}$$

for any $\beta \in \mathbb{Z}[\alpha]$. Put $B = \max\{L(\omega_{2^\ell}), L(\tau), L(\eta_k)\} = \max\{d \cdot 2^\ell, c_3, d(2^k - 2^\ell)\}$. Using (5) we get

$$\begin{aligned}
&\left| \varepsilon_{2^k+t(2^k-2^\ell)} - \left(\tau + \eta_k \sum_{j=1}^{\infty} \alpha^{-j\cdot d(2^k-2^\ell)} \right) \left(\alpha^{d[2^k+t(2^k-2^\ell)]} - 1 \right) \right| \\
= &\left| \omega_{2^\ell} - \eta_k \sum_{j=t+2}^{\infty} \alpha^{-jd(2^k-2^\ell)} + \tau + \eta_k \sum_{j=1}^{\infty} \alpha^{-jd(2^k-2^\ell)} \right| \\
\leq &\ 4 \cdot A|\alpha|^B \left(\frac{|\alpha|}{|\alpha|-1} \right)^2 = c_4
\end{aligned}$$

for any $t \geq 0$ with c_4, which is independent from t. Taking now into consideration (3) we have

$$\gamma = \tau + \eta_k \sum_{j=1}^{\infty} \alpha^{-jd(2^k-2^\ell)},$$

which is a periodic $\alpha\mathcal{N}$-expansion of γ. This proves Theorem 1 in the particular case.

In the second part of the proof we are dealing with the general situation, i.e. if $\gamma = \frac{\beta}{\delta}$ with $\beta, \delta \in \mathbb{Z}[\alpha]$, but $(\delta) + (\alpha) \subset (1)$ in $\mathbb{Z}_{\mathbb{K}}$. Let the prime ideal decomposition of the ideal (α, δ) in $\mathbb{Z}_{\mathbb{K}}$ be

$$(\alpha, \delta) = \mathcal{P}_1^{c_1} \cdots \mathcal{P}_t^{c_t},$$

where $c_1, \ldots, c_t$ are positive integers.

$$(\delta) = \mathcal{P}_1^{a_1} \cdots \mathcal{P}_t^{a_t} \mathcal{Q}_\delta \ \text{ and } \ (\alpha) = \mathcal{P}_1^{b_1} \cdots \mathcal{P}_t^{b_t} \mathcal{Q}_\alpha$$

with $\mathcal{Q}_\delta + (\alpha) = (1)$. Denote by h the class number of $\mathbb{Z}_{\mathbb{K}}$. Then $\mathcal{P}_i^h = (\pi_i),\ i = 1, \ldots, t;\ \ \mathcal{Q}_\delta^h = (\rho_\delta)$ and $\mathcal{Q}_\alpha^h = (\rho_\alpha)$ with $\pi_1, \ldots, \pi_t,\ \rho_\delta,\ \rho_\alpha \in \mathbb{Z}_{\mathbb{K}}$, and we have

$$\delta^h = \pi_1^{a_1} \cdots \pi_t^{a_t} \rho_\delta \eta_\delta \ \text{ and } \ \alpha^h = \pi_1^{b_1} \ldots \pi_t^{b_t} \rho_\alpha \eta_\alpha,$$

where η_δ and η_α are units in $\mathbb{Z}_{\mathbb{K}}$. We may assume without loss of generality, eventually changing ρ_δ and ρ_α, that $\eta_\delta = \eta_\alpha = 1$.

Let s be a positive integer such that $sb_i \geq a_i$ hold for all $i = 1, \ldots, t$ and put

$$\delta_1 = \delta^h \pi_1^{sb_1 - a_1} \cdots \pi_t^{sb_t - a_t} \rho_\alpha^s = \alpha^{sh} \rho_\delta.$$

Then we have

$$\gamma = \frac{\beta \cdot \delta^{h-1} \pi_1^{sb_1 - a_1} \ldots \pi_t^{sb_t - a_t} \rho_\alpha^s}{\alpha^{sh} \rho_\delta} = \frac{\beta_1}{\alpha^{sh} \rho_\delta}.$$

As $(\rho_\delta) + (\alpha) = 1$ there exists a periodic $\alpha\mathcal{N}$-expansion of $\frac{\beta_1}{\rho_\delta}$. Division with α^{sh} does not change the periodicity of this expansion, only the place of the "period", hence γ admits a periodic $\alpha\mathcal{N}$-expansion. Theorem 1 is proved.$\square$

3 Examples

To illustrate how one can compute in the line of the proof of Theorem 1 a periodic $\alpha\mathcal{N}$-expansion we choose α a zero of the cubic polynomial $x^3+9x^2+24x+17$. We proved with B. Kovács in [8] that if $\mathcal{N}=\{0,\dots,16\}$ then $\{\alpha,\mathcal{N}\}$ is a $\mathcal{NS}$ in $\mathbb{Z}[\alpha]$. Put $\gamma=1/2$. It is easy to check that $\alpha^7\equiv 1 \pmod 2$ and $\alpha^j \not\equiv 1 \pmod 2$ for any $0<j<7$. We have

$$\alpha^7 = -7932\alpha^2 - 33326\alpha - 27387.$$

Thus

$$\begin{aligned}
\frac{1}{2} &= \frac{-3966\alpha^2-16663\alpha-13694}{\alpha^7-1} \\
&= \frac{8+12\alpha+13\alpha^2+4\alpha^3+\alpha^4+8\alpha^5+4\alpha^6+\alpha^7}{\alpha^7-1} \\
&= \left(1+\frac{4}{\alpha}+\frac{8}{\alpha^2}+\frac{1}{\alpha^3}+\frac{4}{\alpha^4}+\frac{13}{\alpha^5}+\frac{12}{\alpha^6}+\frac{8}{\alpha^7}\right)\sum_{i=0}^{\infty}\alpha^{-7i} \\
&= \alpha + (4\alpha^6+8\alpha^5+\alpha^4+4\alpha^3+13\alpha^2+12\alpha+9)\sum_{i=1}^{\infty}\alpha^{-7i}.
\end{aligned}$$

Finally we shall show that the periodic $\alpha\mathcal{N}$-expansion is generally not unique. Let $p(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n\in\mathbb{Z}[x]$ such that $1=a_0\le a_1\le \ldots \le a_n,\ a_n\ge 2,\ \alpha$ a zero of $p(x)$ and $\mathcal{N}=\{0,\dots,a_n-1\}$. Then $\{\alpha,\mathcal{N}\}$ is a $\mathcal{NS}$ in $\mathbb{Z}[\alpha]$ by B. Kovács [4], hence $|\alpha|>1$.

Put

$$\gamma = a_n\sum_{i=1}^{\infty}\alpha^{-i} = a_n\frac{\alpha}{\alpha-1}.$$

As $a_n\notin\mathcal{N}$ this is not an $\alpha\mathcal{N}$-expansion of γ, but we can easily find $\alpha\mathcal{N}$-expansions of γ. Indeed, let $0\le j\le n$, then as $p(\alpha)=0$ we have

$$\begin{aligned}
\gamma &= a_n\sum_{i=1}^{\infty}\alpha^{-i} - \frac{p(\alpha)}{\alpha^j}\sum_{i=0}^{\infty}\alpha^{-i(n+1)} + p(\alpha) \\
&= a_0\alpha^n+\ldots+a_{j-1}\alpha^{n-j+1}+(a_j-a_0)\alpha^{n-j}+(a_n-a_{n-j})\alpha^0 \\
&\quad +(a_n-a_{n-j-1})\alpha^{-1}+\ldots+(a_n-a_n)\alpha^{-j} \\
&\quad +\sum_{i=1}^{\infty}\left((a_n-a_0)\alpha^n+\ldots+(a_n-a_n)\right)\alpha^{-i(n+1)-j},
\end{aligned}$$

where $a_{-1}=0$, if $j=0$. It is clear that the coefficients of this power series belong to $\mathcal{N}$, hence γ has at least $n+1$ different, periodic $\alpha\mathcal{N}$-expansions.

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