On the family of Thue equations

$$x^3-(n-1)x^2y-(n+2)xy^2-y^3=k$$

by M. Mignotte, A. Pethő * and F. Lemmermeyer

Abstract : We study the double family of Thue equations $x^3-(n-1)x^2y-(n+2)xy^2-y^3=k$.

1. Introduction

The family of cubic Thue equations $x^3-(n-1)x^2y-(n+2)xy^2-y^3=\pm 1$, with $n\geq 0$, was studied by E. Thomas. He proved in [T1] that this family of equations has only "trivial solutions" except for a finite number of values of the parameter n, explicitly for $n<10^8$. Then, M. Mignotte [M] could solve completely this family of equations and proved that non-trivial solutions occur only for $n\leq 3$.

Then, other infinite families of Thue equations $F(x,y)=\pm 1$ were studied. In the cubic case there are works of Thomas [T2], Mignotte and Tzanakis [MT]; in these cases the family was completely solved except for an explicit finite range for the parameter. In the quartic case, there are works of Pethő [P2], Lettl and Pethő [LeP] and Mignotte, Pethő and Roth [MPR], where the last two studies were completely finished.

Here, we come back to the family of cubics investigated by E. Thomas, the case where $F_n(x,y)=x^3-(n-1)x^2y-(n+2)xy^2-y^3$, but we add a new parameter: the number k on the right hand side. In other words, we study a family of cubic diophantine equations which depend on two parameters.

Our results are explicit bounds for the integer solutions in terms of the two parameters n and k; and we give a special study to the case $x^3-(n-1)x^2y-(n+2)xy^2-y^3=\pm(2n+1)$. This case has some interest since we prove that the diophantine equation $F_n(x,y)=k$ has no solution in the range $1<|k|<2n+1$, except when k is a cube (the existence of a solution in this case is trivial).

We formulate now the three theorems, which are the main results of the present paper.

Theorem 1. *Let $n\geq 1650, k$ be integers. If*

$$|x^3-(n-1)x^2y-(n+2)xy^2-y^3|=k$$

holds for some $x,y\in\mathbf{Z}$, then

$$\log|y|<c_1\log^2(n+2)+c_2\log\, n\log\, k,$$

where

$$c_1=700+476.4\left(1-\frac{1432.1}{n}\right)^{-1}\left(1.501-\frac{1902}{n}\right)<1956.4$$

and

$$c_2=29.82+\left(1-\frac{1432.1}{n}\right)^{-1}\frac{1432}{n\,\log\, n}<30.71.$$

It is well known, see [GyP] and the references therein, that the general estimates of the solutions of a Thue equation depend polynomially in the constant term, but exponentially with respect to the height of

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the form itself, i.e. the maximum of the absolute value of the coefficients of the form corresponding to the Thue equation. In the present case, the dependence is polynomial in k and "nearly" polynomial in the height of $F_n(x, y)$ (which is $n + 2$). Moreover the constants appearing here are much better than in other similar results, see e.g. [BSt].

To illustrate how sharp is the estimate of Theorem 1 let us take $n = 1650$ and $k = 10^9$, then we get

$$|y| < 10^{48698}.$$

This is certainly a bound which is reachable with the present numerical techniques, and in any case such a bound is much smaller than previous bounds obtained for cubic Thue equations.

From Theorem 1, we are able to derive an effective improvement of Liouville's inequality for the rational approximations of the zeros of $F_n(x, 1)$.

Theorem 2. *Let λ be one of the zeros of $F_n(x, 1)$. If $n \geq 1650$ and $\lambda \in]-1, 0[$ or $\lambda \in]-2, -1[$ then*

$$\left|\lambda - \frac{x}{y}\right| > (n+2)^{-c_3} y^{-3+1/(c_2 \log n)}$$

and if $\lambda \in (n, n + 1)$ then

$$\left|\lambda - \frac{x}{y}\right| > (n+2)^{-c_3-1} y^{-3+1/(c_2 \log n)},$$

holds for all $(x, y) \in \mathbf{Z}^2, y \neq 0$, where

$$c_3 = c_1 \cdot \frac{\log(n+2)}{\log n} + 1 < 64.72.$$

We are dealing in the second part of the paper with the inequality

$$|x^3 - (n-1)x^2 y - (n+2)xy^2 - y^3| \leq 2n + 1 \qquad (1).$$

Combining the method of the proof of Theorem 1 with a result of Lemmermeyer and Pethő [LP] we are able to solve the above inequality completely and prove the following.

Theorem 3. *Let n be a nonnegative integer. If $(x, y) \in \mathbf{Z}^2$ is a solution of equation (1), then either $(x, y) = t(u, v)$ with an integer $0 \leq |t| \leq \sqrt[3]{2n+1}$ and $\pm(u, v) \in \{(1, 0), (0, 1), (-1, 1)\}$ or $\pm(x, y) \in \{(-1, -1), (-1, 2), (-1, n + 1), (-n, -1), (n + 1, -n), (2, -1)\}$, except when $n = 2$, in which case (1) has the extra solutions: $\pm(x, y) \in \{(-4, 3), (8, 3), (1, -4), (3, 1), (3, -11)\}$.*

The plan of this paper is the following. Section 2 contains two general lemmas. The next section is devoted to a pure numerical study of the roots of the polynomials $x^3 - (n-1)x^2 - (n+2)x - 1$; these estimates are used in the linear forms of logs which are studied in the sequel. Section 4, which is short and easy, contains the proof of Theorem 2 assuming Theorem 1.

The next section is an instance of the classical Siegel-Baker's reduction of a Thue equation to a linear form in logs. This study is detailed in order to lead to sharp estimates; here we get linear forms in three logs. Then, the linear form in logs obtained in Section 5 is studied a first time in Section 6. In this study, this linear form is considered as a linear form in two logs (we just group two terms). To get sharp estimates, we apply the general result of Laurent-Mignotte-Nesterenko. Of course, this implies a tedious reconstruction of the proof of the lower bound of this linear form; for example we have to choose the parameters of the "auxiliary function" (more exactly, of the interpolation determinant): this is the price to pay. The conclusion of this section is an upper bound on y, in terms of the heights of the terms of the linear form. The very long Section 6 contains a second study of this linear form in "two" logs; first, we prove an estimate of the coefficients of this linear form in terms of y, then we bound these coefficients and (after some computation) at the end we get an explicit upper bound of y, in terms of the initial parameters n and k. This ends the proof of Theorem 1.

Section 8, which is short and easy, give a simple upper bound of y depending only on n, when $k < n^4$; this result is used in the next section.

The last two sections deal with the proof of Theorem 3. Here, a result of Lemmermeyer and Pethő [LP] plays an essential role. Section 9 contains a study of the special linear form in "two" logs which occur in this case; this leads to the proof of Theorem 3 for $n > 1700$. To cover the range $n \leq 1700$, in Section 10 we consider — for the first time — the linear form in logs as a linear form in three logs. This study is classical and we can easily finish the proof of Theorem 3.

2. Preliminary lemmas

This section contains two lemmas on cubic fields and cubic forms which have a general interest.

Lemma 1. *[P1, Theorem 2.] Let $f(X,Y) = X^3 + bX^2Y + cXY^2 + dY^3$ be a cubic form with positive discriminant D_f, and suppose that $f(x,y) = k$, where x, y are rational integers, $y \neq 0$. Let α_i, $i = 1, 2, 3$, be the roots of the polynomial $f(X,1)$. Put $L_i = \alpha_i - \frac{x}{y}$, $i = 1, 2, 3$, and suppose that $|L_1| \leq |L_2| \leq |L_3|$; then*

$$|L_1|^4|L_2|^2 \leq \frac{4|k|^4}{D_f\,|y|^{12}}.$$

Proof Notice the formula

$$D_f = \prod_{i<j}(\alpha_i - \alpha_j)^2 = L_3^4L_2^2\left(1 - \frac{L_2}{L_3}\right)^2\left(1 - \frac{L_1}{L_3}\right)^2\left(1 - \frac{L_1}{L_3}\right)^2.$$

Since the function

$$g(u,v) = (1-u)\,(1-v)\left(1 - \frac{u}{v}\right)$$

satisfies $0 \leq g(u,v) \leq 2$ on the domain $-1 \leq u, v \leq 1$, $|v| \leq |u|$, we have

$$D_f \leq 4\,L_3^4L_2^2.$$

Multiplying both sides of this inequality by $L_2^2L_1^4$ and using the relation $L_1L_2L_3 = k/y^3$, we get the result.

Corollary 1. *Using $|L_1| \leq |L_2|$, we get*

$$|L_1| \leq \left(\frac{4k^4}{D_f}\right)^{1/6} \times \frac{1}{|y|^2}.$$

Lemma 2. *Let $\alpha \in K$, where K is a cubic Galois field. Denote by α, α' and α'' the conjugates of α. Choose such an order that these conjugates satisfy $|\alpha_1| \geq |\alpha_2| \geq |\alpha_3|$. Then the measure of the quotient α/α' satisfies*

$$M(\alpha/\alpha') \leq |\alpha_1|^2|\alpha_2|.$$

Proof Let k be the norm of α. Consider the polynomial

$$k(X - \alpha/\alpha')\,(X - \alpha'/\alpha'')\,(X - \alpha''/\alpha) = (\alpha'X - \alpha)\,(\alpha''X - \alpha')\,(\alpha X - \alpha''),$$

clearly, this polynomial has rational integer coefficients, and α/α' is a root of it. Thus,

$$\begin{aligned} M(\alpha/\alpha') &\leq |k|\,\max\{1, |\alpha/\alpha'|\}\,\max\{1, |\alpha'/\alpha''|\}\,\max\{1, |\alpha''/\alpha|\} \\ &= k\,\frac{|\alpha_1|}{|\alpha_3|} = |\alpha_1|^2|\alpha_2|. \end{aligned}$$

Remark.— It is easy to verify that the previous lemma can be generalized in the following way: Let $\alpha \in K$, where K is a cyclic Galois field of degree d. Let α' be some conjugate of α, with $\alpha' \neq \alpha$. Choose an order on the set $\alpha_1, \ldots, \alpha_d$ of the conjugates of α for which $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_d|$. Then the measure of the quotient α/α' satisfies

$$M(\alpha/\alpha') \leq |\alpha_1|^{d-1}|\alpha_2|^{d-2}\cdots|\alpha_{d-1}|.$$

3. Numerical study of the roots of $x^3-(n-1)x^2-(n+2)x-1$

In this section, we gather several sharp estimates for the roots of the polynomial f associated to the cubic form

$$F(x,y)=F_n(x,y)=x^3-(n-1)x^2y-(n+2)xy^2-y^3,$$

that is

$$f(x)=f_n(x)=x^3-(n-1)x^2-(n+2)x-1.$$

From now on, we shall keep these definitions for f, f_n, F_n and F.

Since $f(-2)<0$, $f(-1)=1$, $f(0)=-1$, $f(n)=-2n-1$ and $f(n+1)=n^2+n+1$, this polynomial has three real roots, say $\lambda=\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$, and we may suppose that

$$\lambda\in\,]-1,0[\,,\qquad \lambda^{(2)}\in\,]-2,-1[\,,\qquad \lambda^{(3)}\in\,]n,n+1[\,.$$

Moreover, the polynomial f is invariant under the change of variable $\sigma: x\mapsto -1/(1+x)$, which satisfies $\sigma^2(x)=-(1+x)/x$ and $\sigma^3(x)=x$. This proves that the field $\mathbf{Q}(\lambda)$ is a Galois field and that

$$\lambda^{(2)}=-\frac{1}{\lambda+1},\quad\text{and}\quad \lambda^{(3)}=-\left(1+\frac{1}{\lambda}\right).$$

More generally,

$$\sigma^j\big(\lambda^{(i)}\big)=\lambda^{(k)},\qquad\text{where}\ \ k\equiv i+j \bmod 3.$$

Since

$$f_n\big(n+\tfrac{2}{n}\big)=1+\frac{4}{n}+\frac{4}{n^2}+\frac{8}{n^3},$$

we see that

$$\lambda^{(3)}<n+\frac{2}{n}.$$

A second computation gives

$$f_n\left(n+\tfrac{2}{n+1}\right)=-1+\frac{4}{n+1}-\frac{4}{(n+1)^2}+\frac{8}{(n+1)^3},$$

which shows that

$$\lambda^{(3)}>n+\frac{2}{n+1}$$

for $n>2$. In the sequel, we suppose $n\geq 3$, and the two cases $n=1$ and $n=2$ will be considered separately.

In terms of $\lambda^{(3)}$, we have

$$\lambda^{(3)}=-\left(1+\frac{1}{\lambda}\right)\implies\lambda=-\frac{1}{\lambda^{(3)}+1},$$

and

$$\lambda^{(2)}=-\frac{1}{1+\lambda}=-\frac{1}{1-\dfrac{1}{\lambda^{(3)}+1}}=-\frac{\lambda^{(3)}+1}{\lambda^{(3)}}.$$

These estimates imply (for $n\geq 3$)

$$-\frac{n+1}{n^2+2n+3}=-\frac{1}{1+n+\frac{2}{n+1}}<\lambda<-\frac{1}{1+n+\frac{2}{n}}=-\frac{n}{n^2+n+2},$$

and

$$-1-\frac{n+1}{n^2+n+2}<\lambda^{(2)}=-\left(1+\frac{1}{\lambda^{(3)}}\right)<-1-\frac{n}{n^2+2}.$$

These estimates lead to estimates for logarithms. Indeed,

$$\log n+\frac{2}{n(n+1)}-\frac{2}{\big(n(n+1)\big)^2}<\log\left(n+\frac{2}{n+1}\right)<\ell_3:=\log\lambda^{(3)}<\log\left(n+\frac{2}{n}\right)<\log n+\frac{2}{n^2},$$

$$\frac{n-\frac{1}{2}}{n^2+2}<\frac{n}{n^2+2}-\frac{n^2}{2\big(n^2+2\big)^2}<\ell_2:=\log\left|\lambda^{(2)}\right|<\frac{n+1}{n^2+n+2},$$

and

$$-\log(n+1)-\frac{2}{n(n+1)}<\ell_1:=\log|\lambda|<-\log(n+1)-\frac{2}{(n+1)^2}+\frac{2}{(n+1)^4}.$$

4. Proof of Theorem 2.

Assuming that Theorem 1 is true we now give the proof of Theorem 2 because it is short and depends only on the estimates of section 3. We consider only the case $\lambda \in]1, 0[$. The proofs of the other cases are similar. Assume that there exists $(x, y) \in \mathbf{Z}^2$, $y \neq 0$ such that

$$\left| \lambda - \frac{x}{y} \right| \leq (n+2)^{-c_3} y^{-3+1/(c_2 \ \log \ n)}.$$

Let λ_2 and λ_3 denote the other zeros of $F_n(x, 1)$ such that $\lambda_2 \in] - 2, -1[$ and $\lambda_3 \in]n, n+1[$. Then

$$\left| \lambda_i - \frac{x}{y} \right| \leq |\lambda - \lambda_i| + (n+2)^{-c_3}$$

for $i = 2, 3$. Using these inequalities and the estimates of Section 3, one can easily prove that

$$|\lambda_2 y - x||\lambda_3 y - x| < (n+2)|y|,$$

hence

$$|F_n(x, y)| < (n+2)^{-(c_3-1)} |y|^{1/(c_2 \ \log \ n)}.$$

Putting $k = (n+2)^{-(c_3-1)} |y|^{1/(c_2 \ \log \ n)}$, Theorem 1 implies

$$\begin{aligned} \log |y| &< c_1 \log^2(n+2) + c_2 \log n \log k \\ &< c_1 \log^2(n+2) + c_2 \log n \left(-c_1 \frac{\log^2(n+2)}{c_2 \log n} + \frac{\log |y|}{c_2 \log n} \right) \\ &= \log |y| \end{aligned}$$

a contradiction, and the Theorem is proved.

5. Reduction of the Thue equation $F_n(x,y)=k$

Let x, y be rational integers, $y>0$, such that

$$F_n(x,y)=x^3-(n-1)x^2y-(n+2)xy^2-y^3=\pm k, \qquad k>0. \tag{2}$$

The condition $k>0$ is not restrictive since $F(-x,-y)=-F(x,y)$.

We know (see [T1]) that λ, $\lambda^{(2)}$ constitute a fundamental system of units for the order $\mathbf{Z}[\lambda]$.

The following lemma is useful for the reduction of the Thue equation in the case $k>1$.

Lemma 3. *Let $0\neq\beta\in\mathbf{Z}[\lambda]$ and let $0<c_1,\, c_2\in\mathbf{R}$. Then, there exist rational integers a_1 and a_2 and some element $\gamma\in\mathbf{Z}[\lambda]$ such that*

$$\beta=\gamma\lambda^{a_1}(\lambda^{(2)})^{a_2}$$

with, for $n\geq 3$,

$$c_i\leq|\gamma^{(i)}|<(n+3)c_i, \qquad i=1,\,2,$$

and

$$\frac{|N(\beta)|}{(n+3)^2c_1c_2}<|\gamma^{(3)}|\leq\frac{|N(\beta)|}{c_1c_2}.$$

Proof Put

$$\tilde{c}_i=c_i\exp\left\{\frac{1}{2}\big|\log|\lambda^{(i)}|\big|+\frac{1}{2}\big|\log|\lambda^{(i+1)}|\big|\right\}, \qquad i=1,\,2,$$

and consider the system of equations

$$\log\big|\beta^{(i)}/\tilde{c}_i\big|=r_1\log|\lambda^{(i)}|+r_2\log|\lambda^{(i+1)}|, \qquad i=1,\,2.$$

As the vectors $(\log|\lambda^{(1)}|,\log|\lambda^{(2)}|,\log|\lambda^{(3)}|)$ and $(\log|\lambda^{(2)}|,\log|\lambda^{(3)}|,\log|\lambda^{(1)}|)$ are linearly independent in $\mathbf{R}^3$, this system has a unique real solution, r_1 and r_2. Put $r_1=a_1+\theta_1$ and $r_2=a_2+\theta_2$, where $a_1,\,a_2\in\mathbf{Z}$, and $|\theta_1|,\ |\theta_2|\leq 1/2$. Then consider the algebraic number $\gamma=\beta\lambda^{-a_1}(\lambda^{(2)})^{-a_2}$. Then $|N\gamma|=|N\beta|$ and $\gamma\in\mathbf{Z}[\lambda]$, and we get

$$\big|\log|\gamma^{(i)}/\tilde{c}_i|\big|\leq\tfrac{1}{2}\big(\big|\log|\lambda^{(i)}|\big|+\big|\log|\lambda^{(i+1)}|\big|\big), \qquad i=1,\,2,$$

which implies

$$\log c_i<\log|\gamma^{(i)}|\leq\log c_i+\big(\big|\log|\lambda^{(i)}|\big|+\big|\log|\lambda^{(i+1)}|\big|\big), \qquad i=1,\,2.$$

By the estimates of the previous section, for $n\geq 3$,

$$\begin{aligned}\big|\log|\lambda^{(1)}|\big|+\big|\log|\lambda^{(2)}|\big|&=\log(\lambda^{(3)}+1)+\log\left(1+\frac{1}{\lambda^{(3)}}\right)=\log\left(\lambda^{(3)}+2+\frac{1}{\lambda^{(3)}}\right)\\&<\log\left(n+\frac{2}{n}+2+\frac{n}{n^2+2}\right)<\log(n+3),\end{aligned}$$

and

$$\big|\log|\lambda^{(2)}|\big|+\big|\log|\lambda^{(3)}|\big|=\log(\lambda^{(3)})+\log\left(1+\frac{1}{\lambda^{(3)}}\right)=\log(\lambda^{(3)}+1)<\log(n+2).$$

This proves the two first inequalities. The third follows immediately.

Corollary 2. *Let $0\neq\beta\in\mathbf{Z}[\lambda]$ with norm k. Then, there exist rational integers a_1 and a_2 and some element $\gamma\in\mathbf{Z}[\lambda]$ such that*

$$\beta=\gamma\lambda^{a_1}(\lambda^{(2)})^{a_2}$$

with

$$M\big(\gamma^{(2)}/\gamma^{(3)}\big)\leq k(n+3)^{5/2},$$

and

$$\left|\log\left|\gamma^{(2)}/\gamma^{(3)}\right|\right| \leq \tfrac{3}{2}\log(n+3).$$

Proof. Put $c_1 = c_2 = \sqrt[3]{k}/\sqrt{n+3}$. The application of Lemma 3 implies the existence of $\gamma \in \mathbf{Z}[\lambda]$, which is associated to β and which satisfies

$$\sqrt[3]{k}/(n+3) < \left|\gamma^{(3)}\right| < \sqrt[3]{k}\cdot(n+3).$$

From Lemma 2, we get

$$M\big(\gamma^{(2)}/\gamma^{(3)}\big) \leq k(n+3)^{5/2}.$$

Finally, as

$$\log\left|\gamma^{(2)}\right| - \log\left|\gamma^{(3)}\right| \leq \frac{1}{3}\log k + \log(n+3) - \frac{1}{3}\log k + \frac{1}{2}\log(n+3) = \frac{3}{2}\log(n+3)$$

and

$$\log\left|\gamma^{(2)}\right| - \log\left|\gamma^{(3)}\right| \geq \frac{1}{3}\log k - \log(n+3) - \frac{1}{3}\log k - \frac{1}{2}\log(n+3) = -\frac{3}{2}\log(n+3),$$

the statement is proved.

If (x, y) is a solution of (2), then there exist $u_1,\, u_2 \in \mathbf{Z}$,

$$x - \lambda y = \gamma\lambda^{u_1}(\lambda^{(2)})^{u_2}, \tag{3}$$

where γ satisfies the conditions of the preceding corollary.

We may assume (see Thomas [T1]) that

$$\left|\frac{x}{y} - \lambda\right| \leq \left|\frac{x}{y} - \lambda^{(2)}\right|,\quad \left|\frac{x}{y} - \lambda^{(3)}\right|.$$

Applying Lemma 1 a first time, we get

$$|L_1| \leq \left(\frac{4k^4}{(n^2+n+7)^2}\right)^{1/6}\frac{1}{y^2},$$

using the fact that the discriminant of the form F_n is equal to $(n^2+n+7)^2$.

We suppose that $n > 0$ and
$$y \geq \max\{3, k, n\}.$$
Then the previous inequality implies $|L_1| \leq \frac{1}{7}$, thus
$$|L_2| \geq |\lambda^{(2)} - \lambda| - |L_1| > \frac{3}{4},$$
and a second application of Lemma 1 gives
$$|x - \lambda y| \leq \left(\frac{64k^4}{9(n^2+n+7)^2}\right)^{1/4} \frac{1}{|y|^2} \leq \frac{0.55}{y} \leq 0.19. \tag{4}$$
Notice also that, for $n \geq 3$,
$$\begin{aligned} y \leq \left|x - \lambda^{(2)} y\right| &\leq y\left(1 + \frac{2}{n^2}\right), \\ y\left(n + \frac{2}{n}\right) \leq \left|x - \lambda^{(3)} y\right| &\leq y\left(n + \frac{4}{n}\right). \end{aligned} \tag{5}$$

Let us give the proofs of these two pairs of inequalities. By the estimates of the preceding section, we have
$$\lambda^{(2)} - \lambda = -\frac{\lambda^{(3)}+1}{\lambda^{(3)}} + \frac{1}{\lambda^{(3)}+1} = -1 - \frac{1}{\lambda^{(3)}(\lambda^{(3)}+1)},$$
hence
$$1 + \frac{1}{(n+2)^2} < \left|\lambda^{(2)} - \lambda\right| < 1 + \frac{1}{n^2}.$$
This leads to the estimate
$$\begin{aligned} y &\leq \left(1 + \frac{1}{(n+2)^2}\right) y - \frac{0.55}{y} \leq |\lambda - \lambda^{(2)}| y - |x - \lambda y| \leq \left|x - \lambda^{(2)} y\right| \\ &\leq |\lambda - \lambda^{(2)}| y + |x - \lambda y| \leq \frac{0.55}{y} + \left(1 + \frac{1}{n^2}\right) y \leq y\left(1 + \frac{2}{n^2}\right). \end{aligned}$$
In the same way,
$$n + \frac{3}{n+1} < n + \frac{2}{n+1} + \frac{n+1}{n^2+n+2} < \lambda^{(3)} - \lambda = \lambda^{(3)} + \frac{1}{\lambda^{(3)}+1} < n + \frac{2}{n} + \frac{n}{n^2+2} < n + \frac{3}{n},$$
which implies
$$\left(n + \frac{2}{n}\right) y < \left|x - \lambda^{(3)} y\right| < \left(n + \frac{4}{n}\right) y.$$
Now, using (5), we can refine (4),
$$|x - \lambda y| = \frac{k}{|x - \lambda^{(2)} y| \cdot |x - \lambda^{(3)} y|} < \frac{k}{n y^2}. \tag{6}$$

Considering the conjugates of (3), we get Siegel's identity

$$\big(\lambda^{(3)}-\lambda\big)\gamma^{(2)}(\lambda^{(2)})^{u_1}(\lambda^{(3)})^{u_2}-\big(\lambda^{(2)}-\lambda\big)\gamma^{(3)}(\lambda^{(3)})^{u_1}\lambda^{u_2}=\big(\lambda^{(3)}-\lambda^{(2)}\big)\gamma\lambda^{u_1}(\lambda^{(2)})^{u_2}.$$

Dividing by $\big(\lambda^{(2)}-\lambda\big)\gamma^{(3)}(\lambda^{(3)})^{u_1}\lambda^{u_2}$ gives

$$\frac{\big(\lambda^{(3)}-\lambda\big)\gamma^{(2)}}{\big(\lambda^{(2)}-\lambda\big)\gamma^{(3)}}\left(\frac{\lambda^{(2)}}{\lambda^{(3)}}\right)^{u_1}\left(\frac{\lambda^{(3)}}{\lambda}\right)^{u_2}-1=\frac{\lambda^{(3)}-\lambda^{(2)}}{\lambda^{(2)}-\lambda}\,\frac{\gamma\lambda^{u_1}(\lambda^{(2)})^{u_2}}{\gamma^{(3)}(\lambda^{(3)})^{u_1}\lambda^{u_2}}.$$

The right-hand side is nonzero and is also equal to

$$\frac{\lambda^{(3)}-\lambda^{(2)}}{\lambda^{(2)}-\lambda}\,\frac{x-\lambda y}{x-\lambda^{(3)}y},$$

thus, by (6) and (5), its absolute value is

$$\leq\left|\frac{\lambda^{(3)}-\lambda^{(2)}}{\lambda^{(2)}-\lambda}\right|\frac{k}{(n^2+2)y^3}.$$

We notice that

$$\frac{\lambda^{(3)}-\lambda^{(2)}}{\lambda^{(2)}-\lambda}=\frac{\lambda^{(3)}+\frac{1}{\lambda^{(3)}+1}}{-\frac{\lambda^{(3)}+1}{\lambda^{(3)}}+\frac{1}{\lambda^{(3)}+1}}=-\lambda^{(3)}\cdot\frac{(\lambda^{(3)})^2+\lambda^{(3)}+1}{(\lambda^{(3)})^2+\lambda^{(3)}+1}=-\lambda^{(3)},$$

hence, the considered right-hand side is

$$\leq\frac{k}{ny^3}.$$

Now put

$$\Lambda=u_1\log\left|\frac{\lambda^{(2)}}{\lambda^{(3)}}\right|+u_2\log\left|\frac{\lambda^{(3)}}{\lambda}\right|+\log\left|\frac{\gamma^{(2)}}{\gamma^{(3)}}\right|+\log\left|\frac{\lambda^{(3)}-\lambda}{\lambda^{(2)}-\lambda}\right|.$$

Using the inequality $|\log z|<1.5\times|z-1|$, which is true for $|z-1|<1/2$, we get

$$0<|\Lambda|<\frac{3k}{2ny^3}.$$

The relation

$$\frac{\lambda^{(3)}-\lambda}{\lambda^{(2)}-\lambda}=\frac{-1-\frac{1}{\lambda}-\lambda}{-\frac{1}{\lambda+1}-\lambda}=\lambda+1=-\frac{1}{\lambda^{(2)}}$$

combined with $\lambda^{(3)}/\lambda=(\lambda^{(3)})^2\lambda^{(2)}$ gives

$$\Lambda=(u_1+u_2-1)\log\big|\lambda^{(2)}\big|+(2u_2-u_1)\log\big|\lambda^{(3)}\big|+\log\left|\frac{\gamma^{(2)}}{\gamma^{(3)}}\right|.$$

6. A first study of the linear form in logs

The previous linear form Λ can be written as

$$\Lambda = V\ell_2 + v\ell_3 + \log\left|\frac{\gamma^{(2)}}{\gamma^{(3)}}\right|,$$

where we have put

$$V = u_1 + u_2 - 1, \qquad v = -u_1 + 2u_2.$$

It satisfies

$$|\Lambda| < \frac{3k}{2ny^3}.$$

Grouping the last two terms we get a linear form of the type (we want to use the results of [LMN], so that we shall use the notations of this paper),

$$\Lambda = b_1 \log\alpha_1 + b_2\log\alpha_2,$$

where

$$b_1 = V, \quad b_2 = 1, \quad \alpha_1 = \lambda^{(2)}, \quad \alpha_2 = \left|(\lambda^{(3)})^v \cdot \gamma^{(2)}/\gamma^{(3)}\right|.$$

Concerning α_1, we have

$$\bigl|\log|\lambda^{(2)}|\bigr| \leq \frac{1}{n}, \qquad M(\alpha_1) = \bigl|\lambda^{(2)}\cdot\lambda^{(3)}\bigr|,$$

which implies

$$h(\alpha_1) = h(\lambda) \leq \tfrac{1}{3}\,\log(n+2).$$

By Corollary 2,

$$\log\Bigl(M\bigl(\gamma^{(2)}/\gamma^{(3)}\bigr)\Bigr) \leq \log k + \frac{5}{2}\log(n+3).$$

Thus,

$$h(\alpha_2) \leq \frac{|v|}{3}\log(n+2) + \frac{1}{3}\Bigl(\log k + \tfrac{5}{2}\log(n+3)\Bigr).$$

For the convenience of the reader, we quote the main result of [LMN].

Theorem A. *Let K and L be integers ≥ 3, R_1, S_1, R_2, S_2 positive integers. Let $\rho > 1$ be a real number. Put $R = R_1 + R_2 - 1$, $S = S_1 + S_2 - 1$,*

$$g = \frac{1}{4} - \frac{KL}{12RS}, \qquad b = \bigl((R-1)b_2 + (S-1)b_1\bigr)\left(\prod_{k=1}^{K-1} k!\right)^{-2/(K^2-K)}.$$

Let a_1, a_2 be real numbers such that

$$a_i \geq \rho|\log\alpha_i| - \log|\alpha_i| + 2Dh(\alpha_i), \quad i = 1,\ 2,$$

where $D = [\mathbf{Q}(\alpha_1,\alpha_2) : \mathbf{Q}]$. Suppose that

$$\begin{gathered} \#\bigl\{\alpha_1^r\alpha_2^s\,;\, 0 \leq r < R_1,\ 0 \leq s \leq S_1\bigr\} \geq L, \\ \#\bigl\{b_1 r + s b_2\,;\, 0 \leq r < R_2,\ 0 \leq s < S_2\bigr\} > (K-1)L \end{gathered} \tag{7}$$

and that

$$K(L-1)\log\rho + (K-1)\log 2 - (D+1)\log(KL) - D(K-1)\log b - gL(Ra_1 + Sa_2) > 0. \tag{8}$$

Then,

$$|\Lambda'| \geq \rho^{-KL+0.5}, \qquad \text{where} \qquad \Lambda' = \Lambda \cdot \max\left\{\frac{LRe^{LR|\Lambda|/(2b_1)}}{2b_1}, \frac{LSe^{LS|\Lambda|/(2b_2)}}{2b_2}\right\}.$$

In our case, we have $D = 3$. The study above shows that one can take

$$a_1 \geq 2\log(n+2) + (\rho - 1)/n$$

and

$$a_2 \geq \Big(2|v| + 5\Big)\log(n+3) + 2\log k + (\rho - 1)\ell_0,$$

where $\ell_0 = |\log\alpha_2|$. For further reference, we notice that (by Corollary 2)

$$\ell_0 \leq \left(|v| + \tfrac{3}{2}\right)\,\log(n+3).$$

The algebraic numbers α_1 and α_2 are multiplicatively independent. In fact, assume that they are not. Then there exist non-zero integers r and s such that

$$1 = \alpha_1^r\alpha_2^s = \lambda^{(2)r}\lambda^{(3)sv}(\gamma^{(2)}/\gamma^{(3)})^s.$$

Thus $\gamma^{(2)}/\gamma^{(3)}$ is a unit in $\mathbf{Z_K}$ and can be written as

$$\gamma^{(2)}/\gamma^{(3)} = \lambda^{(2)t_2}\lambda^{(3)t_3}$$

with suitable $t_2, t_3 \in \mathbf{Z}$. Combining the last two equations and using that $\lambda^{(2)}$ and $\lambda^{(3)}$ are multiplicatively independent we get $s(v + t_3) = 0$ or $t_3 = -v$. This implies

$$\Lambda = (V + t_2)\log|\lambda^{(2)}|,$$

and so either $\Lambda = 0$ or $|\Lambda| > \frac{1}{2n}$. We already noticed that Λ is nonzero, and the second alternative contradicts

$$|\Lambda| < \frac{3k}{2ny^3} < \frac{1}{2ny}.$$

The claim is proved.

Thus, in any case, the first one of conditions (7) holds if we suppose

$$R_1S_1 \geq L,$$

and the choices of these two parameters will satisfy this condition.

To satisfy the second of conditions (7), we take

$$R_2 = \left[\sqrt{(K-1)La_2/a_1}\right] + 1, \quad S_2 = \left[\sqrt{(K-1)La_1/a_2}\right] + 1,$$

so that $R_2S_2 > (K-1)L$.

Now, there are two possibilities: either the numbers $rb_1 + sb_2$, $0 \leq r < R_2$ and $0 \leq s < S_2$ are pairwise different and then the second of conditions (7) holds, or (since $b_2 = 1$) we have $|V| \leq S_2$.

We also take

$$K = [\mu^2 La_1a_2] + 1,$$

where $0 < \mu \leq 0.5$ and L will be chosen later. Now we study the term $g(Ra_1 + Sa_2)$. Since $R = R_1 + R_2 - 1$ and $S = S_1 + S_2 - 1$,

$$\begin{aligned} g(Ra_1 + Sa_2) &= \left(\frac{1}{4} - \frac{KL}{12RS}\right)(Ra_1 + Sa_2) \\ &\leq \frac{1}{4}\big((R_2-1)a_1 + (S_2-1)a_2\big) + \frac{1}{4}(R_1a_1 + S_1a_2) - \frac{KL}{12}\left(\frac{a_1}{(S_2-1)+S_1} + \frac{a_2}{(R_2-1)+R_1}\right). \end{aligned}$$

Then, the definitions of R_2 and S_2 imply

$$g(Ra_1 + Sa_2) = \frac{1}{2}\sqrt{KLa_1a_2} + \frac{1}{4}(R_1a_1 + S_1a_2) - \frac{KL}{12}\left(\frac{a_1}{\sqrt{KLa_1/a_2} + S_1} + \frac{a_2}{\sqrt{KLa_2/a_1} + R_1}\right).$$

Now we use the inequality

$$\frac{a_1}{\sqrt{KLa_1/a_2} + S_1} \leq \left(\frac{a_1}{\sqrt{KLa_1/a_2}}\right)\left(1 - \frac{S_1}{\sqrt{KLa_1/a_2}}\right) = \frac{\sqrt{a_1a_2}}{\sqrt{KL}} - \frac{a_2S_1}{KL},$$

and the similar one with R_1, and get

$$g(Ra_1 + Sa_2) \leq \frac{1}{3}\sqrt{KLa_1a_2} + \frac{1}{3}(a_1R_1 + a_2S_1).$$

Choosing

$$R_1 = \left[\sqrt{La_2/a_1}\right] + 1, \qquad S_1 = \left[\sqrt{La_1/a_2}\right] + 1$$

gives

$$g(Ra_1 + Sa_2) \leq \frac{1}{3}\sqrt{KLa_1a_2} + \frac{2\sqrt{La_1a_2}}{3} + \frac{a_1 + a_2}{3}.$$

By Lemma 9 of [LMN],

$$b \leq \frac{(R-1)b_2 + (S-1)b_1}{K-1} \times \exp\left\{\frac{3}{2} - \frac{\log(2\pi(K-1)/\sqrt{e})}{K-1} + \frac{\log K}{6K(K-1)}\right\},$$

where

$$(R-1)b_2 + (S-1)b_1 \leq \left(\sqrt{(K-1)La_2/a_1} + \sqrt{La_2/a_1} + 1\right) + \left(\sqrt{(K-1)La_1/a_2} + \sqrt{La_1/a_2} + 1\right)V.$$

These remarks show that condition (8) holds if

$$K(L-1)\log\rho+(K-1)\log 2-4\log(KL)-3(K-1)H_0-\frac{3}{2}(K-1)$$
$$+\log\bigl(\pi(K-1)/\sqrt{e}\bigr)-\frac{\log K}{6K}-\frac{L}{3}\sqrt{KLa_1a_2}-\frac{2L^{3/2}a_1a_2}{3}-\frac{L(a_1+a_2)}{3}\geq 0,$$

where we have put

$$H_0=\log\left(\frac{\sqrt{La_2/a_1}\bigl(\sqrt{K-1}+1\bigr)+1+\Bigl(\sqrt{La_1/a_2}\bigl(\sqrt{K-1}+1\bigr)+1\Bigr)V}{K-1}\right).$$

We have

$$\begin{aligned}
&\frac{\sqrt{La_2/a_1}\bigl(\sqrt{K-1}+1\bigr)+1+\Bigl(\sqrt{La_1/a_2}\bigl(\sqrt{K-1}+1\bigr)+1\Bigr)V}{K-1}\\
&\leq\frac{K}{K-1}\times\left(\bigl(1+1/\sqrt{K}\bigr)\frac{\sqrt{La_2/a_1}+V\sqrt{La_1/a_2}}{\sqrt{K}}+\frac{V+1}{K}\right)\\
&\leq\frac{K}{K-1}\times\left(\Bigl(1+\frac{1}{\sqrt{K}}\Bigr)\frac{1}{\mu}\Bigl(\frac{1}{a_1}+\frac{V}{a_2}\Bigr)+\frac{V+1}{\mu^2La_1a_2}\right)\\
&\leq\frac{K}{K-1}\times\frac{1}{\mu}\times\left(\frac{1}{a_1}+\frac{V}{a_2}\right)\left(1+\frac{1}{\sqrt{K}}+\frac{1}{\mu L}\Bigl(\frac{1}{a_1}+\frac{1}{a_2}\Bigr)\right).
\end{aligned}$$

We put $\theta=\log\rho$. Choosing $\mu=2/(3\theta)$, and using the definition $K=[\mu^2La_1a_2]+1$, we get

$$H_0\leq H:=\log\left(\frac{K}{K-1}\right)+\log\left(1+\frac{1}{\sqrt{K}}+\frac{3\theta}{2L}\Bigl(\frac{1}{a_1}+\frac{1}{a_2}\Bigr)\right)+\log^+\left(\frac{1}{\mu a_1}+\frac{V}{\mu a_2}\right),$$

where, as usual, $\log^+x=\max\{0,\log x\}$. Then, we see that condition (8) is satisfied when

$$\Phi:=\frac{2}{3\theta}\left(\frac{2(L-1)}{3}-\frac{2H}{\theta}-\frac{L}{3}+\frac{2\log 2}{3\theta}\right)-\frac{7\log L}{a_1a_2L}-\frac{3\log\bigl(4a_1a_2/(9\theta^2)\bigr)}{a_1a_2L}-\frac{2\sqrt{L}}{3\sqrt{a_1a_2}}-\frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\geq 0.$$

[We divided by La_1a_2.] Now, we choose

$$L=\max\bigl\{16,[\varepsilon+6H/\theta]+3\bigr\},$$

where $\varepsilon>0$ is to be chosen later. This choice implies

$$\begin{aligned}
\Phi&\geq\frac{4\log 2}{9\theta^2}-\frac{7\log L}{a_1a_2L}-\frac{3\log\bigl(4a_1a_2/(9\theta^2)\bigr)}{a_1a_2L}-\frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)+\frac{4\varepsilon L}{9\theta}-\frac{2\sqrt{L}}{3\sqrt{a_1a_2}}\\
&\geq\frac{4\log 2}{9\theta^2}-\frac{1.22}{a_1a_2}-\frac{3\log\bigl(4a_1a_2/(9\theta^2)\bigr)}{16a_1a_2}-\frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)+\sqrt{L}\left(\frac{4\varepsilon\sqrt{L}}{9\theta}-\frac{2}{3\sqrt{a_1a_2}}\right).
\end{aligned}$$

Thus, $\Phi\geq 0$ when both

$$\Theta:=\frac{4\log 2}{9\theta^2}-\frac{1.22}{a_1a_2}-\frac{3\log\bigl(4a_1a_2/(9\theta^2)\bigr)}{16a_1a_2}-\frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\geq 0$$

and

$$\varepsilon\geq\frac{3\theta}{8\sqrt{a_1a_2}}.$$

We take $\rho = 15$, and in this case $\Theta \geq 0$ for

$$a_2 \geq a_1 \geq 15, \qquad a_2 \geq 24, \qquad \varepsilon = 0.054. \tag{9}$$

Then (8) holds, and we get

$$\log|\Lambda'| \geq -(\mu^2 L^2 a_1 a_2 - L)\theta = -\frac{4}{9\theta} L^2 a_1 a_2 - L\theta.$$

The choices of R and S imply $\max\{R, S\} \leq L \max\{a_1, a_2\}$, and the definition of Λ' leads to

$$\log|\Lambda'| \leq \log|\Lambda| + 2\log L + \log\bigl(\max\{a_1, a_2\}\bigr),$$

except maybe when

$$|\Lambda| \geq \frac{1}{L^2 \cdot \max\{a_1, a_2\}}.$$

Hence, we always have

$$\log|\Lambda| \geq -\frac{4}{9\theta} L^2 a_1 a_2 - L\theta - 2\log L - \log\bigl(\max\{a_1, a_2\}\bigr) \geq -\left(\frac{4}{9\theta} L^2 + \frac{0.01}{a_1}\right) a_1 a_2 - (\theta + 0.35)L.$$

Recall that

$$\log|\Lambda| \leq \log\frac{3k}{2n} - 3\log y.$$

Thus, under condition (9), we get

$$\log y \leq \frac{1}{3}\left(\frac{4}{9\theta} + \frac{0.01}{a_1}\right) L^2 a_1 a_2 + \frac{\theta + 0.35}{3} L + \frac{1}{3}\log\frac{3k}{2n}.$$

7. A second study of linear forms and the proof of Theorem 1

Applying conjugations to relation (3) and taking logs, we get the system of relations

$$\begin{cases} u_1\ell_2 + u_2\ell_3 = \log y + \eta_2 \\ u_1\ell_3 + u_2\ell_1 = \log y + \eta_3, \end{cases}$$

where

$$\eta_2 = \log\left|\frac{x}{y} - \lambda^{(2)}\right| - \log|\gamma^{(2)}|$$

and

$$\eta_3 = \log\left|\frac{x}{y} - \lambda^{(3)}\right| - \log|\gamma^{(3)}|.$$

By (5),

$$0 \le \log\left|\frac{x}{y} - \lambda^{(2)}\right| \le \frac{2}{n^2},$$

and

$$\log n \le \log\left|\frac{x}{y} - \lambda^{(3)}\right| \le \log(n+1).$$

Moreover, the choice of γ implies

$$\frac{k^{1/3}}{\sqrt{n+3}} \le |\gamma^{(2)}| \le k^{1/3}\sqrt{n+3}, \qquad \frac{k^{1/3}}{(n+3)} \le |\gamma^{(3)}| \le k^{1/3}(n+3).$$

These estimates lead to

$$-\frac{1}{2}\log(n+3) - \frac{1}{3}\log k \le \eta_2 \le \frac{2}{n^2} - \frac{1}{3}\log k + \frac{1}{2}\log(n+3)$$

and

$$-\log(n+3) + \log n - \frac{1}{3}\log k \le \eta_3 \le -\frac{1}{3}\log k + 2\log(n+3).$$

The above system implies

$$\begin{aligned} u_1(\ell_3^2 - \ell_1\ell_2) &= (\ell_3 - \ell_1)\log y - \ell_1\eta_2 + \ell_3\eta_3, \\ u_2(\ell_3^2 - \ell_1\ell_2) &= (\ell_3 - \ell_2)\log y + \ell_3\eta_2 - \ell_2\eta_3. \end{aligned}$$

Hence,

$$V = u_1 + u_2 - 1 = \frac{1}{\ell_3^2 - \ell_1\ell_2}\Big((2\ell_3 - \ell_1 - \ell_2)\log y + (\ell_3 - \ell_1)\eta_2 + (\ell_3 - \ell_2)\eta_3\Big) - 1.$$

Put

$$\Delta = \ell_3^2 - \ell_1\ell_2,$$

then

$$\Delta(V+1) - (2\ell_3 - \ell_1 - \ell_2)\log y = (\ell_3 - \ell_1)\eta_2 + (\ell_3 - \ell_2)\eta_3.$$

Using the estimates of the ℓ_i's (end of Section 3) and those of η_2 and η_3, we get

$$\begin{aligned} &\Big|\Delta(V+1) - (2\ell_3 - \ell_1 - \ell_2)\log\big(y \,/\, k^{1/3}\big) - (\ell_3 - \ell_2)\log(n+3)\Big| \\ &\quad \le \Big((\ell_3 - \ell_2)\log\big((n+3)^2/n\big) + (\ell_3 - \ell_1)\log(n+3)\Big) \\ &\quad \le \Big(\Big(\log n + \frac{2}{n^2}\Big)\log\big((n+3)^2/n\big) + 2\Big(\log(n+1) + \frac{2}{n^2}\Big)\log(n+3)\Big) \\ &\quad \le 3\,\log(n+2) \times \log(n+3) \end{aligned}$$

for $n \geq 5$. Notice that

$$\Delta \geq (\log n)^2 + \frac{\log(n+1)}{n+1} > \log^2 n$$

and

$$2\ell_3 - \ell_1 - \ell_2 < 2\log n + \frac{4}{n^2} + \log(n+1) + \frac{2}{n(n+1)} - \frac{n - \frac{1}{2}}{n^2+2} \leq 3\,\log(n+1),$$

and also

$$2\ell_3 - \ell_1 - \ell_2 > 2\log n + \frac{4}{(n+1)^2} + \log(n+1) + \frac{2}{(n+1)^2} - \frac{1}{n} > 3\,\log n,$$

for $n \geq 3$.

We notice that this implies $V \geq 0$ for $n \geq 5$ and,

$$\begin{aligned} V + 1 &\leq \frac{1}{\Delta}\Big(3\log(n+1)\log\frac{y}{k^{1/3}} + 4\,\log(n+2) \times \log(n+3)\Big) \\ &\leq \frac{\log(n+2)}{\log^2 n}\Big(3\log\big(y/k^{1/3}\big) + 4\log(n+3)\Big), \end{aligned}$$

for $n \geq 5$.

After this list of estimates, let us bound V. Recall that

$$\log y \leq \frac{1}{3}\left(\frac{4}{9\theta}+0.01\right)L^2 a_1 a_2 + \frac{\theta+0.35}{3}L + \frac{1}{3}\log\frac{3k}{2n}.$$

By the conditions on a_1 and a_2, and since $\rho = 15$ and $L \geq 16$ this implies

$$\log y \leq \frac{1}{3}0.17423 \times L^2 a_1 a_2 + \frac{1}{3}\log\frac{3k}{2n}.$$

Since $a_1 \geq 2\log(n+2)$, we get (for $n \geq 5$)

$$V+1 \leq \frac{\log(n+2)}{\log^2 n}\big(0.17423 \times L^2 a_1 a_2 + 4\log(n+3)\big) \leq \frac{\log(n+2)}{\log^2 n} \times 0.17456 \cdot L^2 a_1 a_2.$$

We consider the two cases

$$L = \begin{cases} 16, \\ [6H/\theta+\varepsilon]+3 \geq 17. \end{cases}$$

In the first case,

$$V+1 \leq \frac{\log(n+2)}{\log^2 n} \times 44.688 \cdot a_1 a_2.$$

Now suppose $L \geq 17$. The definition of H implies

$$H \leq 1.522 + \log^+\left(\frac{|V|}{a_2}\right),$$

and the definition of L, combined with $L \geq 17$, implies

$$\log(V/a_2) \geq 4.77.$$

Then, the inequality

$$\frac{V}{a_2} \leq \frac{\log(n+2)}{\log^2 n} \times 0.17456 \cdot L^2 a_1 \leq 0.17456 \times \frac{a_1\log(n+2)}{\log^2 n}\left[3.054 + \frac{6\big(1.522+\log(V/a_2)\big)}{\theta}\right]^2$$

leads to a contradiction. Thus we have proved that

$$V+1 \leq \frac{\log(n+2)}{\log^2 n} \times 44.688 \cdot a_1 a_2,$$

and that

$$L = 16.$$

We have just obtained an upper bound for $|V|$, using a non trivial lower bound of Λ. Using again Λ, we directly get a lower bound for $|V|$. Indeed,

$$\Lambda = V\ell_2 \pm \ell_0,$$

where $|\ell_2| < (n+1)/(n^2+n+2)$; since $|\Lambda|$ is very small, this implies

$$V \geq n\ell_0.$$

Suppose that $n \geq 1650$, then (in this case $V \geq 0$)

$$V + 1 \leq \frac{\log(n+2)}{\log^2 n} \times 44.688 \cdot a_1 a_2 \leq \frac{44.696}{\log n} \cdot a_1 a_2,$$

where

$$a_1 = \max\bigl\{2\log(n+2) + 14/n, 15\bigr\} < 2.03 \log n,$$

and

$$a_2 = \max\left\{\left(2|v| + \frac{5}{2}\right)\log(n+3) + \log k + 14\ell_0, 24\right\}.$$

Combining these estimates gives

$$V < 89.5\, a_2 \qquad \text{for} \;\; n \geq 1650.$$

If $v \neq 0$ then $a_2 = \left(2|v| + \frac{5}{2}\right)\log(n+3) + \log k + 14\ell_0$ and

$$n\ell_0 < 89.5\left(\left(2|v| + \frac{5}{2}\right)\log(n+3) + \log k + 14\ell_0\right).$$

Hence, if $v \neq 0$,

$$(n - 1253)\ell_0 < \bigl(179|v| + 223.75\bigr)\log(n+3) + 89.5\log k$$

for $n \geq 1650$. Since

$$\ell_0 \geq |v|\log n - \frac{3}{2}\log(n+3) > (|v| - 1.501)\log n,$$

for $n \geq 1650$, we get also

$$(n - 1253 - 179.1)|v| < \bigl(1.501\,(n-1253) + 223.75\bigr) + 89.5\frac{\log k}{\log n},$$

or,

$$|v| < \left(1 - \frac{1432.1}{n}\right)^{-1}\left\{1.501 - \frac{1657}{n} + 89.5\,\frac{\log k}{n\,\log n}\right\}.$$

Now we consider the y term and we suppose $n \geq 1650$. We have

$$\log y \leq \frac{1}{3} 0.17423 \times L^2 a_1 a_2 + \frac{1}{3} \log \frac{3k}{2n},$$

with $a_1 < 2.03 \log n$ and $L = 16$, hence

$$\log y \leq 29.77\, a_2 \log n + \frac{1}{3} \log k.$$

Using the inequality $\ell_0 \leq |v| \log(n+1) + 1.5 \log(n+3)$ and the definition of a_2, we get

$$\log y < 29.77\,(16|v| + 23.5)\ \log^2(n+2) + 29.82\ \log n \times \log k.$$

And the upper bound on $|v|$ gives

$$\log y < \left(700 + 476.4 \left(1 - \frac{1432.1}{n}\right)^{-1} \left\{1.501 - \frac{1902}{n}\right\}\right) \log^2(n+2)$$
$$+ \left(29.82 + \left(1 - \frac{1432.1}{n}\right)^{-1} \frac{1432}{n\ \log n}\right) \log n \times \log k.$$

Theorem 1 is proved.

8. An improved bound, if k is small

In this section we add some hypothesis on k and then get an upper bound for ℓ_0, which leads to upper bounds on $|v|$ and y.

We suppose $n \geq 1650$ and

$$k \leq n^4.$$

Then, the upper bound on v becomes

$$|v| < \left(1 - \frac{1432.1}{n}\right)^{-1} \left\{1.501 - \frac{1657}{n} + \frac{358}{n}\right\},$$

which implies

$$|v| \leq 1 \qquad \text{for} \quad n \geq 2750.$$

Using the definition of a_2, we also obtain

$$\ell_0 < \max\left\{\frac{761 \log n}{n - 1253}, \frac{2148}{n}\right\} = \frac{761 \log n}{n - 1253} \qquad \text{for} \quad n \geq 3150.$$

This implies

$$\ell_0 < \frac{\log n}{n^{1/3}} \qquad \text{for} \quad n \geq 23500.$$

In this case, we choose

$$\rho = n^{1/3};$$

then $\mu = 2/\log n$. Since $a_1 > 2 \log n$ and $a_2 > 2.5 \log n$, taking again $L = 16$, we have $K \geq 360$. It is also easy to verify that

$$H < 4.67 + \log\log n$$

and that we can take $\varepsilon = 0.17$ (the number ε occurs in the definition of the integer L). These estimates prove that the choice $L = 16$ is legitimate. Then applying again Theorem A, for $n \geq 23500$, we get

$$\begin{aligned}
\log y &\leq \frac{1}{3}\left(\frac{4}{9\theta} + \frac{0.01}{a_1}\right) L^2 a_1 a_2 + \frac{\theta + 0.35}{3} L + \frac{1}{3} \log \frac{3k}{2n} \\
&\leq \frac{1}{3}\left(\frac{4}{3} + 0.005\right) 256 \times 2.02 \times 9.5 \times \log n + \frac{\theta + 0.35}{3} L + \frac{1}{3} \log \frac{3k}{2n}.
\end{aligned}$$

Thus, we have proved that

$$\log y < 2180 \log n \qquad \text{if} \quad n \geq 23500 \quad \text{and} \quad k \leq n^4.$$

9. Proof of Theorem 3.

Let us consider the inequality

$$|x^3-(n-1)x^2y-(n+2)xy^2-y^3|\le k$$

with $k=2n+1$. The family of equations $|F_n(x,y)|=1$ was completely solved by Thomas [T1] and Mignotte [M]. Moreover, it was proved by Lemmermeyer and Pethő [LP] that if $\gamma\in \mathbf{Z}[\lambda]$ has norm, in absolute value, less then $2n+1$, then γ is a rational integer multiple of a binomial unit from $\mathbf{Z}[\lambda]$. Thus to prove Theorem 3 it remains to study the equation

$$|x^3-(n-1)x^2y-(n+2)xy^2-y^3|=2n+1. \tag{10}$$

It is easy to see that

$$F_n(1,1)=F_n(1,-2)=F_n(1,-n-1)=F_n(n,1)=F_n(-n-1,n)=F_n(-2,1)=-(2n+1),$$

and

$$F_n(-1,-1)=F_n(-1,2)=F_n(-1,n+1)=F_n(-n,-1)=F_n(n+1,-n)=F_n(2,-1)=2n+1.$$

The continued fraction expansion of λ is

$$\lambda=\left[-1,1,n,\left[\frac{n}{2}\right],\dots\right],$$

if $n>0$, and hence its first convergents are

$$-1,\quad 0,\quad -\frac{1}{n+1},\quad \frac{-\left[\frac{n}{2}\right]}{(n+1)\left[\frac{n}{2}\right]+1}.$$

Let $(x,y)\in\mathbf{Z}^2$ be a solution of (10), such that $y\ge 1$ and

$$|\lambda y-x|<|\lambda^{(2)}y-x|,\ |\lambda^{(3)}y-x|.$$

By Corollary 1, and by $D_f=(n^2+n+7)^2$ we have

$$|L_1|\le\left(\frac{4(2n+1)^4}{(n^2+n+7)^2}\right)^{1/6}\frac{1}{y^2}<\frac{2}{y^2},$$

thus

$$\left|\frac{x}{y}-\lambda\right|<\frac{2}{25},$$

if $|y|\ge 5$. Using the estimations for λ, $\lambda^{(2)}$ and $\lambda^{(3)}$ we get

$$|x-\lambda^{(2)}y|>\left(|\lambda^{(2)}-\lambda|-\frac{2}{25}\right)|y|>\left(1-\frac{2}{25}\right)|y|=\frac{23}{25}|y|$$

and

$$|x-\lambda^{(3)}y|>\left(|\lambda^{(3)}-\lambda|-\frac{2}{25}\right)|y|>\left(n-\frac{2}{25}\right)|y|.$$

Therefore

$$\left|\frac{x}{y}-\lambda\right|<\frac{2n+1}{\frac{23}{25}\left(n-\frac{2}{25}\right)|y|^3}<\frac{1}{2y^2}$$

if $|y| \geq 5$ and $n \geq 3$, which means that in those cases $\frac{x}{y}$ is a convergent of λ. We conclude the same for $n = 1$ when $|y| \geq 8$ and for $n = 2$ when $|y| \geq 6$.

If $n = 0$, then (10) is already solved, see Thomas [T1] and Mignotte [M]. For $n = 1$ with $|y| \leq 7$ and $n = 2$ with $|y| \leq 5$ we get by a direct computation the following solutions.

n	1	2	2	2	2	2	2
x	-1	-1	-3	-4	-1	8	-1
y	2	2	2	3	3	3	4
$f_n(x,y)$	3	5	-5	5	5	5	-5

We remark that exactly the values given in the table serve the non-trivial solutions for $n = 2$.

If $n \geq 3$ and $4 \geq |y| \geq 2$ then $|\lambda| < \frac{2}{9}$ and by using

$$|x - \lambda y| < \frac{2}{|y|}$$

we get

$$-2 < -\frac{2}{|y|} - |\lambda||y| < |x| < \frac{2}{|y|} + |\lambda||y| < 2,$$

but the solutions with $|x| \leq 1$ are listed at the beginning. We did the same for $|y| \leq 1$. Thus apart from the cases listed in the table, the solutions of (10) are coming from convergents of λ, hence $|y| \geq n + 1$. If $y = n + 1$, then

$$F_n(x, n+1) - (2n+1) = (x+1)\big(x^2 - n^2 x - (n^3 + 3n^2 + 5n + 2)\big).$$

If the second factor $q(x)$ is reducible over $\mathbf{Q}[x]$, then it has an integer zero. But a simple calculation shows, that $q(-n-1) < 0$, $q(-n-3) > 0$ and $q(-n-2) = 0$ iff $n = 2$. Thus $q(x)$ is reducible over $\mathbf{Q}[x]$ iff $n = 2$ and these cases are listed in table 1.

Let $q_1(x) = F_n(x, n+1) + 2n + 1$. Then by $q_1(0) < 0$, $q_1(-1) > 0$, $q_1(-n-2) = n^2 + 3n > 0$ and

$$\begin{aligned} q_1(-n-3) &= -n^3 - 3n^2 - 5n - 12 < 0, \\ q_1(n^2+n+1) &= -(n^4 + 4n^3 + 6n^2 + 5n + 1), \\ q_1(n^2+n+2) &= -n^3 + n^2 + n + 7, \\ q_1(n^2+n+3) &= n^4 + 2n^3 + 12n^2 + 13n + 29 \end{aligned}$$

we see that $q_1(x)$ has no integer zero if $n \geq 0$. Thus we may assume $y > n+1$, whence the continued fraction expansion of λ implies $y \geq (n+1)[\frac{n}{2}] + 1 > \frac{n^2}{2}$.

By Theorem 3.1 of Lemmermeyer and Pethő [LP], all elements of $\mathbf{Z}[\lambda]$ of norm $2n+1$ are associated to one of the conjugates of $\lambda - 1$. It is easy to see that $\lambda^{(2)} - 1 = -\frac{\lambda+2}{\lambda+1}$ and $\lambda^{(3)} - 1 = -\frac{2\lambda+1}{\lambda}$ and that $\lambda - 1, \lambda + 2$ and $2\lambda + 1$ satisfy the conclusion of Corollary 2. Hence we may apply the results of sections 6 and 7.

First we prove that if $n \geq 1700$ then (10) has only the trivial solutions. We distiguish three cases and adopt the notations of section 6. We remember that if $n \geq 5$ then $V \geq 0$ and so $v \leq 1$.

Case I. $\gamma = \gamma_1 = \lambda - 1$. Now we have

$$\alpha_2 = \lambda^{(3)v} \left| \frac{\lambda^{(2)} - 1}{\lambda^{(3)} - 1} \right|, \quad h(\alpha_2) \leq \frac{1}{3}(|v| \log(n+2) + \log(n(2n+1))).$$

By the choice $\rho = 15$ we have also $a_2 = 14\ell_0 + 2(|v| + 2)\log(n+2) + 2\log 2$, where

$$|v-1| \log \lambda^{(3)} + \log 2 + \frac{1}{2(n-1)} \leq \ell_0 \leq |v-1| \log \lambda^{(3)} + \log 2 + \frac{3}{2(n-1)}.$$

As $V \geq 0$ and $v \leq 1$ we have $V > n\ell_0 > n(|v-1| \log\, n + \log 2)$. On the other hand

$$V < 89.5a_2 \leq 1253\ell_0 + 179(|v| + 2)\log(n+2) + 179 \log 2.$$

Combining these inequalities we get $n < 8450$ for $v = 1$ and $n < 1620$ for $v < 1$.

Case II. $\gamma = \gamma_2 = \lambda + 2$. We now transform

$$\alpha_2 = \lambda^{(3)v} \left| \frac{\lambda^{(2)} + 2}{\lambda^{(3)} + 2} \right|$$

to a more appropriate algebraic number to prove a sharp bound for n. Indeed, using $\lambda^{(2)} + 2 = (\lambda - n)\lambda$ and $\lambda^{(3)} + 2 = (\lambda^{(2)} - n)\lambda^{(2)}$ we get

$$\frac{\lambda^{(2)} + 2}{\lambda^{(3)} + 2} = \frac{(\lambda - n)\lambda(\lambda+1)^2}{n(\lambda+1)+1} = \frac{n - \lambda}{n(\lambda+1)+1} \frac{1}{\lambda^{(3)}\lambda^{(2)3}}.$$

Thus our linearform becomes

$$\Lambda = (V-3)\log|\lambda^{(2)}| + (v-1)\log|\lambda^{(3)}| + \log\left|\frac{n-\lambda}{n(\lambda+1)+1}\right|.$$

By the estimates of λ, we also have

$$1 - \frac{2}{n^3} < \frac{n-\lambda}{n+n\lambda+1} = 1 - \frac{(n+1)\lambda+1}{n(\lambda+1)+1} < 1 - \frac{1}{n^3}.$$

We proved $|\Lambda| < \frac{3k}{2ny^3}$ in section 5 for all $n \geq 3$ and $k \in \mathbf{N}$. With our special value $k = 2n+1$ using the estimate $y > \frac{n^2}{2}$ we get

$$|\Lambda| < \frac{12(2n+1)}{n^7} < \frac{1}{n^3}$$

if $n \geq 4$ immediately. This inequality remains true for $n = 3$ too, which one can verify by a direct computation. Hence $v = 1$ is impossible and if $v < 1$ then

$$|(V-3)\log|\lambda^{(2)}|| > |v-1|\log|\lambda^{(3)}| - |\Lambda| - \left|\log\frac{n-\lambda}{n+n\lambda+1}\right| > |v-1|\log n,$$

which implies

$$V - 3 > |v - 1| n \log n.$$

On the other hand

$$V < 89.5 a_2$$

with

$$a_2 = (16|v| + 18)\log(n+2) + 2\log 2.$$

These inequalities imply again

$$n < 1620.$$

Case III. $\gamma = \gamma_3 = 2\lambda + 1$. This case is similar to case I, and therefore we have given only the most important data:

$$\alpha_2 = \lambda^{(3)v} \left| \frac{2\lambda^{(2)} + 1}{2\lambda^{(3)} + 1} \right|, \quad h(\alpha_2) \leq \frac{1}{3}((|v| + 2)\log(n+2) + 2\log 2)$$

and

$$|v - 1| \log\ n - \log 2 < \ell_0 < |v - 1| \log(n+2) - \log 2.$$

Thus

$$a_2 < (16|v| + 18)\log(n+2) - 10\log 2.$$

If $v = 1$, then $n < 8640$ and if $v < 1$, then $n < 1700$.

We have proved that equation (10) has just the "trivial solutions" for $n > 8640$; moreover $v = 1$ if $1700 \leq n \leq 8640$ and this can occur only in cases I and III. We shall prove now the impossibility of $v = 1$.

From $|\Lambda| < \frac{12(2n+1)}{n^7}$ we get

$$\left| V + v \frac{\log|\lambda^{(3)}|}{\log|\lambda^{(2)}|} + \frac{\log|\gamma_i^{(2)}| - \log|\gamma_i^{(3)}|}{\log|\lambda^{(2)}|} \right| < \frac{1}{n^5}$$

for $n \geq 30$ and $i = 1, 2, 3$. Putting

$$\delta_1 = \frac{\log|\lambda^{(3)}|}{\log|\lambda^{(2)}|} \quad \text{and} \quad \delta_{2i} = \frac{\log|\gamma_i^{(2)}| - \log|\gamma_i^{(3)}|}{\log|\lambda^{(2)}|}$$

this can be reformulated to

$$||v\delta_1 + \delta_{2i}|| < \frac{1}{n^5}, \tag{11}$$

$i = 1, 2$ or 3, where $||\ \ ||$ denotes the distance to the nearest integer.

For $1700 \leq n \leq 8640$ and $i = 1$ and 3 we tested (11) with $v = 1$, but did not found any n satisfying (11) even with the much larger right hand side 10^{-5}. We performed the same test with $v = 1$ for the values $1 \leq i \leq 3$ and $0 \leq n < 1700$ with the same result. Thus it remains to prove Theorem 3 only in the range $0 \leq n \leq 1700$ and with $v \neq 1$.

10. The case $0 \le n \le 1700$ and finish the proof of Theorem 3.

In the range $0 \le n \le 1700$ application of linear forms in two logarithms fails to work because we are not able to prove an upper bound for $|v|$ and V. Therefore we apply the following more general theorem of Baker and Wüstholz [BW]:

Theorem B. *For a linear form $\Lambda \neq 0$ in logarithms of m real algebraic numbers $\alpha_1, \dots, \alpha_m$ with rational integer coefficients $b_1, \dots, b_m$ we have*

$$\log|\Lambda| \ge -18(m+1)!m^{m+1}(32D)^{m+2}h(\alpha_1)\dots h(\alpha_m)\log B,$$

where $B = \max\{|b_1|, \dots |b_m|\}$, and where D is the degree of the number field generated by $\alpha_1, \dots, \alpha_m$.

In the actual case we have $m = 3, D = 3$,

$$h(\lambda^{(2)}) = h(\lambda^{(3)}) \le \frac{1}{3}\log(n+2), \quad h\left(\frac{\gamma_i^{(2)}}{\gamma_i^{(3)}}\right) \le \frac{1}{3}\log\big(n(2n+1)\big) \quad i = 1,2,3, \quad B = V,$$

and we get

$$\begin{aligned}\log|\Lambda| &\ge -18 \cdot 4! \cdot 3^6 \cdot 32^5 \cdot \log^2(n+2)\log\big(n(2n+1)\big)\log V \\ &> -1.06 \cdot 10^{13}\log^2(n+2)\log(n(2n+1))\log V.\end{aligned}$$

On the other hand, if $|y| \ge 2$ and $n \ge 5$ then

$$\log|\Lambda| < -(V+1)\frac{\log^2 n}{\log(n+2)} + 4\log(n+3) - \log n + \log\frac{3}{2}.$$

A comparison of these estimates leads to

$$V < 10^{17}.$$

This bound is much worse than that we got for $n > 1650$ and demonstrates that the estimates for linear forms in two logarithms are much more suitable for solving diophantine problems completely than the general estimates. Unfortunately, estimates for two logs are not always applicable.

Despite the large bound for V we can prove that our problem has no non-trivial solution in the actual range too. For this purpose we use a variant of the Baker-Davenport lemma [BD], which was helpful in [MPR] too. We remember that

$$|\Lambda_i| = |V + v\delta_1 + \delta_{2i}| < \frac{3(2n+1)}{2ny^3\log|\lambda^{(2)}|} < \exp\left\{-(V+1)\frac{\log^2 n}{\log(n+2)} + 4\log(n+3) + \log 3\right\} \tag{12}$$

$(i = 1,2,3)$, where the numbers δ_1 and δ_{2i} were defined in the previous section.

Lemma 4. *Suppose that $n \ge 5$, $1 \le i \le 3$, $V < 10^{17}$, $v \ne 1$ and*

$$|\Lambda_i| < 10^{-50}$$

and that $\tilde{\delta}_1$ and $\tilde{\delta}_{2i}$ are rational numbers such that

$$|\delta_1 - \tilde{\delta}_1| < 10^{-50}, |\delta_{2i} - \tilde{\delta}_{2i}| < 10^{-50}.$$

If there exists a convergent p/q in the continued fraction expansion of $\tilde{\delta}_1$ such that $q \le 10^{25}$ and

$$q||q\tilde{\delta}_{2i}|| > \frac{5 \cdot 10^{17}}{n\log n}$$

then (12) *cannot hold for* $v, V \in \mathbf{Z}$.

Proof Assume that there exist $v, V \in \mathbf{Z}$ which satisfy (12) and $|\Lambda_i| < 10^{-50}$. We remember that (12) implies $V > |v-1|n(\log n - \log 2)$. Let us fix i.

Let p/q be a convergent of $\tilde{\delta}_1$ with the properties given in the lemma. Multiplying (12) by q and inserting δ_1 and δ_{2i} we get

$$q|\Lambda_i| = |q\tilde{\delta}_2 + q(\delta_{2i} - \tilde{\delta}_{2i}) + v(q\tilde{\delta}_1 - p) + vq(\delta_1 - \tilde{\delta}_1) + vp + Vq| < 10^{-25}.$$

Thus

$$||q\tilde{\delta}_{2i}|| < 2 \cdot 10^{-25} + |v||q\tilde{\delta}_1 - p| + |v|10^{-25}$$

and

$$q||q\tilde{\delta}_{2i}|| < 2 + |v|q|q\tilde{\delta}_1 - p| + |v| < \frac{5V}{n \log n} < \frac{5 \cdot 10^{17}}{n \log n},$$

which is a contradiction. The lemma is proved.

As $|v-1| \neq 0$ we have $V > n \log(n/2)$, which implies $|\Lambda_i| < 10^{-50}$ for $n \geq 25$ immediately. The same is true for $n \geq 8$ and $|v| > 10$. In these cases we computed δ_1 and δ_{2i} with 50 decimal digit precision and found for all n, where $8 \leq n \leq 1700$, a convergent p/q of $\tilde{\delta}_1$ with the properties given in the lemma. For those values (12) can not hold. We tested (11) for $8 \leq n \leq 25$ and $|v| \leq 10$ separately.

In the remaining cases $0 < n \leq 7$ we used the method from Pethő und Schulenberg [PSch], but did not found any more solutions. Thus, Theorem 3 is proved.

For the computation we used PARI-GP on a notebook. The total computation time took some hour.

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Maurice Mignotte
Département de Mathématique et Informatique
Université Louis Pasteur
7, rue René Descartes
67084 Strasbourg Cedex
France

Attila Pethő
Laboratory of Informatics
University of Medicine
Nagyerdei Krt. 98
H–4032 Debrecen
Hungary

Franz Lemmermeyer
Erwin-Rohde-Str. 19
D–69120 Heidelberg,
Germany