Acta Scientiarum Mathematicarum manuscript No.
(will be inserted by the editor)

Multiple common expansions in non-integer bases

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Version of January 23, 2016

Abstract We investigate the existence of simultaneous representations of real numbers x in bases $1 < q_1 < \cdots < q_r$, $r \geq 2$ with a finite digit set $A \subset \mathbb{R}$. We prove that if A contains both positive and negative digits, then each real number has infinitely many common expansions. In general the bases depend on x. If A contains the digits $-1, 0, 1$, then there exist two non-empty open intervals I, J such that for any fixed $q_1 \in I$ each $x \in J$ has common expansions for some bases $q_1 < \cdots < q_r$.

Keywords simultaneous Rényi expansions · interval filling sequences

Mathematics Subject Classification (2000) 11A63 · 11B83

1 Introduction

Given a finite *alphabet* or *digit set* A of real numbers and a real *base* $q > 1$, by an *expansion* of a real number x we mean a sequence $(c_i) \in A^\infty$ satisfying

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The third author was partially supported by the OTKA grants No. 100339 and 104208.

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the equality

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

This concept was introduced by Rényi [16] as a generalization of the familiar integer base expansions.

In order to have an expansion x must belong to the interval

$$J_{A,q} := \left[\frac{\min A}{q-1}, \frac{\max A}{q-1} \right],$$

and the endpoints of $J_{A,q}$ always have unique expansions.

In the familiar integer base case any number has at most two expansions. The general case is much more complex. Consider for example the two-letter alphabet $A = \{0, 1\}$:

- If $q \in (1, \varphi)$, where $\varphi := (1 + \sqrt{5})/2 \approx 1.618$ denotes the Golden Ratio, then each interior element of $J_{A,q}$ has $2^{\aleph_0}$ expansions by [8, Theorem 3].
- If $q \in [\varphi, 2)$, then almost all $x \in J_{A,q}$ have $2^{\aleph_0}$ expansions by Sidorov [17], Dajani and de Vries [4] (see also [7, Theorem 2.3.2]), and infinitely many numbers have $\aleph_0$ expansions by [6, Theorem 1.4 (iii)].
- If $q = 2$, then the diadically rational interior elements of $J_{A,q}$ have two expansions, and all other elements have a unique expansion.
- If $q > 2$, then no number has more than one expansion.

Turning back to the general case, we may ask whether certain numbers may have the same expansions in different bases, say

$$\sum_{i=1}^{\infty} \frac{c_i}{q_1^i} = \sum_{i=1}^{\infty} \frac{c_i}{q_2^i} = x. \tag{1}$$

Except the trivial case $x = 0$ if $0 \in A$, this may only occur for alphabets having both positive and negative elements:

- If $A = \{-1, 0, 1\}$, $1 < q_1 < q_2$ and $q_1 \le 2$, then (1) holds for infinitely many numbers x. This is a special case of [14, Theorem 1].
- If $A = \{-1, 1\}$ and $1 < q_1 < q_2 < 1 + \sqrt[3]{\sqrt{10} - 2} \approx 1.05$, then (1) holds for all $x \in [-\delta, \delta]$ for some $\delta = \delta(q_1, q_2) > 0$. This is a special case of [5, Theorem 1.1] of Dajani et al.

In this paper we investigate the existence of non-trivial common expansions in more than two bases. For this we need a different approach.

If no base is fixed in advance, then a very general result holds for all alphabets containing both positive and negative digits:

Theorem 1 *Given two real numbers $a < 0 < b$, there exist $2^{\aleph_0}$ sequences $(c_i) \in \{a, b\}^{\infty}$ such that for each $x \in \mathbb{R}$ the equality*

$$\sum_{i=1}^{\infty} \frac{c_i}{q_j^i} = x$$

holds for infinitely many bases q_j.

The problem is more difficult if the bases are prescribed. Given an alphabet A and an integer $r \geq 2$, a finite set of bases

$$q_1 < \cdots < q_r \tag{2}$$

is said to have the *simultaneous expansion property* if the relations

$$\sum_{i=1}^{\infty} \frac{c_i}{q_1^i} = \cdots = \sum_{i=1}^{\infty} \frac{c_i}{q_r^i} = x \tag{3}$$

hold for all numbers x belonging to some non-degenerate interval, with $(c_i) \in A^\infty$ depending on x.

Conjecture 1 If the alphabet contains both positive and negative digits, then there exists $\delta_r > 0$ such that all sets of bases (2) in $(1, 1 + \delta_r)$ have the simultaneous expansion property.

In the following theorem we may fix one base.

Theorem 2 *We consider the alphabet $A = \{-1, 0, 1\}$ and an integer $r \geq 2$. There exist two non-empty open intervals I, J such that for any fixed $q_1 \in I$ and $x \in J$ there exist bases $q_2, \ldots, q_r$ satisfying the relations* (2) *and* (3).

The proof will provide intervals I arbitrarily close to 1 and intervals J containing 1.

The next two sections are devoted to the proofs of the theorems. We conclude our paper with some comments and open questions.

2 Proof of Theorem 1

We will construct a sequence of integers $0 < n_1 < n_2 < \cdots$ and a sequence $p_1 > p_2 > \cdots$ of real numbers converging to 1 such that setting

$$(c_i) := a^{n_1} b^{n_2 - n_1} a^{n_3 - n_2} b^{n_4 - n_3} \cdots$$

the following inequalities are satisfied:

$$\sum_{i=1}^{n_k} \frac{c_i}{p_k^i} + \sum_{i=n_k+1}^{\infty} \frac{b}{p_k^i} < -k \quad \text{for} \quad k = 1, 3, 5, \ldots, \tag{4}$$

$$\sum_{i=1}^{n_k} \frac{c_i}{p_k^i} + \sum_{i=n_k+1}^{\infty} \frac{a}{p_k^i} > k \quad \text{for} \quad k = 2, 4, 6, \ldots. \tag{5}$$

Observe that setting

$$f(q) := \sum_{i=1}^{\infty} \frac{c_i}{q^i}$$

the left side of (4) is greater than $f(p_k)$, and the left side of (5) is smaller than $f(p_k)$; therefore

$$\liminf_{q\to 1+} f(q) = -\infty \quad \text{and} \quad \limsup_{q\to 1+} f(q) = \infty. \tag{6}$$

Since f is continuous in $(1,\infty)$ (the defining series is locally uniformly convergent), hence f takes each real value x infinitely many times.

Now we turn to the construction. Choose $n_0 \ge 1$ and $p_0 > 1$ arbitrarily. Assume that $n_0 < \cdots < n_{k-1}$ and $p_0 > \cdots > p_{k-1}$ have already been defined for some $k \ge 1$.

If k is odd, then choose $p_k \in (1, p_{k-1})$ satisfying $p_k < (k+1)/k$ such that

$$\sum_{i=1}^{n_{k-1}} \frac{c_i}{p_k^i} + \sum_{i=n_{k-1}+1}^{\infty} \frac{a}{p_k^i} < -k$$

(this is possible because the left side tends to $-\infty$ as $p_k \to 1+$), and then choose a sufficiently large $n_k > n_{k-1}$ such that

$$\sum_{i=1}^{n_{k-1}} \frac{c_i}{p_k^i} + \sum_{i=n_{k-1}+1}^{n_k} \frac{a}{p_k^i} + \sum_{i=n_k+1}^{\infty} \frac{b}{p_k^i} < -k.$$

This inequality coincides with (4).

If k is even, then choose $p_k \in (1, p_{k-1})$ satisfying $p_k < (k+1)/k$ such that

$$\sum_{i=1}^{n_{k-1}} \frac{c_i}{p_k^i} + \sum_{i=n_{k-1}+1}^{\infty} \frac{b}{p_k^i} > k$$

(this is possible because the left side tends to ∞ as $p_k \to 1+$), and then choose a sufficiently large $n_k > n_{k-1}$ such that

$$\sum_{i=1}^{n_{k-1}} \frac{c_i}{p_k^i} + \sum_{i=n_{k-1}+1}^{n_k} \frac{b}{p_k^i} + \sum_{i=n_k+1}^{\infty} \frac{a}{p_k^i} > k.$$

This inequality coincides with (5).

Finally we observe that during the construction of the sequence (n_k) we had in each step more than one choice (in fact, infinitely many choices). Hence there are $2^{\aleph_0}$ such sequences.

3 Proof of Theorem 2

Let h be a polynomial with coefficients in $\{-1,0,1\}$ and having a unique, simple zero ψ in $(0,1)$. For example, if $p\in(1,2)$ is a Pisot number, then by [9, Theorem 1] (see also [10] and [1] for the converse statements) there exists a polynomial g with coefficients in $\{-1,0,1\}$ whose unique zero in $(1,2)$ is p. Then we may take

$$h(t):=t^{\deg g}g(1/t).$$

For the Golden Ratio $p=\varphi$ we may take $g(t)=t^2-t-1$ and $h(t)=1-t-t^2$.

Changing h to $-h$ if necessary, we may assume that $h(0)=1$. Fix an integer $D>\deg h$, choose a sufficiently large integer $n\ge -1$ satisfying

$$\psi^{D^{-n-r}}>\frac{1}{3}, \tag{7}$$

and introduce the polynomial

$$f(t):=-\prod_{k=n+1}^{n+r}h\left(t^{D^k}\right)=\sum_{i=0}^{\deg f}c_it^i.$$

Then f has r simple roots in $(0,1)$:

$$0<\psi^{D^{-n-1}}<\cdots<\psi^{D^{-n-r}}<1,$$

and all other roots of f belong to $\mathbb{C}\setminus(0,1)$.

Observe that $c_0=f(0)=-1$, and $(c_i)\subset\{-1,0,1\}$. Indeed, developing the product all terms have different degrees by our assumption $D>\deg h$.

Write

$$\alpha_k:=\psi^{D^{-n-k}}$$

for brevity, and fix real numbers β_k satisfying the inequalities

$$0<\beta_0<\alpha_1<\beta_1<\alpha_2<\cdots<\beta_{r-1}<\alpha_r<\beta_r<1. \tag{8}$$

Since each root α_k is simple,

$$f(\beta_0),\dots,f(\beta_r)\quad\text{have alternating non-zero signs.} \tag{9}$$

Now set

$$\varepsilon:=\frac{1}{2}\min\{|f(\beta_0)|,\dots,|f(\beta_r)|\}\,(>0),$$

choose a large positive integer $N\ge\deg f$ satisfying the inequality

$$\sum_{i=N+1}^{\infty}\beta_r^i<\varepsilon,$$

and define

$$\varepsilon_N:=\sum_{i=N+1}^{\infty}\left(\frac{\alpha_r}{2}\right)^i\,(<\varepsilon).$$

Since f is continuous and $f(\alpha_r) = 0$, we may fix a small positive number δ such that

$$|t - \alpha_r| < \delta \Longrightarrow |f(t)| < \frac{\varepsilon_N}{2}. \tag{10}$$

We may assume (using also (7)) that

$$0 < \delta < \frac{\alpha_r}{2}, \quad \beta_{r-1} < \alpha_r - \delta < \alpha_r + \delta < \beta_r \quad \text{and} \quad \alpha_r - \delta \geq \frac{1}{3}.$$

Now fix two arbitrary real numbers

$$x \in \left(1 - \frac{\varepsilon_N}{2}, 1 + \frac{\varepsilon_N}{2}\right) \quad \text{and} \quad \gamma_r \in (\alpha_r - \delta, \alpha_r + \delta).$$

Then there exists a sequence $(c_i)_{i=N+1}^{\infty} \in \{-1, 0, 1\}^{\infty}$ satisfying

$$f(\gamma_r) + 1 + \sum_{i=N+1}^{\infty} c_i \gamma_r^i = x.$$

Indeed, using (10) we have

$$|x - 1 - f(\gamma_r)| \leq |x - 1| + |f(\gamma_r)| < \frac{\varepsilon_N}{2} + \frac{\varepsilon_N}{2} = \varepsilon_N$$

and

$$\sum_{i=N+1}^{\infty} \gamma_r^i \geq \sum_{i=N+1}^{\infty} (\alpha_r - \delta)^i \geq \sum_{i=N+1}^{\infty} \left(\frac{\alpha_r}{2}\right)^i = \varepsilon_N,$$

so that

$$x - 1 - f(\gamma_r) \in \left[-\sum_{i=N+1}^{\infty} \gamma_r^i, \sum_{i=N+1}^{\infty} \gamma_r^i\right].$$

It remains to observe that

$$\left\{\sum_{i=N+1}^{\infty} c_i \gamma_r^i \ : \ (c_i) \in \{-1, 0, 1\}^{\infty}\right\} = \left[-\sum_{i=N+1}^{\infty} \gamma_r^i, \sum_{i=N+1}^{\infty} \gamma_r^i\right].$$

This follows from a classical theorem of Kakeya [12], [13] (see also [15, Part 1, Exercise 131] or [14, Proposition 3]), because

$$\gamma_r > \alpha_r - \delta \geq \frac{1}{3},$$

and therefore the sequence (γ_r^i) satisfies Kakeya's condition: each element is less than or equal to the sum of the smaller elements.

Now let us introduce the function

$$g(t) := -x + \sum_{i=1}^{\deg f} c_i t^i + \sum_{i=N+1}^{\infty} c_i t^i = -x + 1 + f(t) + \sum_{i=N+1}^{\infty} c_i t^i. \tag{11}$$

We have $g(\gamma_r) = 0$ by definition. Furthermore, the following estimates hold for each $k = 0, \dots, r$:

$$\begin{aligned}
|g(\beta_k) - f(\beta_k)| &\le |x-1| + \sum_{i=N+1}^{\infty} \beta_k^i \\
&\le |x-1| + \sum_{i=N+1}^{\infty} \beta_r^i \\
&< \frac{\varepsilon_N}{2} + \varepsilon \\
&< \frac{3\varepsilon}{2} \\
&< |f(\beta_k)| .
\end{aligned}$$

Therefore $g(\beta_k) \ne 0$, and $g(\beta_k)$ and $f(\beta_k)$ have equal signs for each k. Using (9) it follows that

$$g(\beta_0), \dots, g(\beta_r) \quad \text{have alternating non-zero signs.}$$

Applying Bolzano's theorem and using (8) we conclude that there exist real numbers $\gamma_1, \dots, \gamma_{r-1}$ satisfying the inequalities

$$0 < \beta_0 < \gamma_1 < \beta_1 < \gamma_2 < \cdots < \beta_{r-1} < \gamma_r < \beta_r < 1$$

and the equalities

$$g(\gamma_1) = \cdots = g(\gamma_r) = 0.$$

Setting $c_i := 0$ for $i = (\deg f) + 1, \dots, N$, $q_k := 1/\gamma_{r+1-k}$ for $k = 1, \dots, r$, and using the definition (11) of $g(t)$, these equalities may be rewritten in the form (3), and the theorem follows with

$$I = \left(\frac{1}{\alpha_r + \delta}, \frac{1}{\alpha_r - \delta} \right) \quad \text{and} \quad J = \left(1 - \frac{\varepsilon_N}{2}, 1 + \frac{\varepsilon_N}{2} \right).$$

4 Concluding remarks and open questions

Concerning Theorem 1 the following natural questions can be asked:

(i) How slowly may the sequence (q_j) converge to 1?
(ii) How large is the set of sequences (c_i) in $\{-1,0,1\}^\infty$ for which the corresponding function f satisfies the relations (6)?

Concerning Theorem 2 we may ask whether we may choose $I = (1,p)$ with some $p = p_r > 1$ (depending on r). One way to prove this would be to generalize the construction of the polynomial f in the proof of Theorem 2. We started with a polynomial $h(t)$ with coefficients in $\{-1,0,1\}$, equal to 1 in zero, and having a unique, simple root in $(0,1)$. Instead of the last property it is sufficient to assume that h has $n \ge 1$ roots of odd multiplicity in $(0,1)$.

Then for any integers $r \geq 1$ and $D > \deg h$ the coefficients of the polynomial $f(t) = \prod_{k=1}^{r} h(t^{D^k})$ belong to $\{-1, 0, 1\}$. Furthermore, if D is sufficiently large, then it has at least rn simple real roots in $(0, 1)$. Indeed, if $0 < m < M < 1$ are two numbers such that all roots of odd multiplicity of h in $(0, 1)$ lie in $[m, M]$, then it suffices to choose $D > \deg h$ satisfying $M^D < m$ by an elementary computation.

The *Fekete polynomials*

$$F_p(t) = \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) t^j$$

where p is an odd prime and $\left(\frac{j}{p}\right)$ denotes the Legendre symbol, are well known examples of polynomials with coefficients in $\{-1, 0, 1\}$. Baker and Montgomery [2], see also [3], proved that $F_p(t)$ may have arbitrary many real roots, provided p is large enough. Dividing them by $\pm t$ they will be equal to 1 in zero. Unfortunately, we have no information about the multiplicity of their real roots.

However, some concrete examples may be found. The polynomial $F_{163}(t)$ has two simple real roots in $(0, 1)$ by Exercise 46 in [15, Section 5]. Furthermore, a straightforward computation with MAPLE shows that the same holds for $F_{43}(t)$, too, and that $F_{547}(t)$ has four simple real roots in the same interval. Hence these polynomials can be chosen for $h(t)$ in the construction of $f(t)$.

Acknowledgements. We are grateful to the referee for his suggestions leading to the present strengthened form of Theorem 1.

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