COMMON EXPANSIONS IN NONINTEGER BASES

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Dedicated to Professor Zoltán Daróczy on his 75th birthday.

Abstract. In this paper we study the existence of simultaneous representations of real numbers in bases $p > q > 1$ with the digit set $A = \{-m, \dots, 0, \dots, m\}$. Among other results, we prove if $m = 1$ and $q < 2$, then there is a continuum of sequences $(c_i) \in A^\infty$ satisfying $\sum_{i=1}^\infty c_i q^{-i} = \sum_{i=1}^\infty c_i p^{-i}$. On the other hand, if $m = 1$ and $q \ge 2 + \sqrt{2}$, then only the trivial sequence $(c_i) = 0^\infty$ satisfies the former equality.

1. Introduction

Given a finite *alphabet* or *digit set* A of real numbers and a real *base* $q > 1$, by an *expansion* of a real number x we mean a sequence $c = (c_i) \in A^\infty$ satisfying the equality

$$\sum_{i=1}^\infty \frac{c_i}{q^i} = x.$$

This concept was introduced by Rényi [10] as a generalization of the radix representation of integers.

Given two different bases p, q we wonder whether there exist real numbers having the same expansions in both bases:

$$\sum_{i=1}^\infty \frac{c_i}{q^i} = x = \sum_{i=1}^\infty \frac{c_i}{p^i}. \tag{1}$$

In case $0 \in A$ a trivial example is $x = 0$ with $(c_i) = 0^\infty$. If the alphabet A contains no pair of digits with opposite signs, then this is the only such example. Indeed, if for instance all digits are nonnegative and

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$0^\infty \neq (c_i) \in A^\infty$, then for $p > q$ we have

$$\sum_{i=1}^{\infty} \frac{c_i}{p^i} < \sum_{i=1}^{\infty} \frac{c_i}{q^i}$$

by an elementary monotonicity argument.

Even if the alphabet A contains digits of opposite signs, the existence of *common expansions* (1) seems to be a rare event.

Similar phenomena appears with the common radix representation. Indlekofer, Kátai and Racskó [4] called $\mathbf{a} \in \mathbb{Z}^d$ *simultaneously representable* by $\mathbf{q} \in \mathbb{Z}^d$, if there exist integers $0 \leq m_0, \dots, m_\ell < Q := |q_1 \cdots q_d|$ such that

$$a_i = \sum_{j=0}^{\ell} m_j q_i^j, \quad i = 1, \dots, d.$$

If $q_1, \dots, q_d > 0$ then apart from the zero vector no integer vectors are simultaneously representable by $\mathbf{q}$. If, however, some of the base numbers are negative, then simultaneous representations may appear. For example take $q_1 = -2$ and $q_2 = -3$ then we have $(101)_{10} = (1431335045)_{-2} = (1431335045)_{-3}$. Changing the sign of the "digits" with odd position we get a common representation of 101 in bases 2 and 3 with digits from $\{-6, \dots, 0, \dots, 6\}$. Pethő [8] gave a criterion of simultaneous representability on the one hand with the Chinese reminder theorem and, on the other hand with CNS polynomials. A similar result was proved by Kane [7].

No results on simultaneous representability of real numbers in non-integer bases seem to have appeared in the literature. In this paper we start such a study by investigating the case of the special alphabets $A = \{-m, \dots, 0, \dots, m\}$ for some given integer $m \geq 1$. Let us denote by $C(p, q)$ the set of sequences $(c_i) \in A^\infty$ satisfying

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = \sum_{i=1}^{\infty} \frac{c_i}{p^i}. \tag{2}$$

We call $C(p, q)$ *trivial* if its only element is the null sequence.

Our main result is the following:

Theorem 1. *Let $p > q > 1$.*

(i) *If $q < (1 + \sqrt{8m+1})/2$, then $C(p, q)$ has the cardinality of the continuum.*
(ii) *If $(1 + \sqrt{8m+1})/2 \leq q \leq m + 1$, then $C(p, q)$ is infinite.*
(iii) *Let $m + 1 < q \leq 2m + 1$.*

(a) *If*

$$p \le \frac{(m+1)(q-1)}{q-m-1}, \tag{3}$$

then $C(p,q)$ is nontrivial.

(b) *If*

$$p > \frac{(m+1)(q-1)}{q-m-1}, \tag{4}$$

then $C(p,q)$ is trivial.

(iv) *Let $2m+1 < q < m+1+\sqrt{m(m+1)}$.*
   (a) *$C(p,q)$ is a finite set.*
   (b) *There is a continuum of values $p > q$ for which $C(p,q)$ is nontrivial.*
   (c) *If $p > q$ satisfies (4), then $C(p,q)$ is trivial.*

(v) *If $q \ge m+1+\sqrt{m(m+1)}$, then $C(p,q)$ is trivial.*

Remark 2.

(i) The proof of (iii) (a) will also show that if $m+1 < q \le 2m+1$, and

$$\frac{1}{m} \le \frac{1}{q-1} + \left(\frac{1}{q-1} - \frac{1}{p-1}\right)\frac{q^n}{p^n - q^n} \tag{5}$$

for some positive integer n, then $C(p,q)$ has at least $n+1$ elements. For $n=1$ this condition reduces to (3).

Furthermore, we show in Remark 7 that the right side of this inequality is a decreasing function of p, so that the solutions p of the inequality form a half-closed interval, say $(q, p_n]$. (We have clearly $p_1 > p_2 > \cdots$.)

(ii) The proof of (iv) (a) will show more precisely that if $q > 2m+1$ and

$$\frac{1}{2m} > \frac{1}{q-1} + \left(\frac{1}{q-1} - \frac{1}{p-1}\right)\frac{q^n}{p^n - q^n}$$

for some positive integer n, then $C(p,q)$ has at most $(2m+1)^n$ elements.

2. Proofs

We begin by establishing some auxiliary results.

Interval filling sequences play an important role in establishing the existence of various kinds of representations of real numbers; see, e.g., Daróczy, Járai and Kátai [1], Daróczy and Kátai [2]. We also need such a result here: a variant of a classical theorem of Kakeya [5], [6] (see also [9, Part 1, Exercise 131]).

Proposition 3. *Let* $A = \{-m, \dots, 0, \dots, m\}$ *and let* $\sum_{k=1}^{\infty} r_k$ *be a convergent series of positive numbers, satisfying the inequalities*

$$r_n \le 2m \sum_{k=n+1}^{\infty} r_k \tag{6}$$

for all $n = 1, 2, \dots$. *Then the sums*

$$\sum_{k=1}^{\infty} c_k r_k, \quad (c_k) \in A^{\infty} \tag{7}$$

fill the interval

$$\left[-m \sum_{k=1}^{\infty} r_k, m \sum_{k=1}^{\infty} r_k \right]. \tag{8}$$

Proof. It is clear that all sums (7) belong to the interval (8). Conversely, for each given x in this interval we define a sequence $(c_k) \in A^{\infty}$ by the following greedy algorithm. If $c_1, \dots, c_{n-1}$ are already defined (no assumption if $n = 1$), then let c_n be the largest element of A such that

$$\left(\sum_{k=1}^{n} c_k r_k \right) - m \left(\sum_{k=n+1}^{\infty} r_k \right) \le x.$$

Letting $n \to \infty$ it follows that $\sum_{k=1}^{\infty} c_k r_k \le x$. It remains to prove the converse inequality. This is obvious if $c_k = m$ for all $k \in \mathbb{N}$ because then

$$\sum_{k=1}^{\infty} c_k r_k = m \sum_{k=1}^{\infty} r_k \ge x$$

by the choice of x.

If $c_n < m$ for infinitely many indices, then

$$\left(\sum_{k=1}^{n-1} c_k r_k \right) + m r_n - m \left(\sum_{k=n+1}^{\infty} r_k \right) > x$$

for all such indices, and letting $n \to \infty$ we conclude that $\sum_{k=1}^{\infty} c_k r_k \ge x$.

The proof will be complete if we show that (c_k) cannot have a last term $c_n < m$, i.e., an index n such that $c_n = j < m$, and $c_k = m$ for all $k > n$. Assume on the contrary that there exists such an index n. Then we have

$$\left(\sum_{k=1}^{n-1} c_k r_k \right) + j r_n + m \left(\sum_{k=n+1}^{\infty} r_k \right) \le x$$

and

$$\left(\sum_{k=1}^{n-1} c_k r_k\right) + (j+1)r_n - m\left(\sum_{k=n+1}^{\infty} r_k\right) > x$$

by construction. Hence

$$r_n > 2m \sum_{k=n+1}^{\infty} r_k,$$

contradicting (6). □

We also need two technical lemmas.

Lemma 4. *If* $1 < q < (1+\sqrt{8m+1})/2$ *and* $p > q$, *then the sequence* $(r_k)_{k\in\mathbb{N}} := (q^{-i} - p^{-i})_{i\in\mathbb{N}\setminus n\mathbb{N}}$ *satisfies* (6) *for all sufficiently large integers* n.

Proof. Fix a sufficiently large integer n such that

$$\frac{1}{2m} < \frac{1}{q(q-1)} - \frac{1}{q(q^n-1)}.$$

This is possible by our assumption on q, because we have the following equivalences for $m > 0$ and $q > 1$:

$$\begin{aligned}\frac{1}{2m} < \frac{1}{q(q-1)} &\iff 4q(q-1) < 8m\\ &\iff (2q-1)^2 < 8m+1\\ &\iff 2q-1 < \sqrt{8m+1}\\ &\iff q < \left(1+\sqrt{8m+1}\right)/2.\end{aligned}$$

Now, if

$$r_{h'} = q^{-h} - p^{-h} = q^{-h}\left(1-(q/p)^h\right)$$

for some $h' \geq 1$, then

$$\sum_{k=h'+1}^{\infty} r_k = \sum_{i\in\mathbb{N}\setminus n\mathbb{N}, i>h} (q^{-i} - p^{-i}) = \sum_{i\in\mathbb{N}\setminus n\mathbb{N}, i>h} q^{-i}\left(1-(q/p)^i\right).$$

Since $(1-(q/p)^i) > \bigl(1-(q/p)^h\bigr)$ for all $i>h$, it follows that (we use the choice of n in the last step)

$$\begin{aligned}\frac{\sum_{k=h'+1}^{\infty} r_k}{r_{h'}} &\ge \sum_{i\in\mathbb{N}\setminus n\mathbb{N}, i>h} q^{h-i}\\ &= \left(\sum_{i=1}^{\infty} q^{-i}\right) - \left(\sum_{i>\frac{h}{n}} q^{h-in}\right)\\ &\ge \left(\sum_{i=1}^{\infty} q^{-i}\right) - \left(\sum_{i=0}^{\infty} q^{-1-in}\right)\\ &= \left(\sum_{i=2}^{\infty} q^{-i}\right) - \left(\sum_{i=1}^{\infty} q^{-1-in}\right)\\ &= \frac{1}{q(q-1)} - \frac{1}{q(q^n-1)} > \frac{1}{2m}.\end{aligned}$$

□

Lemma 5. *Let $p>q>1$. The sequence*

$$\left(\frac{\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i})}{q^{-n}-p^{-n}}\right)_{n=1}^{\infty}$$

is strictly decreasing, and tends to $1/(q-1)$.

Proof. Since $1>(q/p)^n \searrow 0$, the results follow from the identity

$$\frac{\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i})}{q^{-n}-p^{-n}} = \frac{1}{q-1} + \frac{p-q}{(q-1)(p-1)}\frac{(q/p)^n}{1-(q/p)^n}. \tag{9}$$

Setting $x=q/p$ for brevity, the identity is proved as follows:

$$\begin{aligned}\frac{\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i})}{q^{-n}-p^{-n}} &= \frac{q^{-n}}{(q-1)(q^{-n}-p^{-n})} - \frac{p^{-n}}{(p-1)(q^{-n}-p^{-n})}\\ &= \frac{1}{(q-1)(1-x^n)} - \frac{x^n}{(p-1)(1-x^n)}\\ &= \frac{1-x^n+x^n}{(q-1)(1-x^n)} - \frac{x^n}{(p-1)(1-x^n)}\\ &= \frac{1}{q-1} + \frac{x^n}{1-x^n}\left(\frac{1}{q-1}-\frac{1}{p-1}\right)\\ &= \frac{1}{q-1} + \frac{p-q}{(q-1)(p-1)}\frac{x^n}{1-x^n}.\end{aligned}$$

□

Remark 6. Let us note for further reference the following equivalent form of (9), obtained during the proof:

$$\frac{\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i})}{q^{-n}-p^{-n}}=\frac{1}{q-1}+\left(\frac{1}{q-1}-\frac{1}{p-1}\right)\frac{q^n}{p^n-q^n}. \tag{10}$$

Now we are ready to prove our theorem.

Proof of Theorem 1 (i). We adapt the proof of Theorem 3 in [3], which states that if $1<q<(1+\sqrt{5})/2$, then every x satisfying $q<x<1/(q-1)$ has a continuum of expansions in base q with digits 0 or 1.

Applying Lemma 4 we fix a large positive integer n such that the sequence $(r_k)_{k\in\mathbb{N}}:=(q^{-i}-p^{-i})_{i\in\mathbb{N}\setminus n\mathbb{N}}$ satisfies (6). Next we fix a large positive integer N such that

$$\begin{aligned}\left[-m\sum_{i=N}^{\infty}(q^{-in}-p^{-in}),m\sum_{i=N}^{\infty}(q^{-in}-p^{-in})\right]&\\ \subset\left[-m\sum_{i\in\mathbb{N}\setminus n\mathbb{N}}(q^{-in}-p^{-in}),m\sum_{i\in\mathbb{N}\setminus n\mathbb{N}}(q^{-in}-p^{-in})\right]&.\end{aligned}\tag{11}$$

This is possible because the right side interval contains 0 in its interior. The sets

$$\begin{aligned}&B:=\mathbb{N}\setminus n\mathbb{N},\\ &C:=\{in\ :\ i=N,N+1,\dots\},\\ &D:=\{in\ :\ i=1,\dots,N-1\}\end{aligned}$$

form a partition of $\mathbb{N}$.

Choose an arbitrary sequence $(c_i)_{i\in C}\in A^C$; there is a continuum of such sequences because C is an infinite set. Since

$$-\sum_{i\in C}c_i(q^{-i}-p^{-i})$$

belongs to the left side interval in (11), applying Proposition 3 there exists a sequence $(c_i)_{i\in B}\in A^B$ such that

$$\sum_{i\in B\cup C}c_i(q^{-i}-p^{-i})=0.$$

Setting $c_i=0$ for $i\in D$ we obtain a sequence $(c_i)_{i\in\mathbb{N}}\in C(p,q)$. ☐

Proof of Theorem 1 (ii). We show that for each positive integer n there exists a sequence $(c_i)\in C(p,q)$, beginning with $c_1=\cdots=c_{n-1}=0$

and $c_n = -1$. Indeed, since $q \le m+1$, by Lemma 5 we have

$$0 < q^{-n} - p^{-n} < (q-1)\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i}) \le m\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i}).$$

Since $q \le 2m+1$, Lemma 5 also shows that the condition (6) of Proposition 3 is satisfied for the alphabet $A = \{-m, \dots, m\}$ and the sequence $r_k := q^{-k-n} - p^{-k-n}$, $k = 1, 2, \dots$. Hence there exists a sequence $(c_i)_{i=n+1}^{\infty} \in A^{\infty}$ satisfying

$$q^{-n} - p^{-n} = \sum_{i=n+1}^{\infty} c_i(q^{-i}-p^{-i});$$

setting $c_1 = \dots = c_{n-1} = 0$ and $c_n = -1$ this yields (2). □

Proof of Theorem 1 (iii) (a). We show that there is a sequence $(c_i) \in C(p,q)$, beginning with $c_1 = -1$. Since $q \le 2m+1$, by Proposition 3 and Lemma 5 it is sufficient to show that

$$(0<)q^{-1} - p^{-1} \le m\sum_{i=2}^{\infty}(q^{-i}-p^{-i}).$$

By (9) this is equivalent to the inequality

$$\frac{1}{m} \le \frac{1}{q-1} + \frac{p-q}{(p-1)(q-1)}\frac{\frac{q}{p}}{1-\frac{q}{p}} = \frac{1}{q-1} + \frac{q}{(p-1)(q-1)},$$

i.e., to $p \le (m+1)(q-1)/(q-m-1)$. Indeed, since $m>0$, $q>1$ and $p > m+1$, we have

$$\begin{aligned}\frac{1}{m} \le \frac{1}{q-1} + \frac{q}{(p-1)(q-1)} &\iff (p-1)(q-1) \le m(p-1) + mq\\ &\iff p(q-m-1) \le (m+1)(q-1)\\ &\iff p \le \frac{(m+1)(q-1)}{q-m-1}.\end{aligned}$$

□

Remark 7. Now we prove our statement in Remark 2 (i). If $m+1 < q \le 2m+1$ and $p > q$ is closer to q so that

$$(0<)q^{-n} - p^{-n} \le m\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i})$$

or equivalently (see (10))

$$\frac{1}{m} \le \frac{1}{q-1} + \left(\frac{1}{q-1} - \frac{1}{p-1}\right)\frac{q^n}{p^n-q^n}$$

for some positive integer n, then the adaptation of the preceding proof shows that for each $k = 1, \dots, n$ there exists a sequence $(c_i) \in C(p,q)$, beginning with $c_1 = \dots = c_{k-1} = 0$ and $c_k = -1$.

The right side of the above inequality is a decreasing function of p because the function

$$f(p) := \left(\frac{1}{q-1} - \frac{1}{p-1}\right)\frac{1}{p^n - q^n}$$

has a negative derivative for all $p > q$.

Indeed, we have

$$f'(p) = \frac{1}{(p-1)^2(p^n - q^n)} - \left(\frac{1}{q-1} - \frac{1}{p-1}\right)\frac{np^{n-1}}{(p^n - q^n)^2},$$

whence

$$\frac{(p-1)^2(p^n-q^n)^2}{p-q}f'(p) = \frac{p^n - q^n}{p-q} - np^{n-1}\frac{p-1}{q-1} < \frac{p^n - q^n}{p-q} - np^{n-1}.$$

We conclude by noticing that $\frac{p^n-q^n}{p-q} = nr^{n-1}$ by the Lagrange mean value theorem for some $q < r < p$ and therefore

$$\frac{p^n-q^n}{p-q} - np^{n-1} = n\left(r^{n-1} - p^{n-1}\right) \le 0.$$

Proof of Theorem 1 (iv) (a). Since $1/(q-1) < 1/2m$, by Lemma 5 we have

$$q^{-n} - p^{-n} > 2m \sum_{i=n+1}^{\infty} (q^{-i} - p^{-i})$$

for all sufficiently large integers n, say for all $n > N$.[1] This implies that if two different sequences $(c_i), (c_i') \in A^\infty$ satisfy $c_i = c_i'$ for $i = 1, \dots, N$, then

$$\sum_{i=1}^{\infty} c_i(q^{-i} - p^{-i}) \ne \sum_{i=1}^{\infty} c_i'(q^{-i} - p^{-i}).$$

[1]If this inequality holds for some n, then it also holds for all larger integers by the monotonicity property of Lemma 5.

Indeed, if n is the first index for which $c_n \neq c'_n$, then $n > N$, and therefore

$$\begin{aligned}
&\left|\sum_{i=1}^{\infty}(c_i - c'_i)(q^{-i} - p^{-i})\right| \\
&\qquad \geq |c_n - c'_n| \, (q^{-n} - p^{-n}) - \sum_{i=n+1}^{\infty} |c_i - c'_i| \, (q^{-i} - p^{-i}) \\
&\qquad \geq (q^{-n} - p^{-n}) - 2m \sum_{i=n+1}^{\infty} (q^{-i} - p^{-i}) \\
&\qquad > 0.
\end{aligned}$$

It follows that if two different sequences $(c_i), (c'_i) \in A^\infty$ satisfy

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} - \sum_{i=1}^{\infty} \frac{c_i}{p^i} = \sum_{i=1}^{\infty} \frac{c'_i}{q^i} - \sum_{i=1}^{\infty} \frac{c'_i}{p^i} = 0,$$

then already their beginning words $c_1 \dots c_N$ and $c'_1 \dots c'_N$ must differ. We conclude that there are at most $(2m+1)^N$ sequences $(c_i) \in A^\infty$ satisfying (2). ☐

Proof of Theorem 1 (iv) (b). Thanks to (a) it is sufficient to exhibit a continuum of sequences $(c_i) \in A^\infty$ such that each sequence satisfies (2) for at least one base $p > q$.

Our assumption $q < m + 1 + \sqrt{m(m+1)}$ implies the inequality

$$\frac{1}{q^2} < m \sum_{i=2}^{\infty} \frac{i}{q^{i+1}}. \tag{12}$$

Indeed, differentiating the identity

$$\sum_{i=1}^{\infty} \frac{1}{q^i} = \frac{1}{q-1}$$

we get

$$\sum_{i=1}^{\infty} \frac{i}{q^{i+1}} = \frac{1}{(q-1)^2},$$

so that, since $m > 0$ and $q > 1$, (12) is equivalent to

$$\frac{m+1}{q^2} < \frac{m}{(q-1)^2}.$$

This inequality can be rewritten as

$$q^2 - 2q(m+1) + m + 1 < 0. \tag{13}$$

The polynomial $x^2-2x(m+1)+m+1$ has exactly one root, which is larger than one, namely $x=m+1+\sqrt{m(m+1)}$. Thus (12) holds if and only if $q<m+1+\sqrt{m(m+1)}$.

In view of (12) we may choose a sufficiently large positive integer N such that

$$\frac{1}{q^2}<m\sum_{i=2}^{N}\frac{i}{q^{i+1}}. \tag{14}$$

Now fix an arbitrary sequence $(c_i)\in A^\infty$ satisfying

$$c_1=-1,\ c_2=\cdots=c_N=m \quad \text{and } c_i\ge 0 \text{ for all } i>N. \tag{15}$$

(There is a continuum of such sequences.) We are going to prove that (2) holds for at least one base $p>q$.

It is sufficient to show that

$$\sum_{i=1}^{\infty}c_i(q^{-i}-p^{-i})<0$$

if $p>q$ is large enough, and

$$\sum_{i=1}^{\infty}c_i(q^{-i}-p^{-i})>0$$

if $p>q$ is close enough to q. Indeed, then we will have equality for some intermediate value of p by continuity.

The first property will follow from the stronger relation

$$\lim_{p\to\infty}\sum_{i=1}^{\infty}c_i(q^{-i}-p^{-i})<0,\quad \text{i.e.,}\quad \sum_{i=1}^{\infty}\frac{c_i}{q^i}<0.$$

The proof is straightforward: since $c_1=-1$ and $q>m+1$, we have

$$\sum_{i=1}^{\infty}\frac{c_i}{q^i}\le\frac{-1}{q}+\sum_{i=2}^{\infty}\frac{m}{q^i}=\frac{-1}{q}+\frac{m}{q(q-1)}<\frac{-1}{q}+\frac{1}{q}=0.$$

Since $c_i\ge 0$ for all $i>N$, the second property is weaker than the inequality

$$\sum_{i=1}^{N}c_i(q^{-i}-p^{-i})>0$$

for all $p>q$ close enough to q, and this is weaker than the relation

$$\lim_{p\to q}\frac{1}{p-q}\sum_{i=1}^{N}c_i(q^{-i}-p^{-i})>0.$$

The last property follows by using (14) and (15):

$$\lim_{p\to q}\frac{1}{p-q}\sum_{i=1}^{N}c_i(q^{-i}-p^{-i})=\sum_{i=1}^{N}\frac{ic_i}{q^{i+1}}=-\frac{1}{q^2}+m\sum_{i=2}^{N}\frac{i}{q^{i+1}}>0. \qquad \square$$

Proof of Theorem 1 (iii) (b), (iv) (c) and (v). If $p>q>m+1$ satisfy (4), then the proof of (iii) (a) shows that

$$q^{-1}-p^{-1}>m\sum_{i=2}^{\infty}(q^{-i}-p^{-i}).$$

Then by Lemma 5 we also have, more generally,

$$q^{-n}-p^{-n}>m\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i})$$

for all positive integers n.

Now if a sequence $(c_i)\in A^\infty$ has a first nonzero term c_n, then

$$\left|\sum_{i=n+1}^{\infty}c_i(q^{-i}-p^{-i})\right|\le\sum_{i=n+1}^{\infty}m(q^{-i}-p^{-i})$$
$$<q^{-n}-p^{-n}\le\left|c_n(q^{-n}-p^{-n})\right|,$$

so that (2) cannot hold. This completes the proof of (iii) (b) and (iv) (c).

For the proof of (v) it remains to check that in case $q\ge m+1+\sqrt{m(m+1)}$ the condition (4) holds for all $p>q$. This is equivalent to

$$q\ge\frac{(m+1)(q-1)}{q-m-1},$$

which can be rewritten as

$$q^2-2q(m+1)+m+1\ge 0.$$

By our observation after (13) this inequality holds if and only if $q\ge m+1+\sqrt{m(m+1)}$. $\square$

We end this paper by formulating some open questions:

(1) Find the optimal conditions on p and q in Theorem 1. In particular,
   (a) Can $C(p,q)$ be infinite for some $p>q>m+1$?
   (b) In case $2m+1<q<m+1+\sqrt{m(m+1)}$ is $C(p,q)$ nontrivial for *all* $p>q$ sufficiently close to q?

(2) Construct an alphabet and three (or more) different bases such that a continuum of (or infinitely many) real numbers have identical expansions in all three bases.

(3) Given two bases $p > q > 1$ investigate the set of points of the form

$$\sum_{i=1}^{\infty} c_i(p^{-i} - q^{-i}), \quad (c_i) \in A^{\infty}.$$

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