Nearly linear recursive sequences, especially SRS

Attila Pethő
University of Debrecen, Hungary

Based on a joint works with Shigeki Akiyama, Tsukuba, Japan Jan-Hendrik Evertse, Leiden, The Netherlands, Jörg Thuswaldner and Mario Weitzer, Leoben, Austria

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1. Introduction

Let $A_0, \ldots, A_{d-1} \in \mathbb{C}$. Let (a_n) be a sequence of complex numbers and define the error sequence (e_n) by the initial terms $e_0 = \cdots = e_{d-1} = 0$ and by the equations

$$e_{n+d} = a_{n+d} + A_{d-1} a_{n+d-1} + \cdots + A_0 a_n$$

for $n \geq 0$. We call (a_n) a *nearly linear recursive sequence*, in shortcut nlrs, if the sequence $(|e_n|)$ is bounded.

Homogenous and inhomogenous linear recursive sequences, and srs are examples for nlrs.

Two aspects:

1) (Masculine) (a_n) is given, find $A_0, \ldots, A_{d-1} \in \mathbb{C}$ and (e_n). Classify the nlrs's. Topic of the first part.

2) (Feminine) $A_0, \ldots, A_{d-1} \in \mathbb{C}$ and (e_n) or a rule for the generation of (e_n) is given. Find all (a_n) satisfying these requirements. Topic of the second part.

Let $a_n = 1$ and $b_n = 2^n$ for all n. $e_{n+1} = a_{n+1} + 0 \cdot a_n = 1$ as well as $0 = e_{n+1} = a_{n+1} - 2a_n$ show that $(a_n), (b_n)$ are nlrs, but they have different "characteristic" polynomials.

Both $(a_n), (b_n)$ satisfy the recursion $0 = a_{n+2} - 3a_{n+1} + 2a_n$.

2. Characteristic polynomial of nlrs

Let (a_n) be a nlrs. The set of polynomials $\sum_{j=0}^{l} B_j x^j \in \mathbb{C}[x]$ such that the sequence $(\sum_{j=0}^{l} B_j a_{n+j})$ is bounded is an ideal of $\mathbb{C}[x]$ called *the ideal* of (a_n). There is a unique, monic polynomial generating the ideal of (a_n), called *the characteristic polynomial* of (a_n).

The characteristic polynomial of a linear recurrence sequence (lrs) (a_n) may be different from the characteristic polynomial of (a_n) when viewed as an nlrs.

For instance, the Fibonacci sequence (F_n) given by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ has characteristic polynomial $x^2 - x - 1$ when viewed as an lrs, but characteristic polynomial $x - \theta$ with $\theta = \frac{1+\sqrt{5}}{2}$ when viewed as an nlrs.

Lemma 1. *Suppose (a_n) is an nlrs with characteristic polynomial $P(x)$. Then*

(i) the roots of $P(x)$ all have modulus ≥ 1.
(ii) Assume further that $a_n = O(n)$ holds. Then the roots of $P(x)$ all lie on the unit circle.

3. The Binet-like formula for nlrs

Define the function

$$c(z) = \sum_{j=1}^{\infty} e_{d+j-1} z^{-j}.$$

Since (e_n) is bounded, $c(z)$ is convergent for all complex z with $|z| > 1$.

Denote $(\widehat{a}_n)$ the lrs having the initial terms $\widehat{a}_0 = \cdots = \widehat{a}_{d-2} = 0, \widehat{a}_{d-1} = 1$ and satisfying the recursion

$$\widehat{a}_{n+d} + A_{d-1}\widehat{a}_{n+d-1} + \cdots + A_0 \widehat{a}_n = 0. \tag{1}$$

The lrs $(\tilde{a}_n)$ is defined by the same recursion (1) with the initial terms $\tilde{a}_j = a_j$ $(j = 0, \ldots, d-1)$.

Denote

$$P(x) = x^d + A_{d-1}x^{d-1} + \cdots + A_0$$

the common characteristic polynomial of $(\widehat{a}_n), (\tilde{a}_n)$ and of (a_n). Assume that $P(x)$ has d distinct roots $\alpha_1, \ldots, \alpha_d$. Then there exists uniquely defined complex numbers $\widehat{g}_j, j = 1, \ldots, d$ with

$$\widehat{a}_n = \widehat{g}_1\alpha_1^n + \cdots + \widehat{g}_d\alpha_d^n \tag{2}$$

for all n.

Theorem 2. *Assume that the zeroes of $P(x)$ are ordered as*

$$|\alpha_1| \geq \cdots \geq |\alpha_{r_1}| > 1 = |\alpha_{r_1+1}| = \cdots = |\alpha_{r_1+r_2}|,\ r_1 + r_2 = d.$$

Denote by $\tilde{g}_j, \widehat{g}_j$ the (constant) coefficients of $\alpha_j^n, j = 1, \ldots, d$ in the expression (2) of $\tilde{a}_n$ and $\widehat{a}_n$ respectively. Then

(i) *if $r_1 > 0$ and $r_2 = 0$ then*

$$a_n = (\tilde{g}_1 + \widehat{g}_1 c(\alpha_1))\alpha_1^n + \cdots + (\tilde{g}_{r_1} + \widehat{g}_{r_1} c(\alpha_{r_1}))\alpha_{r_1}^n + O(1),$$

(ii) *if $r_1 > 0$ and $r_2 > 0$ then*

$$a_n = (\tilde{g}_1 + \widehat{g}_1 c(\alpha_1))\alpha_1^n + \cdots + (\tilde{g}_{r_1} + \widehat{g}_{r_1} c(\alpha_{r_1}))\alpha_{r_1}^n + O(n),$$

and in both cases $(\tilde{g}_j + \widehat{g}_j c(\alpha_j)) \neq 0, j = 1 \ldots, r_1$.

(iii) *if $r_1 = 0$ and $r_2 > 0$ then*

$$a_n = O(n),$$

The proof combines the identity

$$a_n = \tilde{a}_n + \sum_{j=1}^{n-d+1} \widehat{a}_{n-j} e_{d-1+j}$$

with Binet-formulae for $(\tilde{a}_n)$ and $(\widehat{a}_n)$.

- The assertions (iii) of Theorem 2 remains true with simple modifications for nlrs with inseparable characteristic polynomial.

- The error term in (ii) and (iii) is best possible.

- The converse of (i) is also true.

4. On the growth of nlrs

Theorem 3. *Assume that* $r \geq 2$.
(i) *Let* $\eta_1, \ldots, \eta_r$ *be any pairwise distinct complex numbers lying on the unit circle, and* $\gamma_1, \ldots, \gamma_r$ *any non-zero complex numbers. Then there exists a constant* $d_1 > 0$ *such that*

$$|\gamma_1 \eta_1^n + \cdots + \gamma_r \eta_r^n| > d_1 \tag{3}$$

holds for infinitely many positive integers n.
(ii) *Let* $\eta_1, \ldots, \eta_r$ *be any pairwise distinct complex numbers lying on the unit circle such that at least one of* η_j / η_r, $1 \leq j < r$ *is not a root of unity. Let* $\gamma_1, \ldots, \gamma_{r-1} \neq 0$ *complex numbers. Then for all* $d_2 > 1$ *there exists* γ_r *such that the inequality*

$$|\gamma_1 \eta_1^n + \cdots + \gamma_r \eta_r^n| < d_2^{-n} \tag{4}$$

holds for infinitely many positive integers n.

Theorem 3 implies that general linear recursive sequences may have surprising big fluctuation.

Corollary 1. *Let $r \geq 2$ be an integer and $d > 1$ be a real number. There exists a non zero lrs u_n of degree r such that $|u_n| \gg d^n$ for infinitely many n and $|u_n| \ll d^{-n}$ for infinitely many n.*

Remark that as a consequence of the p-adic Subspace Theorem of Schmidt and Schlickewei, if $\gamma_1, \ldots, \gamma_r$, $\eta_1, \ldots, \eta_r$ are all algebraic, non-zero and η_j/η_r is not a root of unity for at least one $1 \leq j < r$, then

$$|\gamma_1 \eta_1^n + \cdots + \gamma_r \eta_r^n| \ll d^{-n}$$

holds for every $d > 1$ for only finitely many positive integers n.

The Skolem-Mahler Lech theorem asserts that if (a_n) is a lrs, then the set of n with $a_n = 0$ is either finite or contains an infinite arithmetic progression. We show that there is no analogue for nlrs.

Theorem 4. *There exist nlrs with integer terms (a_n) such that $\limsup(a_n) = \infty$, but having infinitely many zero terms. Moreover the set of indices of the zero terms does not contain infinite arithmetic progressions.*

Let $P(x) = x^2 + A_1 x + A_2 \in \mathbb{Z}[x]$ with roots $\alpha, \bar{\alpha}$ such that $|\alpha| = |\bar{\alpha}| > 1$, and $\alpha/\bar{\alpha}$ is not a root of unity. Denote $p > 3, p \equiv -1 \pmod 4$ an odd prime. Setting $A_1 = 1, A_2 = \frac{p+1}{4}$ we get such polynomials.

Choosing $d_2 = 2$ there exists by Theorem 3 (ii) a complex number γ such that if $a_{1,n} = \alpha^n + \gamma\bar{\alpha}^n, n = 0, 1, \ldots$ then

$$|a_{1,n}| < 2^{-n}$$

holds for infinitely many n. If $a_{1,n} \in \mathbb{R}$ for all n then let $a_{2,n} = a_{1,n}$, otherwise let

$$a_{2,n} = a_{1,n} + \bar{a}_{1,n} = (1 + \bar{\gamma})\alpha^n + (1 + \gamma)\bar{\alpha}^n.$$

Plainly the sequence $(a_{2,n})$ contains only real numbers, it satisfies the recursion

$$a_{2,n+2} + A_1 a_{2,n+1} + A_2 a_{2,n} = 0,\ n = 0, 1, \ldots \tag{5}$$

and there are infinitely many n such that $|a_{2,n}| < 2^{-(n-1)}$. Further we have $\limsup a_{2,n} = \infty$ by Theorem 3 (i).

For a real number x let $\lfloor x \rceil := [x + 1/2]$. With this notation let

$$a_n = \lfloor a_{2,n} \rceil, \; n = 0, 1, \ldots$$

and $e_{2,n} = a_{2,n} - a_n$. Then (a_n) is a sequence of integers and as $|e_{2,n}| \leq 1/2$ we have $\limsup a_n = \infty$. Moreover, for those $n > 2$ with $|a_{2,n}| < 2^{-(n-1)}$ we have $a_n = 0$, i.e. (a_n) has infinitely many zero terms.

Finally it is easy to prove that (a_n) is a nlrs such that the set of indices of its zero terms does not contain infinite arithmetic progressions.

Now we compare the nlrs (a_n) and its corresponding lrs analogue $(\tilde{a}_n)$. Although $a_n = \tilde{a}_n$ for $0 \leq n < d$, the difference can not be bounded under a mild condition:

Corollary 2. *Under the same assumptions as in Theorem 2 set*

$$R = \{\alpha_i \mid i = 1, \ldots, r_1 \textit{ and } c(\alpha_i) \neq 0\}.$$

Assume that $R \neq \emptyset$. If among the elements of R there is exactly one of maximal modulus, then $\lim_{n\to\infty} |a_n - \tilde{a}_n| = \infty$*, otherwise* $\limsup_{n\to\infty} |a_n - \tilde{a}_n| = \infty$*.*

The last corollary deals with shift radix systems.

Corollary 3. *If the dominating root of the characteristic polynomial of the nlrs* (s_n) *is real, greater than one and not an algebraic integer, then the sequence* $(s_n - \tilde{s}_n)$ *diverges.*

5. Common values of nlrs

Common values of lrs's with algebraic terms are quite well investigated. Thanks to the theory of S-unit equations, Laurent (1989) characterized those sequences, which may have infinitely many common terms. These results are not effective.

If the characteristic polynomials of the sequences have dominating simple roots, which are multiplicatively independent then Mignotte (1978) gave an effective upper bound for the index of possible common values. In the above mentioned results the Binet formula plays central role.

The next result implies that the situation for nlrs is quite different from that of lrs.

Theorem 5. *Let α, β be two multiplicatively independent real numbers > 1. Then there exist nlrs (a_n), (b_n) with integer terms, having characteristic polynomials with dominating roots α, β, respectively, such that there are infinitely many pairs of non-negative integers (k, m) with $a_k = b_m$. This set of pairs (k, m) has finite intersection with every rational line.*

Something remains true from the diophantine theory of lrs.

Theorem 6. *Let* (a_n) *be an nlrs. Assume that its characteristic polynomial is separable, and has a dominant root* α *with* $|\alpha| > 1$*. Then the equation*

$$a_k = a_m \tag{6}$$

has only finitely many solutions with $k \neq m$*.*

6. On bounded nlrs

Theorem 7. *Let $d \in \mathbb{N}$ and $\beta_1, \ldots, \beta_d \in \mathbb{C}$ such that $|\beta_1| \leq \cdots \leq |\beta_r| < 1 < \left|\beta_{r+1}\right| \leq \cdots \leq |\beta_d|$. Furthermore, let $(a_n)_{n\in\mathbb{N}}, (e_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ with $|e_n| \leq E$ for all $n \in \mathbb{N}$, such that*

$$a_{n+d} + p_{d-1}a_{n+d-1} + \cdots + p_0 a_n = e_n \tag{7}$$

for all $n \in \mathbb{N}$, where $\prod_{j=1}^{d}(x - \beta_j) = x^d + p_{d-1}x^{d-1} + \cdots + p_1 x + p_0$.

- *If $r = d$, or*
- *if $r < d$ and $(a_n)_{n\in\mathbb{N}}$ is bounded, then for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that*

$$|a_n| < \frac{E}{\prod_{j=1}^{d}(1 - \left|\beta_j\right|)} + \varepsilon$$

for $n \geq n_0$.

7. Bounded orbits of expansive srs

If $\mathbf{r} = (p_0, \dots, p_{d-1}) \in \mathbb{R}^d$ and $\mathbf{a} = (a_0, \dots, a_d) \in \mathbb{Z}^d$ and $\tau_{\mathbf{r}}^n(\mathbf{a}) = (a_{n+d-1}, \dots, a_n)$ then

$$a_{n+d} + p_{d-1} a_{n+d-1} + \cdots + p_0 a_n = e_n$$

with $e_n \in [0,1]$, i.e. (a_n) is an nlrs associated to the srs $\tau_{\mathbf{r}}$. For such sequences Theorem 7 implies

Corollary 4. *Assume that the sequence of integers* $(a_n)_{n\in\mathbb{N}}$ *is associated to the srs* $\tau_{\mathbf{r}}$*. Order* $\beta_1, \ldots, \beta_d$ *as in Theorem 7. Then,*

(i) if $r = d$*, or*

(ii) if $r < d$ *and* $(a_n)_{n\in\mathbb{N}}$ *is bounded,*

then $(a_n)_{n\in\mathbb{N}}$ *is ultimately periodic and*

$$|a_n| \leq \frac{1}{\prod_{j=1}^{d} \left|1 - \left|\beta_j\right|\right|}$$

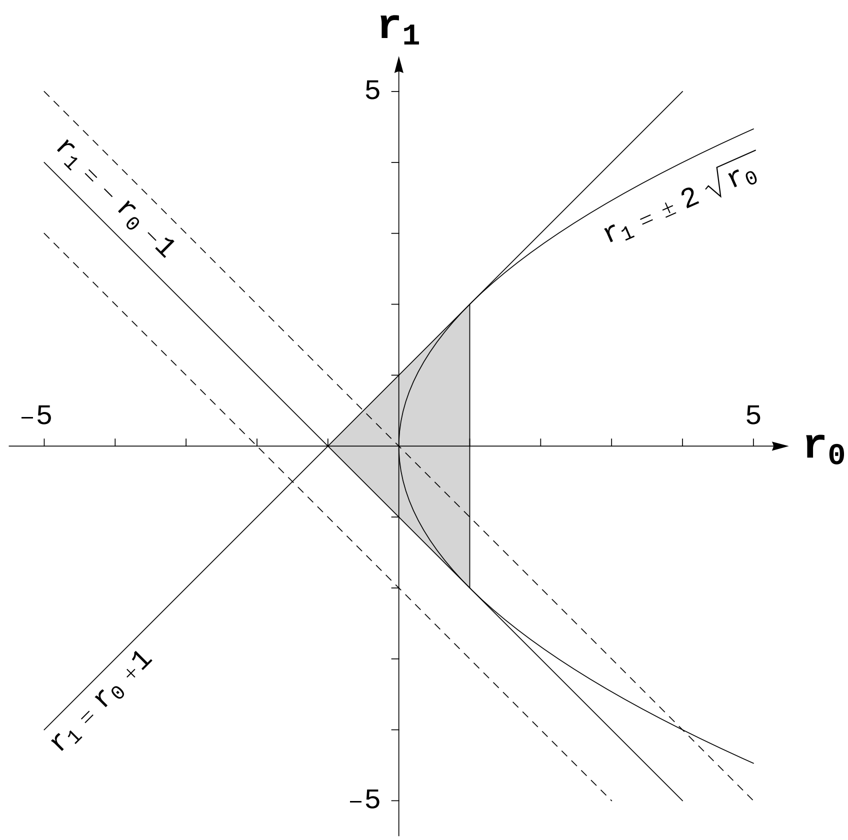
holds for all elements of the cycle.

7. Bounded orbits of expansive 2-dim. srs

For all $\mathbf{r} = (r_0, r_1) \in \mathbb{R}^2$ the zero sequence is an orbit of the srs $\tau_{\mathbf{r}}$. It is called trivial. If $x^2 + r_1 x + r_0$ is expansive then by Theorem 2 the orbits of $\tau_{\mathbf{r}}$ grow exponentially. In some cases there are bounded non-trivial orbits as well.

- If $-2 \le r_0 + r_1 < -1$ then (1),
- if $-1 \le r_0 + r_1 < 0$ then (-1),
- if $r_0 - r_1 = 1$ then $(1, -1)$,
- if $-2 < r_0 \le -1$ and $-1 < r_1 \le 0$ then $(0, -1)$, and
- if $-1 \le r_0 < 0$ and $0 \le r_1 < 1$ then $(0, 1)$

are non-trivial orbits.



Theorem 8. *If $r_1 > r_0 + 1$ or $r_1 < -r_0 - 2$ or $r_0 > \frac{3}{2} + \sqrt{2}$ and $\mathbf{r} = (r_0, r_1)$ does not lie in one of the boxes $-2 < r_0 \leq -1, -1 < r_1 \leq 0$ and $-1 \leq r_0 < 0, 0 \leq r_1 < 1$, then $\tau_{\mathbf{r}}$ has no non-trivial orbit.*