General shift radix systems and discrete rotation

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1. CNS and SRS

Let $p = p_n X^n + \ldots + p_1 X + p_0 \in \mathbb{Z}[X], p_n = 1, |p_0| > 1$ and $\mathcal{D} = \{0, 1, \ldots, |p_0| - 1\}$. The pair $(p, \mathcal{D})$ is called *canonical number system polynomial* - CNS - if there exist for all $0 \neq a \in \mathbb{Z}[X]$ integers ℓ and $a_0, \ldots, a_\ell \in \mathcal{D}$ such that

$$a \equiv a_0 + \ldots + a_\ell X^\ell \pmod{p}.$$

- This is a generalization of radix representation of integers. It was initiated by D. Knuth, and developed further by Penney, I. Kátai, J. Szabó, B. Kovács, etc.
- Not all $(p, \mathcal{D})$ is a CNS! For example $\left(X^2 + 2X + 2, \{0.1\}\right)$ is, but $\left(X^2 - 2X + 2, \{0.1\}\right)$ is not a CNS. Characterization of CNS is a hard problem.

To $\mathbf{r} \in \mathbb{R}^n$ associate the nearly linear mapping $\tau_{\mathbf{r}} \quad \mathbb{Z}^n \mapsto \mathbb{Z}^n$ such that if $(a_1, \ldots, a_n) = \mathbf{a} \in \mathbb{Z}^n$ then

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \ldots, a_n, -[\mathbf{ar}]),$$

where [.] denotes the integer part, and $\mathbf{ar}$ the inner product.

Akiyama et al., 2005, called $\tau_{\mathbf{r}}$ a *shift radix system* - SRS - if the orbit $\tau_{\mathbf{r}}^k(\mathbf{a}), k = 0, 1, \ldots$ is eventually zero for all $\mathbf{a} \in \mathbb{Z}^n$.

They proved: $(p, \mathcal{D})$ is a CNS iff for $\mathbf{r} = \left(\frac{1}{p_0}, \frac{p_{n-1}}{p_0}, \ldots, \frac{p_n}{p_0}\right)$ the mapping $\tau_{\mathbf{r}}$ is a SRS.

Found relation between SRS and β-expansions too.

2. Generalized number system - GNS

Let $\mathcal{O}$ denote an order, that is a commutative ring with 1 whose additive group is free abelian of rank d. Identify $m \in \mathbb{Z}$ with $m \cdot 1$, and thus assume $\mathbb{Z} \subset \mathcal{O}$.

The order $\mathcal{O}$ may be given explicitly by a basis $\{1 = \omega_1, \omega_2 \ldots \omega_d\}$ and a multiplication table

$$\omega_i \omega_j = \sum_{l=1}^{d} a_{ijl} \omega_l \quad (i, j = 2 \ldots d) \quad \text{with } a_{ijl} \in \mathbb{Z}, \tag{1}$$

satisfying the commutativity and associativity rules.

A *generalized number system* over $\mathcal{O}$ (GNS over $\mathcal{O}$ for short) is a pair $(p, \mathcal{D})$, where $p \in \mathcal{O}[X]$ is a monic polynomial such that $p(0)$ is not a zero divisor of $\mathcal{O}$, and where $\mathcal{D}$ is a (necessarily finite) complete residue system of $\mathcal{O}$ modulo $p(0)$ containing 0.

An element $a \in \mathcal{O}[X]$ is *representable in* $(p, \mathcal{D})$ if there exist an integer $L \geq 0$ and $a_0, \ldots, a_L \in \mathcal{D}$ such that

$$a \equiv \sum_{j=0}^{L} a_j X^j \pmod{p}. \tag{2}$$

The set of in $(p, \mathcal{D})$ representable elements is $R(p, \mathcal{D})$. If $R(p, \mathcal{D}) = \mathcal{O}$ then $(p, \mathcal{D})$ is called *GNS with finiteness property*.

A GNS $(p, \mathcal{D})$ over $\mathcal{O}$ may be viewed as a matrix number system introduced by Vince, 1993, with lattice $\Lambda = \mathcal{O}[x]/(p)$, the linear mapping $\varphi : f \pmod{p} \mapsto x \cdot f \pmod{p}$, and digit set $D = \mathcal{D}$.

We may view $\mathcal{O}[X]$ as a free $\mathbb{Z}[X]$-module of finite rank, and $a \mapsto p \cdot a$ as a $\mathbb{Z}[X]$-linear map from $\mathcal{O}[X]$ to itself. The determinant of this $\mathbb{Z}[X]$-linear map is a monic polynomial in $\mathbb{Z}[X]$, which we denote by Np.

Theorem 1 (Evertse, Győry, Pethő, Thuswaldner, 2019). *Let $(p, \mathcal{D})$ be a GNS over $\mathcal{O}$ with $\deg p = n \geq 1$. Then there is an effectively computable number C'', depending on $\mathcal{O}$, p and $\mathcal{D}$, such that the following are equivalent:*
(i) $(p, \mathcal{D})$ has the finiteness property;
(ii) the polynomial Np is expansive, and every $a \in \mathcal{O}[X]$ with $\|a\| \leq C''$, $\deg a < n$ belongs to $R(p, \mathcal{D})$.

3. Families of GNS

We view $\mathcal{O}$ as a full rank sublattice of the $\mathbb{R}$-algebra $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}$. We recall that *$\theta \in \mathcal{O}$ is not a zero divisor of $\mathcal{O}$ if and only if it is invertible in $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}$.*

A fundamental domain for $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}/\mathcal{O}$ is a subset of $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}$ containing precisely one element from every residue class of $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}$ modulo $\mathcal{O}$. For a fundamental domain $\mathcal{F}$ for $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}/\mathcal{O}$ with $0 \in \mathcal{F}$ and $\theta \in \mathcal{O}$ which is not a zero divisor, we define

$$\mathcal{D}_{\mathcal{F},\theta} := \theta\mathcal{F} \cap \mathcal{O} = \{\alpha \in \mathcal{O} :\ \theta^{-1}\alpha \in \mathcal{F}\}.$$

Lemma 1. *Let $\mathcal{F}$ be a fundamental domain for $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}/\mathcal{O}$ with $0 \in \mathcal{F}$ and $\theta \in \mathcal{O}$ not a zero divisor. Then $\mathcal{D}_{\mathcal{F},\theta}$ is a complete residue system for $\mathcal{O}$ modulo θ containing* 0*.*

If $\mathcal{F}$ is a fundamental domain for $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{R}/\mathcal{O}$ with $0 \in \mathcal{F}$ and $p \in \mathcal{O}[X]$ runs through the monic polynomials such that $p(0)$ is not a zero divisor then $(p, \mathcal{D}_{\mathcal{F},p(0)})$ is a family of GNS over $\mathcal{O}$.

For example put $\mathcal{O} = \mathbb{Z}$, $\mathcal{F} = [0, 1)$ and $p \in \mathbb{Z}[X]$ with $p(0) > 1$ then $\mathcal{D}_{\mathcal{F},p(0)} = \{0, 1, \ldots, p(0) - 1\}$, thus $(p, \mathcal{D})$ is exactly the CNS.

4. Generalization of SRS

Let $0 \in \mathcal{F}$ be a fundamental domain for $\mathbb{R}^d$. For any $\mathbf{v} \in \mathbb{R}^d$ there exist a unique $\mathbf{a} \in \mathbb{Z}^d$, such that $\mathbf{v} - \mathbf{a} \in \mathcal{F}$, which will be denoted by $\lfloor \mathbf{v} \rfloor_{\mathcal{F}}$.

For fixed matrices $R_1, \ldots, R_n \in \mathbb{R}^{d \times d}$ define the sequence of integer vectors by the initial terms $\mathbf{a}_1, \ldots \mathbf{a}_n \in \mathbb{Z}^d$ and for $m > n$ by the nearly linear recursive relation

$$\mathbf{a}_m = -\left\lfloor \sum_{\ell=1}^{n} R_\ell \mathbf{a}_{m-n+\ell-1} \right\rfloor_{\mathcal{F}}. \tag{3}$$

With the n-tuple $\mathbf{R} = (R_1, \ldots, R_n)$ of matrices define the mapping $\tau_{\mathbf{R}} : \mathbb{Z}^{d\times n} \mapsto \mathbb{Z}^{d\times n}$ such that if $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{Z}^{d\times n}$ then

$$\tau_{\mathbf{R}}(\mathbf{A}) = (\mathbf{a}_2, \ldots, \mathbf{a}_n, \mathbf{a}_{n+1}), \tag{4}$$

where $\mathbf{a_{n+1}}$ is defined by (3) with $m = n + 1$. The mapping $\tau_{\mathbf{R}}$ is called *generalized SRS, or short GSRS*.

For $k = 1$ identify the matrices $R_1, \ldots, R_n$, with their real entries and $\mathbf{R}$ with a n-dimensional real vector. Similarly $\mathbf{A}$ can be identified with a n-dimensional integer vector. Further in (3) in the bracket stays the inner product of $\mathbf{R}$ and $\mathbf{A}$. Choosing finally $\mathcal{F} = [0, 1)$ we get the familiar definition of SRS.

5. Relation between GNS and GSRS

We show similar relation between GNS and GSRS as between CNS and SRS.

Let $p \in \mathcal{O}[X]$ of degree n be monic, such that $p(0)$ is not a zero divisor and consider the GNS $(p, \mathcal{D})$, where $\mathcal{D} = \mathcal{D}_{\mathcal{F},p(0)}$. Let $a \in \mathcal{O}_n[X]$, where $\mathcal{O}_n[X]$ denotes the elements of $\mathcal{O}[X]$ of degree at most $n-1$.

Let $T_p : \mathcal{O}_n[x] \mapsto \mathcal{O}_n[x]$ be the *backward division mapping*, which is defined as

$$T_p(a)(X) = \frac{a(X) - qp(X) - d}{X},$$

where $d \in \mathcal{D}$ is the unique element of $\mathcal{D}$ with $d \equiv a(0) \pmod{p_0}$ and $q = \frac{a(0)-d}{p_0}$. This means $q = \left\lfloor \frac{a(0)}{p_0} \right\rfloor_{\mathcal{F}}$.

Iterating T_p for h-times we obtain $d_0, \ldots, d_{h-1} \in \mathcal{D}$, and $r \in \mathcal{O}[X]$ such that

$$a(X) = \sum_{j=0}^{h-1} d_j X^j + X^h T_p^h(a)(X) + r(X)p(X). \tag{5}$$

Clearly, $a \in R(p, \mathcal{D})$ if and only if there exists h_0 such that $T_p^h(a) = 0$ for all $h \geq h_0$.

The mapping T_p acts essentially on the coefficient vector $\mathbf{a} = (a_0, \ldots, a_{n-1})$ of $a = \sum_{i=0}^{n-1} a_i X^i$ by the rule

$$T_p(\mathbf{a}) = (a_1 - qp_1, \ldots, a_{n-1} - qp_{n-1}, -qp_n) = T\mathbf{a} - q\mathbf{p},$$

where $q = \left\lfloor \frac{a(0)}{p_0} \right\rfloor_{\mathcal{F}}$, $\mathbf{p} = (p_1, \ldots, p_n)$ and $T = (t_{ij})_{i,j=1,\ldots,n}$ is the matrix with $t_{i,i+1} = 1, i = 1, \ldots, n-1$ and all other entries are zero.

Choosing a different basis for $\mathcal{O}_n[x]$, as in Brunotte (2001) or Scheicher and Thuswaldner (2003)

$$w_j = \sum_{m=1}^{j} p_{n-j+m} X^{m-1},\ j = 1, \ldots, n$$

we get a different form of this transformation. Writing

$$a = \sum_{j=0}^{n-1} a_j X^j = \sum_{j=1}^{n} c_j w_j$$

then

$$T_p(\mathbf{c}) = \left(c_2, \ldots, c_n, -\left\lfloor \mathbf{c}\mathbf{p}' \right\rfloor_{\mathcal{F}}\right),$$

where $\mathbf{c} = (c_1, \ldots, c_n)$ and $p' = \left(\frac{p_n}{p_0}, \ldots, \frac{p_1}{p_0}\right)$.

Now we prove that T_p is a special case of $\tau_{\mathbf{R}}$.

Write $c_j = \mathbf{c}_j(\omega_1, \ldots, \omega_d)$ with $\mathbf{c}_j \in \mathbb{Z}^d, j = 1, \ldots, d$

The multiplication with any fixed element of $\mathcal{O}$ is a linear mapping of $\mathcal{O}$ into itself. As $p(0)$ is not a zero divisor, $1/p(0)$ is a well defined element of $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{R}/\mathcal{O}$. One can extend the multiplication to $1/p(0)$ such that it is again a linear mapping on $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{R}/\mathcal{O}$. Thus there exist $M_1, \ldots, M_n \in \mathbb{Q}^{d \times d}$ associated to the multiplication by $p_n/p_0, \ldots, p_1/p_0$. Thus

$$\mathbf{c}\mathbf{p}' = \sum_{j=1}^{n} M_j \mathbf{c_j}, \text{ i.e, } T_p(\mathbf{c}) = \tau_{M_1, \ldots, M_n}(\mathbf{c_1}, \ldots, \mathbf{c_n}).$$

which proves the claim.

Theorem 2. *Let $p \in \mathcal{O}[X]$ be monic and such that $p(0)$ is not a zero divisor. Let $\mathcal{F}$ be, a fundamental domain for $\mathbb{R}^d$. Then $a\mathcal{O}[X]$ is representable in $(p, \mathcal{D}_{\mathcal{F},p(0)})$ if and only if the orbit of $\tau^k_{(M_1,\ldots,M_n)}(\mathbf{c_1}, \ldots, \mathbf{c_n})$ is ultimately zero.*

An example

Let $\mathcal{O} = \sqrt{-7}, \omega_1 = 1, \omega_2 = \frac{1+\sqrt{-7}}{2}, \omega = (\omega_1, \omega_2)$ and $\mathcal{F} = [0,1)^2$. Then

$$\omega_1 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \omega_2 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}.$$

Let $p = X + \frac{-1+\sqrt{-7}}{2}$. Then

$$\frac{1}{p(0)} = -\frac{1+\sqrt{-7}}{4} = -\frac{\omega_2}{2}.$$

Hence $\mathcal{D} = \{0, -1\}$ and the matrix associated to the multiplication by $1/p(0)$ is $M = \begin{pmatrix} 0 & -1/2 \\ 1 & -1/2 \end{pmatrix}$.

Finally the searched dynamical system is

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto \begin{pmatrix} \lfloor -a_2/2 \rfloor \\ \lfloor a_1 - a_2/2 \rfloor \end{pmatrix},$$

where $a_1, a_2 \in \mathbb{Z}$.

6. Closer look at the case $n = 1$

In the case $n = 1$ the GSRS simplifies to

$$\mathbf{a}_m = \tau_R(\mathbf{a}_{m-1}) = -\lfloor R\mathbf{a}_{m-1} \rfloor_{\mathcal{F}}, \text{ for } m \geq 1,$$

where $R \in \mathbb{R}^{d \times d}$ and $\mathbf{a}_0 \in \mathbb{Z}^d$.

Theorem 3. *If all orbits of τ_R are periodic then the spectral radius of R is at most 1, consequently $|\det R| \leq 1$.*

Theorem 4. *If the spectral radius of R is less than 1 then all orbits of τ_R are periodic.*

Notice that the above properties are independent from $\mathcal{F}$. In the sequel $\mathcal{F} = [0, 1)^d$.

What happens when all eigenvalues of R lie on the unit circle?

6.1. Discrete rotation on the plane

We consider the case $n = 1, d = 2$ and $R \in \mathbb{R}^{2\times 2}$, which has two different eigenvalues on the unit circle. (Only ± 1 can be multiple eigenvalues.) A convenient representation of R is

$$R = TA_\varphi T^{-1}, \qquad A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

where $T \in \mathbb{R}^{2\times 2}$ is an invertible matrix and $0 \leq \varphi < 2\pi$.

The points $R^k(a, b)^T$, $(a, b) \in \mathbb{Z}^2$ form a **bounded set**; generally they lie on an ellipse, in the case $T = E$, i.e., $R = A_\varphi$ on the unit circle.

Akiyama, Brunotte, Pethő and Steiner (2006) studied the case $n = 2, d = 1$, when $a_{m+1} = -\lfloor \lambda a_m + a_{m-1} \rfloor$ with $|\lambda| < 2$.

We can write

$$\begin{aligned} a_{m+1} &= -\lfloor \lambda a_m + a_{m-1} \rfloor \\ a_m &= -\lfloor -a_m \rfloor. \end{aligned}$$

Putting $R = \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix}$ we obtain

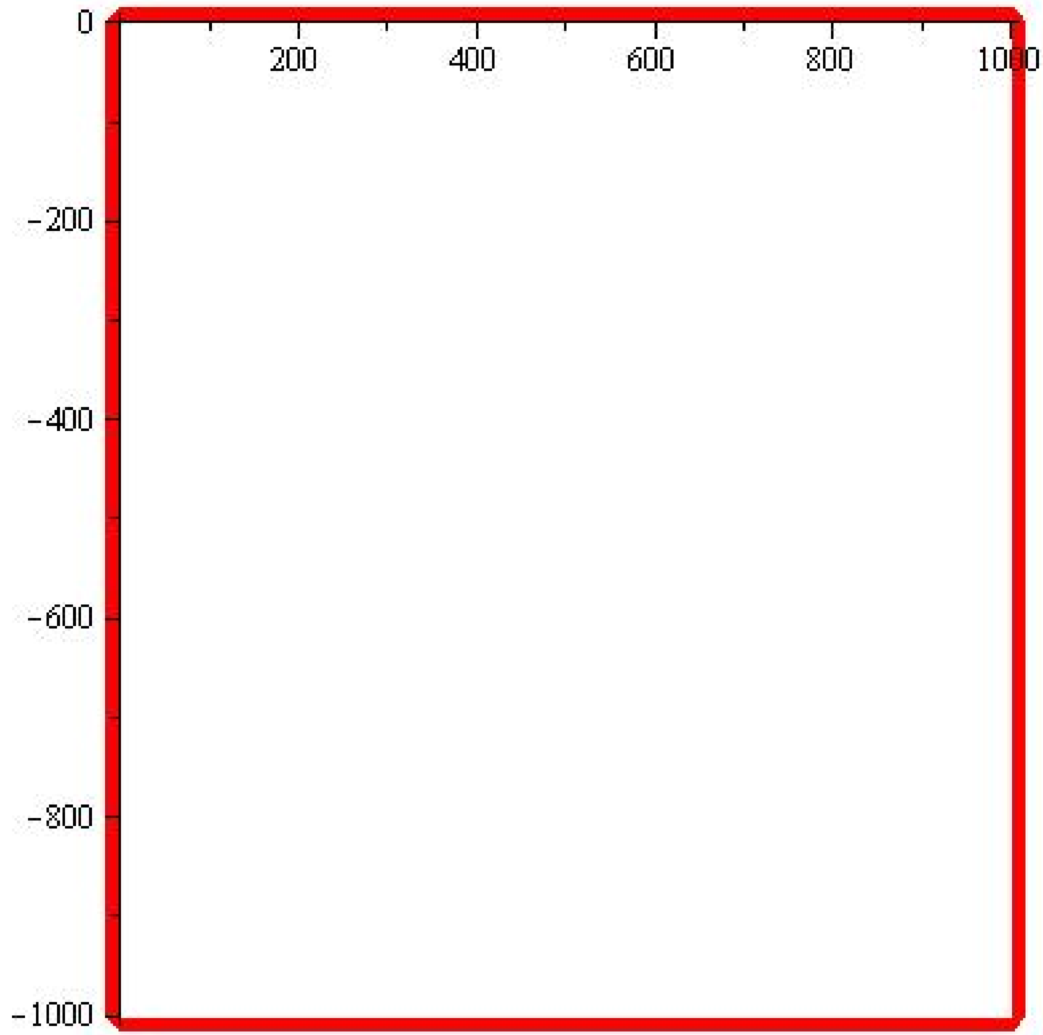
$$\begin{pmatrix} a_{m+1} \\ a_m \end{pmatrix} = -\left\lfloor R \begin{pmatrix} a_m \\ a_{m-1} \end{pmatrix} \right\rfloor.$$

Thus $n = 2, d = 1$ is a special case of $n = 1, d = 2$.

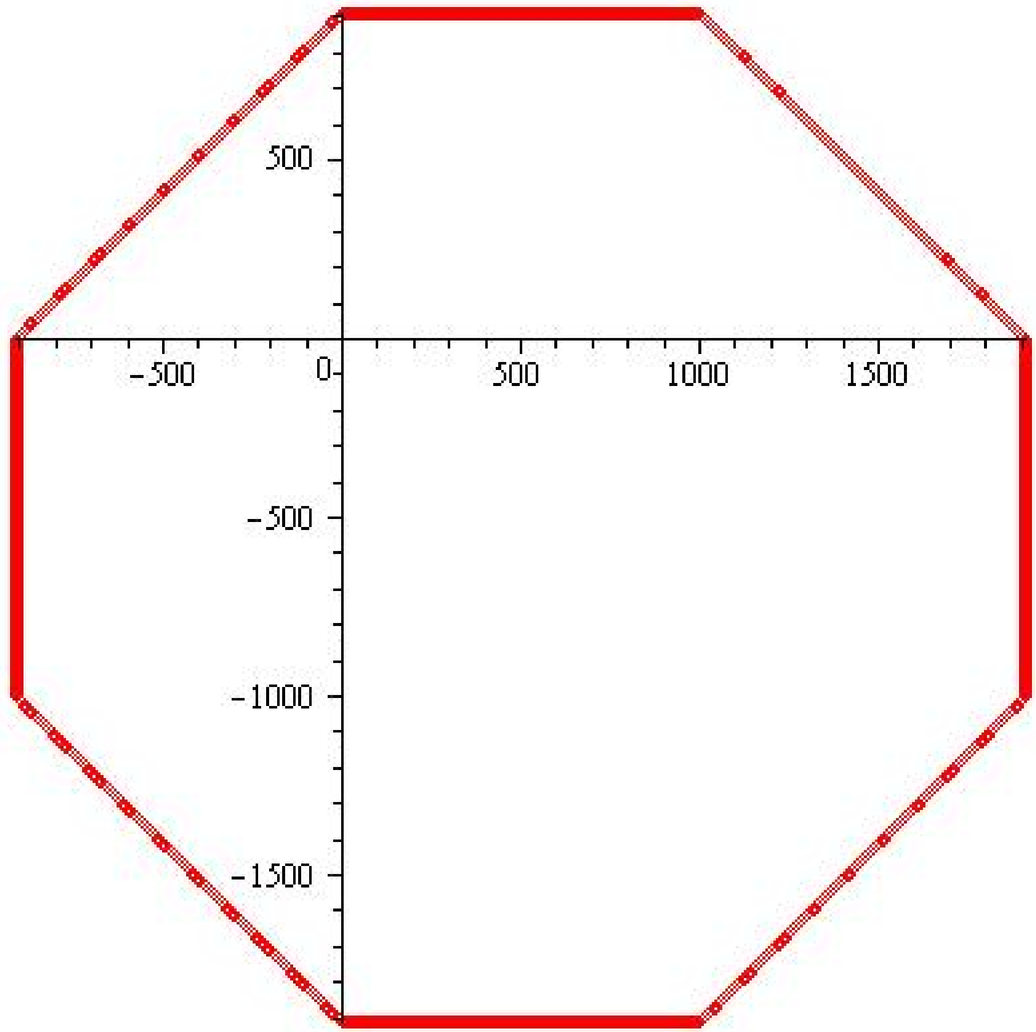
Akiyama et al. conjecture that the sequence (a_m) is always periodic.

In 2008 they verified this conjecture for $\lambda = \pm\sqrt{2}, \pm\frac{1\pm\sqrt{5}}{2}$.

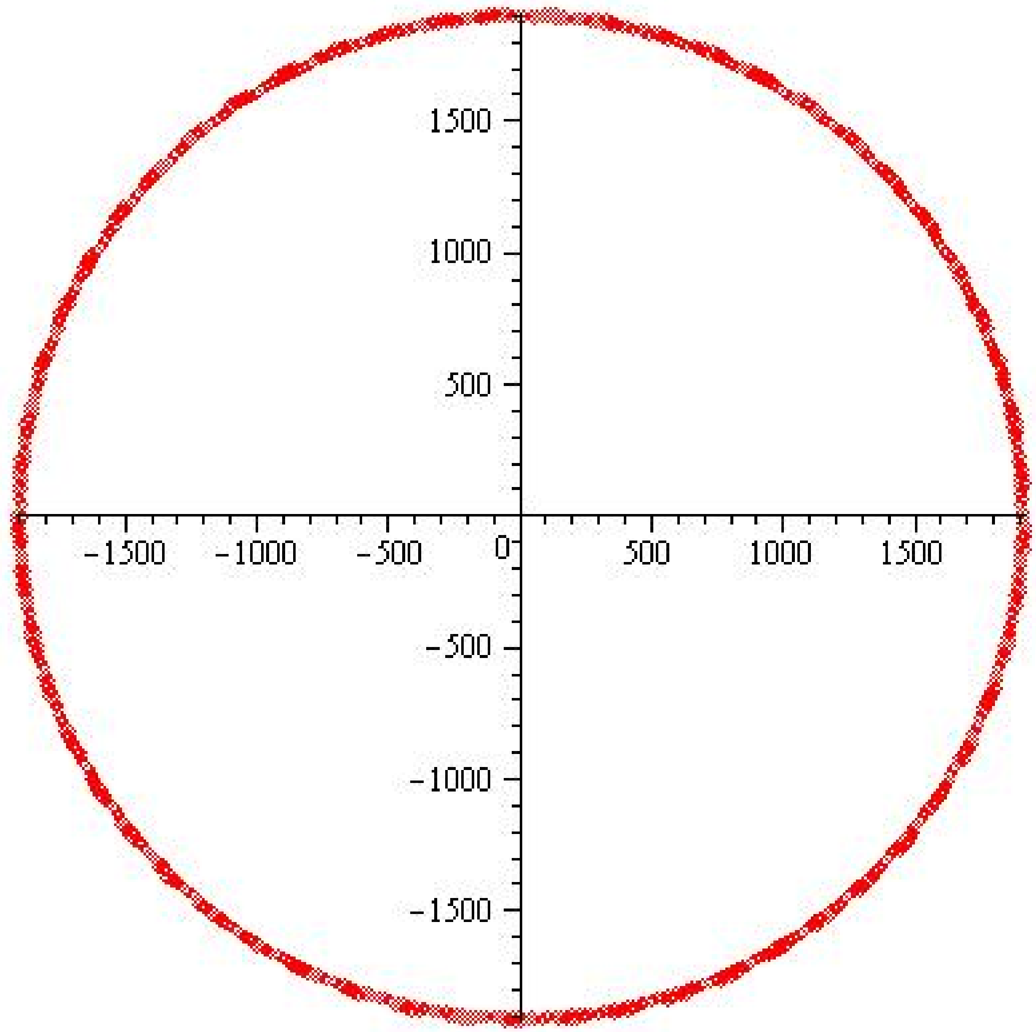
Akiyama and Pethő proved (2013) that for any λ there are infinitely many starting values a_0, a_1 such that (a_m) is periodic.



Staring value $(10, 9), \varphi = 0.001$.



Staring value $(0, 910), \varphi = 0.001$.



Staring value $(1904, 0), \varphi = 0.11.$

Theorem 5. *There are infinitely many* $(a, b) \in Z^2$ *such that the sequence* $\mathbf{x}_0 = (a, b), \mathbf{x}_{m+1} = \lfloor A_{\pi/4}\mathbf{x}_m, m = 0, 1, \ldots$ *is periodic of length* 8.

The proof is tiering computation with the integer part function. Its essence is:

Lemma 2. *Let* $a \in \mathbb{N}$, $\omega = \lfloor \frac{1}{\sqrt{2}} a \rfloor$ *and suppose* $\lfloor \sqrt{2}\omega \rfloor = a - 1$. *If* $\{\frac{1}{\sqrt{2}} a\} \in \left[1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$, *and* $\{\sqrt{2}\omega\} \in \left[1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$, *then* $A_\varphi^8(a, 0) = (a, 0)$.

Thank you for the attention!