GENERAL SHIFT RADIX SYSTEMS AND DISCRETE ROTATION

ATTILA PETHŐ
FACULTY OF INFORMATICS, UNIVERSITY OF DEBRECEN
PETHO.ATTILA@INF.UNIDEB.HU

Abstract

My talk is based on joint research with Carolin Hannusch.

Let $\mathcal{F}$ be a bounded fundamental domain of the action of $\mathbb{Z}^k$ on $\mathbb{R}^k$. For any $\mathbf{v} \in \mathbb{R}^k$ there exists a unique $\mathbf{a} \in \mathbb{Z}^k$, such that $\mathbf{v} - \mathbf{a} \in \mathcal{F}$, which is denoted by $\lfloor \mathbf{v} \rfloor_{\mathcal{F}}$.

For fixed matrices $R_1, \dots, R_n \in \mathbb{R}^{k \times k}$ define the sequence of integer vectors by the initial terms $\mathbf{a}_1, \dots \mathbf{a}_n \in \mathbb{Z}^k$ and for $m > n$ by the nearly linear recursive relation

$$\mathbf{a}_m = - \left\lfloor \sum_{\ell=1}^{n} R_\ell \mathbf{a}_{m-n+\ell-1} \right\rfloor_{\mathcal{F}}. \tag{1}$$

With the n-tuple $\mathbf{R} = (R_1, \dots, R_n)$ of matrices define the mapping $\tau_{\mathbf{R}} : \mathbb{Z}^{k \times n} \mapsto \mathbb{Z}^{k \times n}$ such that if $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{Z}^{k \times n}$ then

$$\tau_{\mathbf{R}}(\mathbf{A}) = (\mathbf{a}_2, \dots, \mathbf{a}_n, \mathbf{a}_{n+1}), \tag{2}$$

where $\mathbf{a_{n+1}}$ is defined by (1) with $m = n + 1$. The mapping $\tau_{\mathbf{R}}$, which is a discrete dynamical system, is called *general srs, or short gsrs.*

In the first part we show that gsrs is a generalization not only of srs introduced by Akiyama, Borbély, Brunotte, Pethő and Thuswaldner in 2005, but also of GNS over general orders, studied very recently by Evertse, Győry, Pethő and Thuswaldner.

In the second part we concentrate on the special case $n = 1$ and $\mathcal{F} = [0, 1)^k$, i.e., when $\mathbf{a}_0 \in \mathbb{Z}^k$ and

$$\tau_R(\mathbf{a}_m) = \lfloor R\mathbf{a}_m \rfloor, \ m = 0, 1, \dots$$

with a $k \times k$ real matrix R and the integer part function is meant coordinate wise. It is easy to see that every orbits of τ_R is bounded, i.e, ultimately periodic, if all eigenvalues of R lie in the unit circle. In the other hand there exist unbounded orbits if some eigenvalue lie outside the unit circle. The fascinating question is, like for srs, what happens if all eigenvalues of R lie on the unit circle? This class includes the rotations of the plane, in which case τ_R is a digital rotation. We performed lot of experiments with different values of R and $\mathbf{a}_0$ and we found only ultimately periodic orbits. If the angle of the rotation is $2\pi/3, \pi/2$ and $\pi/4$ we show infinite classes of initial vectors for which the orbits are indeed ultimately periodic.

We show under general conditions on R that τ_R is not surjective.