Systems of linear equations (SLE)

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Solving an SLE

Example

The SLE

$$\begin{aligned} -2x_1 - x_2 + 4x_3 &= 3 \\ 2x_1 + 3x_2 - x_3 &= 1 \\ -4x_1 - 10x_2 - 5x_3 &= -12 \end{aligned}$$

can be written in matrix form $Ax = b$, where

$$A = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 1 \\ -12 \end{pmatrix}$$

Gaussian elimination (without row-swapping):

$$\left(\begin{array}{rrr|r} -2 & -1 & 4 & 3 \\ 2 & 3 & -1 & 1 \\ -4 & -10 & -5 & -12 \end{array}\right) \xrightarrow[\ell_{31}=2]{\ell_{21}=-1} \left(\begin{array}{rrr|r} -2 & -1 & 4 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & -8 & -13 & -18 \end{array}\right)$$

$$\xrightarrow{\ell_{32}=-4} \left(\begin{array}{rrr|r} -2 & -1 & 4 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & -1 & -2 \end{array}\right)$$

The back substitution:

$$\begin{aligned} -x_3 &= -2 & \rightarrow \quad x_3 &= 2 \\ 2x_2 + 3x_3 &= 4 & \rightarrow \quad x_2 &= -1 \\ -2x_1 - x_2 + 4x_3 &= 3 & \rightarrow \quad x_1 &= 3 \end{aligned}$$

Gaussian elimination (without row-swapping): $Ax = b$, where $A \in \mathbb{R}^{n \times n}$.
The `augmented` matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & \vdots & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right)$$

Suppose that $a_{11} \neq 0$. Subtracting $\ell_{i1} = \frac{a_{i1}}{a_{11}}$ times the 1-st row from the i-th one, we get:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\ \vdots & \vdots & & \vdots & \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} & b_n^{(2)} \end{array}\right)$$

Using the $a_{ij}^{(1)} := a_{ij}$ notation:

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \ell_{i1} a_{1j}^{(1)} \qquad \ell_{i1} = \frac{a_{i1}^{(1)}}{a_{11}^{(1)}}$$

$$b_i^{(2)} = b_i^{(1)} - \ell_{i1} b_1^{(1)} \qquad \begin{matrix} i = 2, \dots, n, \\ j = 2, \dots, n \end{matrix}$$

If $a_{22}^{(2)} \neq 0$ then subtracting $\ell_{i2} = \frac{a_{i2}^{(2)}}{a_{22}^{(2)}}$ times the 2-nd row from the i-th one:

$$\left(\begin{array}{ccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} & b_3^{(3)} \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & a_{n3}^{(3)} & \cdots & a_{nn}^{(3)} & b_n^{(3)} \end{array} \right)$$

In the k-th phase, (if $a_{kk}^{(k)} \neq 0$):

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \ell_{ik} a_{kj}^{(k)} \qquad \ell_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}$$

$$b_i^{(k+1)} = b_i^{(k)} - \ell_{ik} b_k^{(k)} \qquad \begin{matrix} i = k+1, \ldots, n, \\ j = k+1, \ldots, n \end{matrix}$$

If we have $a_{11}^{(1)} \neq 0$, $a_{22}^{(2)} \neq 0$, ... , $a_{n-1,n-1}^{(n-1)} \neq 0$, then after the $n-1$-th phase:

$$\left(\begin{array}{ccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} & b_3^{(3)} \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & a_{nn}^{(n)} & b_n^{(n)} \end{array}\right)$$

If $a_{nn}^{(n)} \neq 0$, then we can begin the back-substitution.

The Gaussian-elimination without row-swapping can be done, if and only if:

$$a_{11}^{(1)} \neq 0, \;\; a_{22}^{(2)} \neq 0 \;\; \dots a_{nn}^{(n)} \neq 0$$

Complexity:

Counting only the multiplications ($n \geq 2$):

$$\sum_{k=1}^{n-1} (n-k)^2 + (n-1) = \frac{(n-1)n(2n-2)}{6} + (n-1) \leq \frac{n^3}{3}$$

For the back-substitution we need additional:

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \mathcal{O}(n^2)$$

operations, so the final complexity:

$$\frac{n^3}{3} + \mathcal{O}(n^2)$$

LU-factorization

Example

$$A = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{pmatrix}$$

The Gaussian elimination:

$$\begin{pmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{pmatrix} \xrightarrow[\ell_{31} = 2]{\ell_{21} = -1} \begin{pmatrix} -2 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & -8 & -13 \end{pmatrix}$$

In matrix form:

$$\begin{pmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & -8 & -13 \end{pmatrix}$$

Gaussian elimination:

$$\begin{pmatrix} -2 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & -8 & -13 \end{pmatrix} \xrightarrow{\ell_{32} = -4} \begin{pmatrix} -2 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

In matrix form:

$$\begin{pmatrix} -2 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & -8 & -13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \begin{pmatrix} -2 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

The steps can be combined into one matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix},$$

We have:

$$A = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}}_{L:=} \underbrace{\begin{pmatrix} -2 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}}_{U:=}$$

The `LU`-factorization of matrix A:

$$A = LU$$

where L is a `lower-triangle` matrix with full of 1-s in its main diagonal and U is an `upper-triangle` matrix.

The LU-factorization

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \ell_{21} & 1 & 0 & \dots & 0 \\ \ell_{31} & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ \ell_{n1} & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & & & \\ 0 & a_{n2}^{(2)} & \dots & a_{nn}^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ \ell_{21} & 1 & 0 & 0 & \dots & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 & \dots & 0 \\ \ell_{41} & \ell_{42} & 0 & 1 & \dots & 0 \\ \vdots & & & & & \\ \ell_{n1} & \ell_{n2} & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \dots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \dots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{n3}^{(3)} & \dots & a_{nn}^{(3)} \end{pmatrix}$$

After the $(n-1)$-th phase:

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \ell_{21} & 1 & 0 & \dots & 0 \\ \ell_{31} & \ell_{32} & 1 & \dots & 0 \\ \vdots & & & & \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \dots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \dots & a_{3n}^{(3)} \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & a_{nn}^{(n)} \end{pmatrix}$$

$$A = LU$$

We call it the `LU`-factorization of matrix A.

Let us turn back to our original problem, solving the SLE:

$$Ax = b \Leftrightarrow LUx = b$$

After the factorization done, the solution can be obtained in two (similar) steps:

1. $Ly = b$
2. $Ux = y$

Both systems are in triangular form, so the back-substitution can be used.

The determinant of A

If $A = LU$, then by the properties of determinant:

$$\det(A) = \det(L) \cdot \det(U)$$

For matrices in triangle form the determinant is the product of all elements in their main diagonal, so:

$$\det(A) = \det(L)\det(U) = 1\det(U) = \det(U) =$$

$$u_{11} \cdot u_{22} \cdot \ldots \cdot u_{nn}$$

The determinant of A is the product of elements in U's main diagonal.

The LU-factorization

Example continued

$Ax = b$, where

$$A = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 1 \\ -12 \end{pmatrix}$$

The factorization of A:

$$A = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} -2 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}}_{U}$$

Solving $Ly = b$:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -12 \end{pmatrix}$$

Back-substituting from up to down:

$$\begin{aligned} y_1 &= 3 & & \\ -y_1 + y_2 &= 1 & \quad\rightarrow\quad & y_2 = 4 \\ 2y_1 - 4y_2 + y_3 &= -12 & \quad\rightarrow\quad & y_3 = -2 \end{aligned}$$

Solving $Ux = y$.

$$\begin{pmatrix} -2 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

Back-substitute upwards:

$$\begin{aligned} -x_3 = -2 \quad &\rightarrow \quad x_3 = 2 \\ 2x_2 + 3x_3 = 4 \quad &\rightarrow \quad x_2 = -1 \\ -2x_1 - x_2 + 4x_3 = 3 \quad &\rightarrow \quad x_1 = 3 \end{aligned}$$

The determinant of A:

$$\det(A) = \det(U) = (-2) \cdot 2 \cdot (-1) = 4$$

The complexity of `LU` factorization is the same as of the Gaussian-elimination. For the factorization we need: $\frac{n^3}{3} + \mathcal{O}(n^2)$ operations, which does not change with the back-substitutions, because they require n^2 operations. It follows that, computing the determinant can be done with $\approx \frac{n^3}{3}$ operations.

The space complexity of the LU-factorzation:

$$A = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & -8 & -13 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix} \begin{pmatrix} -2 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

It can be written in a more compact form:

$$\begin{pmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & -1 & 4 \\ -1 & 2 & 3 \\ 2 & -8 & -13 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -2 & -1 & 4 \\ -1 & 2 & 3 \\ 2 & -4 & -1 \end{pmatrix}$$

Example

Solve the LSE $Ax = b$ with LU-factorization!

$$A = \begin{pmatrix} 2 & -1 & 1 & 3 \\ -4 & 0 & 2 & -5 \\ 6 & -1 & -2 & 6 \\ 4 & 2 & -10 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -3 \\ -5 \\ 2 \\ 19 \end{pmatrix}$$

$$A \to \begin{pmatrix} \boxed{2} & -1 & 1 & 3 \\ -2 & -2 & 4 & 1 \\ 3 & 2 & -5 & -3 \\ 2 & 4 & -12 & -7 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 & 3 \\ -2 & \boxed{-2} & 4 & 1 \\ 3 & -1 & -1 & -2 \\ 2 & -2 & -4 & -5 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & -1 & 1 & 3 \\ -2 & -2 & 4 & 1 \\ 3 & -1 & \boxed{-1} & -2 \\ 2 & -2 & 4 & 3 \end{pmatrix}$$

So:

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 2 & -2 & 4 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & -1 & 1 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}}_{U}$$

The back-substitutions:

1. $Ly = b$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 2 & -2 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \\ 2 \\ 19 \end{pmatrix}$$

$$\begin{aligned} y_1 &= -3 & & \\ -2y_1 + y_2 &= -5 & \quad\rightarrow\quad & y_2 = -11 \\ 3y_1 - y_2 + y_3 &= 2 & \quad\rightarrow\quad & y_3 = 0 \\ 2y_1 - 2y_2 + 4y_3 + y_4 &= 19 & \quad\rightarrow\quad & y_4 = 3 \end{aligned}$$

The back-substitutions:

2. $Ux = y$

$$\begin{pmatrix} 2 & -1 & 1 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ -11 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{aligned} 3x_4 = 3 \quad &\rightarrow \quad x_4 = 1 \\ -x_3 - 2x_4 = 0 \quad &\rightarrow \quad x_3 = -2 \\ -2x_2 + 4x_3 + x_4 = -11 \quad &\rightarrow \quad x_2 = 2 \\ 2x_1 - x_2 + x_3 + 3x_4 = -3 \quad &\rightarrow \quad x_1 = -1 \end{aligned}$$

The determinant:

$$\det(A) = \det(U) = 2 \cdot (-2) \cdot (-1) \cdot 3 = 12$$

Exercise 1

Solve $Ax = b$ with LU-factorization! Compute the determinant of A!

1.
$$A = \begin{pmatrix} -2 & 2 & 1 \\ 6 & -3 & -4 \\ -4 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ -8 \\ 4 \end{pmatrix},$$

2.
$$A = \begin{pmatrix} -1 & -2 & 0 & -3 \\ 1 & -1 & 2 & 2 \\ 2 & -2 & 6 & 5 \\ 0 & -6 & -2 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 5 \\ 14 \\ 0 \end{pmatrix},$$

3.
$$A = \begin{pmatrix} -2 & 1 & 4 & -2 \\ 2 & -4 & -1 & 1 \\ -4 & 8 & 6 & -3 \\ -6 & 3 & 8 & -3 \end{pmatrix}, \qquad b = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 16 \end{pmatrix}.$$

Exercise 2

Solve $Ax = b$ and $Ax = c$ with LU-factorization!

$$A = \begin{pmatrix} -3 & 1 & 0 \\ 12 & -6 & 2 \\ -6 & 4 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 5 \\ -28 \\ 12 \end{pmatrix}, \qquad c = \begin{pmatrix} -6 \\ 18 \\ -15 \end{pmatrix}.$$

Remark

If we have more than one constant vectors (possibly in the future) but one coefficient matrix A – the LU factorization of A can be and should be used multiple times! It is one of the main application of the factorizations.

Cholesky-factorization

If A is symetric and positive definit then it can be written in the form of LL^T, where L is a lower triangular matrix, what is called the `Cholesky-factorization` of A. Here we do not need to compute and store L^T, so the time and space complexity is approximately the half of the complexity of Gaussian elimination.

Cholesky-factorization

The elements of L can be computed column-wise:

$$\ell_{kk} = \left(a_{kk} - \sum_{j=1}^{k-1} \ell_{kj}^2 \right)^{1/2},$$

$$\ell_{ik} = \left(a_{ik} - \sum_{j=1}^{k-1} \ell_{ij} \ell_{kj} \right) / \ell_{kk}, \quad i = k+1, \ldots, n.$$

The back substitutions:
1. $Ly = b$ (overwriting b with y)

$$b_i = \left(b_i - \sum_{j=1}^{i-1} l_{ij} b_j \right) / l_{ii}, \quad i = 1, \ldots, n$$

2. $L^T x = y$ (overwriting y with x)

$$b_i = \left(b_i - \sum_{j=i+1}^{n} l_{ji} b_j \right) / l_{ii}, \quad i = n, \ldots, 1$$

Example

$$A = \begin{pmatrix} \boxed{9} & & \\ -3 & 5 & \\ -6 & 4 & 9 \end{pmatrix}, \quad L = \begin{pmatrix} 3 & & \\ -1 & X & \\ -2 & X & X \end{pmatrix}$$

$$A = \begin{pmatrix} 9 & & \\ -3 & \boxed{5} & \\ -6 & 4 & 9 \end{pmatrix}, \quad L = \begin{pmatrix} 3 & & \\ -1 & 2 & \\ -2 & 1 & X \end{pmatrix}$$

$$A = \begin{pmatrix} 9 & & \\ -3 & 5 & \\ -6 & 4 & \boxed{9} \end{pmatrix}, \quad L = \begin{pmatrix} 3 & & \\ -1 & 2 & \\ -2 & 1 & 2 \end{pmatrix}$$

that is,

$$A = \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & 1 & 2 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}_{L^T}$$

Exercise 3

Compute the Cholesky-factorization and the determinant!

$$A = \begin{pmatrix} 4 & -2 & -4 \\ -2 & 10 & 5 \\ -4 & 5 & 9 \end{pmatrix}$$

$$B = \begin{pmatrix} 4 & -4 & -4 & 0 \\ -4 & 13 & 7 & 6 \\ -4 & 7 & 6 & -1 \\ 0 & 6 & -1 & 17 \end{pmatrix}$$

$$C = \begin{pmatrix} 9 & -3 & -3 \\ -3 & 5 & 5 \\ -3 & 5 & 9 \end{pmatrix}$$

Exercise 4

Solve $Ax = b$ with Cholesky-factorization!

1.

$$A = \begin{pmatrix} 9 & -6 & -6 & 3 \\ -6 & 8 & 6 & -6 \\ -6 & 6 & 9 & -10 \\ 3 & -6 & -10 & 18 \end{pmatrix}, \quad b = \begin{pmatrix} 36 \\ -38 \\ -47 \\ 58 \end{pmatrix}$$

2.

$$A = \begin{pmatrix} 9 & -6 & 3 \\ -6 & 20 & -14 \\ 3 & -14 & 14 \end{pmatrix}, \quad b = \begin{pmatrix} -21 \\ 34 \\ -34 \end{pmatrix}$$

PLU-factorization (GE with row-swapping)

Permutation matrix: swapping some rows of the identity matrix
For example, swapping the i-th and j-th rows ($i < j$):

$$P = \begin{pmatrix} 1 & 0 & \dots & & & & 0 & 0 \\ 0 & 1 & \dots & & & & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & & & & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_j \\ \vdots \\ e_i \\ \vdots \\ e_n \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow i. \\ \\ \leftarrow j. \\ \\ \\ \end{matrix}$$

Facts

PA: swaps the i-th and j-th `rows` of A

AP: swaps the i-th and j-th `columns` of A

Gaussian-elimination:

1. if $a_{11} \neq 0 \quad \rightarrow \quad$ then the 1-st phase can be performed
2. if $a_{11} = 0$ and $a_{i1} = 0$ for all $i = 2, \ldots, n \quad \rightarrow \quad$, nothing to eliminate in the 1-st column $\quad \rightarrow \quad$ go to phase 2
3. if $a_{11} = 0$, and there an i, with $a_{i1} \neq 0 \quad \rightarrow \quad$ swap the 1-st and the i-th row, and proceed the 1-st phase. If P_1 swaps the 1-st and the i-th rows, then:

$$A = P_1 A_1$$

and the 1-st phase can be performed for A_1. After the 1-st phase, we have:

$$A = P_1 A_1 = P_1 L_1 A^{(2)}$$

2-nd phase:

1. if $a_{22}^{(2)} \neq 0 \quad \rightarrow \quad$ then the 2-nd phase can be performed
2. if $a_{22}^{(2)} = 0$ and $a_{i2}^{(2)} = 0$ for all $i = 3, \ldots, n \quad \rightarrow \quad$ nothing to eliminate $\quad \rightarrow \quad$ go to phase 3
3. if $a_{22}^{(2)} = 0$ and $a_{i2}^{(2)} \neq 0$ for some $i > 2 \quad \rightarrow \quad$ swap the rows 2 and i, and proceed the 2-nd phase.

If P_2 swaps the 2-nd and the i-th rows, then:

$$A^{(2)} = P_2 A_2$$

$$A = P_1 L_1 P_2 A_2$$

$$L_1 P_2 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \ell_{21} & 1 & 0 & \dots & 0 \\ \ell_{31} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \\ \ell_{i1} & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & & \\ \ell_{n1} & 0 & 0 & \dots & 1 \end{pmatrix} P_2 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ \ell_{21} & 0 & 0 & \dots & 1 & \dots & 0 \\ \ell_{31} & 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & & & & \\ \ell_{i1} & 1 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & & & & \\ \ell_{n1} & 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

$$L_1 P_2 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \ell_{21} & 1 & 0 & \dots & 0 \\ \ell_{31} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \\ \ell_{i1} & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & & \\ \ell_{n1} & 0 & 0 & \dots & 1 \end{pmatrix} P_2 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ \ell_{21} & 0 & 0 & \dots & 1 & \dots & 0 \\ \ell_{31} & 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & & & & \\ \ell_{i1} & 1 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & & & & \\ \ell_{n1} & 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

$$= P_2 \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \ell_{i1} & 1 & 0 & \dots & 0 \\ \ell_{31} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \\ \ell_{21} & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & & \\ \ell_{n1} & 0 & 0 & \dots & 1 \end{pmatrix} =: P_2 \widetilde{L}_1$$

$$A = P_1 P_2 \widetilde{L}_1 A_2$$

$\rightarrow$ continue the factorization...
After $(n-1)$-th phase:

$$A = \underbrace{P_1 P_2 \cdots P_{n-1}}_{P:=} LU$$

that is,

$$A = PLU$$

where
P: is a permutation matrix
L: is a lower triangle matrix, (with 1-s in its main diagonal)
U: is an upper triangle matrix

Solving $Ax = b$ is a two step procedure:

$$A = PLU$$

$$LUx = P^{-1}b$$

1. solve $Ly = P^{-1}b$
2. solve $Ux = y$

Facts

1. $P^{-1} = P^T$
2. instead of P it is more convenient to maintain P^T
3. no need to store P or P^{-1} as a matrix, it can be represented as a vector.

Example

$$A = \begin{pmatrix} 0 & 1 & -2 & 4 \\ 1 & -3 & 0 & 2 \\ 4 & 2 & -28 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} -6 \\ -11 \\ -2 \\ -1 \end{pmatrix}$$

1-st phase (after swap(1,2)):

$$P = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & -2 & 4 \\ 4 & 2 & -28 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -3 & 0 & 2 \\ 0 & 1 & -2 & 4 \\ 4 & 14 & -28 & -7 \\ -1 & -3 & 1 & 3 \end{pmatrix}$$

2-nd phase (no swap):

$$\rightarrow \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & \boxed{1} & -2 & 4 \\ 4 & 14 & 0 & -63 \\ -1 & -3 & -5 & 15 \end{pmatrix}$$

3-rd phase (after swap(3,4)):

$$P = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & -2 & 4 \\ -1 & -3 & \boxed{-5} & 15 \\ 4 & 14 & 0 & -63 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -3 & 1 & 0 \\ 4 & 14 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -5 & 15 \\ 0 & 0 & 0 & -63 \end{pmatrix}$$

$$b = \begin{pmatrix} -6 \\ -11 \\ -2 \\ -1 \end{pmatrix}, \qquad P^{-1}b = \begin{pmatrix} -11 \\ -6 \\ -1 \\ -2 \end{pmatrix}$$

1. $Ly = P^{-1}b \quad \to \quad y_1 = -11,\ y_2 = -6,\ y_3 = -30,\ y_4 = 126$
2. $Ux = y \quad \to \quad x_4 = -2,\ x_3 = 0,\ x_2 = 2,\ x_1 = -1$

Linear algebra with Octave/Matlab

Example

Solve $Ax = b$ if

$$A = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \\ -12 \end{pmatrix}$$

Solution

Use the `backslash` operator!

```
>>A=[-2 -1 4; 2 3 -1; -4 -10 -5];
>>b=[3; 1; -12];
>>x=A\b
x=
   3
  -1
   2
```

Note, that b must be a column vector.

Apply the function `rref` to the augmented matrix of the LSE:

```
>>rref([A b])

ans=

1 0 0 -3
0 1 0 -1
0 0 1 -2
```

One can from the output that b is a linear combination of the column vectors of A, and the columns of A is `linearly independent` system, so the solution is unique.

Example

Solve the system $Ax = b$, if

$$A = \begin{pmatrix} -4 & -4 & 2 \\ -2 & -7 & 3 \\ 2 & 12 & -5 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 6 \\ -13 \end{pmatrix}$$

Solution

Using the `backslash`, we get

```
>>A=[-4 -4 2; -2 -7 3; 2 12 -5];
>>b=[-2; 6; -13];
>>x=A\b
Warning: Matrix is singular to working precision
x=

   1.93162
  -1.27350
   0.31624
```

The matrix is singular ($\det(A) = 0$), but we can assure that $||Ax - b||$ is about machine precision, so x is acceptable solution.

Using the `rref`, we get:

```
>> rref( [A, b] )
ans =

   1.00000   0.00000  -0.10000   1.90000
   0.00000   1.00000  -0.40000  -1.40000
   0.00000   0.00000   0.00000   0.00000
```

We can read from the output that the column vectors form a *dependent* system. We see also another solution. Actually the system has infinitely many solutions.

By using the function `null`, all solutions to the system can be obtained. The function returns a base of the null-space of A:

```
>> format rat
>> null(A)
ans =

  -60/649
  -240/649
  -600/649
```

So, the general solution of the system is:

$$\begin{pmatrix} 1.9 \\ -1.4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -60/649 \\ -240/649 \\ -600/649 \end{pmatrix}$$

ahol $\lambda \in \mathbb{R}$.

Multiple right hand sides

Example

Solve the following two systems: $Ax = b$ and $Ax = c$, if

$$A = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 3 & -1 \\ -4 & -10 & -5 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \\ -12 \end{pmatrix}, \quad c = \begin{pmatrix} 17 \\ 1 \\ -42 \end{pmatrix}$$

Solution

We can solve them in one run:

```
>>A=[-2 -1 4; 2 3 -1; -4 -10 -5];
>>b=[3; 1; -12]; c=[17; 1; -42];
>>x=A\[b c]
x=
   3  -2
  -1   3
   2   4
```