Norms, condition numbers

Ágnes Baran, Csaba Noszály

Example

Compare the solutions of the linear systems below:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Observation: `small` change in the right-hand side - `big` change in the solution. Explanation?

Exercise 1

Implement a function that returns the $n \times n$ matrix $A = (a_{ij})$:

$$a_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -1, & \text{if } i < j, \\ 0, & \text{otherwise.} \end{cases}$$

Hint

Solution

Exercise 2

Using the 100×100 matrix A of the previous exercise, compare the solutions of the linear systems below:

$Ax = b_1$	$Ay = b_2$

where $b_1 = -98 : 1$ and b_2 is a "perturbed" version of b_1, that is

$$b_2 = b_1$$
$$b_2(100) = b_2(100) + 1e - 6$$

Use the `backslash` operator (`\`) !

Hint

Solution

Suppose that $A \in \mathbb{R}^{n\times n}$ is invertible, $b \in \mathbb{R}^n$, $b \neq 0$. We are searching for the solution of $Ax = b$. In practice the right hand side (observations, measurements) is not exactly known, it contains errors (δb), so we *have* to solve $Ay = b + \delta b$.

The question: how big can be the vector $y - x$? We want to measure vectors, difference of vectors, so we introduce the notion of `norm`.

Norm

Let X a linear vector space over $\mathbb{R}$. The map $\|.\| : X \to \mathbb{R}$ is a norm on X if:

1. $\|x\| \geq 0$ for all $x \in X$
2. $\|x\| = 0 \iff x = 0$
3. $\|\lambda x\| = |\lambda| \|x\|$, for all $\lambda \in \mathbb{R}$ and $x \in X$
4. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ triangle inequality

Norms on $\mathbb{R}^n$

Let $X = \mathbb{R}^n$

1. 1-norm (or octahedron, or Manhattan)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

2. 2-norm (or euclidean):

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

3. ∞-norm (or maximum):

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

Example

For

$$x = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

we have:

$\|x\|_1 = |-3| + |0| + |1| = 4$

$\|x\|_2 = \left(|-3|^2 + |0|^2 + |1|^2\right)^{1/2} = \sqrt{10}$

$\|x\|_\infty = \max\{|-3|, |0|, |1|\} = 3$

Exercise 3

Depict the set $\{x \in \mathbb{R}^2 : \|x\| = 1\}$ for 1,2 and ∞-norm!

Exercise 4

Implement functions for computing the 1,2 and ∞ norm!

Exercise 5

Compute the ∞-norm of vectors b_1, b_2 and x, y!

Exercise 6

Read the help of the function `norm`!

Matrix-norm

Let $\|\cdot\|$ be a vector-norm on $\mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is a matrix. Then the quantity

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

defines a norm on $\mathbb{R}^{n \times n}$. It is the matrix norm induced by the given vector norm.

Remark

It can be verified that $\|.\|$ is a norm.

Properties of the matrix-norm

1. $\|E\| = 1$
2. $\|Ax\| \leq \|A\| \cdot \|x\|$ for all $x \in \mathbb{R}^n$
3. $\|AB\| \leq \|A\| \cdot \|B\|$ for all $A, B \in \mathbb{R}^{n \times n}$

Remark

The (induced) matrix norm can be defined as the smallest number M, for which

$$\|Ax\| \leq M \cdot \|x\|$$

holds, for all $x \in \mathbb{R}^n$

Computational rules for $1, 2, \infty$ matrix norms

1. 1-norm (column norm):

$$\boxed{\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|}$$

2. ∞ norm (row norm):

$$\boxed{\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|}$$

3. 2-norm (spectral norm):

$$\boxed{\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}}$$

where $\lambda_{\max}(A^T A)$ is the largest eigenvalue of $A^T A$.

Example

$$A = \begin{pmatrix} -3 & 0 & 4 \\ 1 & -1 & 2 \\ -2 & 1 & -2 \end{pmatrix}, \qquad \|A\|_1 =? \quad \|A\|_\infty =?$$

$$\begin{array}{cl} \begin{pmatrix} -3 & 0 & 4 \\ 1 & -1 & 2 \\ -2 & 1 & -2 \end{pmatrix} & \begin{array}{l} \leftarrow 7 \\ \leftarrow 4 \\ \leftarrow 5 \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ 6 & 2 & 8 \end{array} & \end{array}$$

$\|A\|_1 = 8$ and $\|A\|_\infty = 7$

Example

$$A = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}, \qquad \|A\|_2 = ?$$

$$A^T A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 13 & -7 \\ -7 & 5 \end{pmatrix}$$

the eigenvalues of $A^T A$:

$$\begin{vmatrix} 13-\lambda & -7 \\ -7 & 5-\lambda \end{vmatrix} = (13-\lambda)(5-\lambda) - 49 = \lambda^2 - 18\lambda + 16 = 0$$

$$\lambda_{1,2} = \frac{18 \pm \sqrt{18^2 - 64}}{2} = 9 \pm \sqrt{65}$$

$\|A\|_2 = \sqrt{9 + \sqrt{65}} \approx 4.13$

Exercise 7

Implement functions for computing the induced $1, \infty$ matrix norms!

Hint

The condition number

Suppose that $A \in \mathbb{R}^{n\times n}$ is invertible, $b \in \mathbb{R}^n$, $b \neq 0$. We are searching for the solution of $Ax = b$. Suppose that we have error in the right hand side, so we *have* to solve the $A(x + \delta x) = b + \delta b$ system. Then, on the one hand:

$$A(x + \delta x) = b + \delta b$$

$$\underline{Ax} + A \cdot \delta x = \underline{b} + \delta b$$

$$A \cdot \delta x = \delta b$$

$$\delta x = A^{-1}\delta b$$

$$\|\delta x\| = \|A^{-1}\delta b\| \leq \|A^{-1}\| \cdot \|\delta b\|$$

On the other hand: $Ax = b \implies$

$$\|b\| = \|Ax\| \leq \|A\| \cdot \|x\|$$

$$\frac{1}{\|x\|} \leq \|A\| \cdot \frac{1}{\|b\|}$$

Putting them together:

$$\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\|A\| \cdot \|A^{-1}\|}_{cond(A):=} \frac{\|\delta b\|}{\|b\|}$$

$$\boxed{\frac{\|\delta x\|}{\|x\|} \leq cond(A) \frac{\|\delta b\|}{\|b\|}}$$

Definition

Let A be an invertible matrix. Then

$$\boxed{cond(A) = \|A\| \cdot \|A^{-1}\|}$$

is called the `condition number` of A.

Properties

1. its value *does* depend on the norm used
2. $cond(A) \geq 1$
3. if A is orthogonal ($A^T A = E$), then $cond_2(A) = 1$
4. $$\left|\frac{\lambda_{\max}}{\lambda_{\min}}\right| \leq cond(A)$$
   where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues in absolute value of A
5. for $c \neq 0$, $cond(cA) = cond(A)$

Remark

Let $C = cond(A)$. The condition number tells us, that the relative error in the right-hand side can get C-times large in the solution. Note that it is the worst case scenario.

Example

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1.0001 \end{pmatrix}, \quad cond_\infty(A) = ?$$

Then $\det(A) = 10^{-4}$,

$$A^{-1} = \begin{pmatrix} 10001 & -10000 \\ -10000 & 10000 \end{pmatrix}$$

$\|A\|_\infty = 2.0001$ and $\|A^{-1}\|_\infty = 20001$,

$cond_\infty(A) = 2.0001 \cdot 20001 \approx 40000.$

The error in the solution can be as large as 40000 times the error in the right hand side.

Remark

The condition number does not depend on the determinant!

Exercise 2 cont.

Compute the relative error of the right-hand side, the solution and the condition number! Use 1 and ∞ norm!

Review

Inverse of a 2×2 matrix: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to A^{-1} = \frac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{det(A)}$

Exercise 8

Solve the system $Ax = b$ where

$$A = \begin{pmatrix} 1 & 0.99 \\ 0.99 & 0.98 \end{pmatrix}, \qquad b = \begin{pmatrix} 1.99 \\ 1.97 \end{pmatrix}.$$

Now, suppose that instead of b, we have

$$b + \delta b = \begin{pmatrix} 1.98 \\ 1.98 \end{pmatrix}$$

Solve the system $Ay = b + \delta b$! Also, compute the relative error of the right-hand side and the solution in ∞-norm. What is $cond_\infty(A)$?

Hilbert matrix

$$H_n = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots & 1/n \\ 1/2 & 1/3 & 1/4 & \cdots & 1/(n+1) \\ 1/3 & 1/4 & 1/5 & \cdots & 1/(n+2) \\ \vdots & & & & \\ 1/n & 1/(n+1) & 1/(n+2) & \cdots & 1/(2n-1) \end{pmatrix}$$

Exercise 9a

Implement a function that computes the $n \times n$ Hilbert-matrix!

Exercise 9b

Compute the condition number of the Hilbert matrix of size 6×6! Use `hilb` and `cond`!

Exercise 9c

What is the condition numbers of a random matrix of size 6×6? Experiment with different instances, use `rand` !

Remark

For computing the condition numbers of large matrices use the `condest` function! It computes an estimation of $cond_1(A)$, without the expensive A^{-1} computations.

Suppose for the relative error of b:

$$\boxed{\frac{\|\delta b\|}{\|b\|} \approx \varepsilon_1}$$

If, in addition we have:

$$cond(A) \geq \frac{1}{\varepsilon_1}$$

then

$$cond(A)\frac{\|\delta b\|}{\|b\|} \geq 1,$$

that is, the error in the solution can be as large as the solution itself. It is bad. Such matrices (or linear systems) are called ill-conditioned .

In order to get at least 1 exact digit in the solution, we need smaller condition number. For example, if

$$cond(A) \leq \frac{1}{a\varepsilon_1}$$

then

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{1}{a}$$