1. Introduction to Logic and Computer Science
2. Foundations of Computer Science
3. Logic in Computer Science

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September 10, 2017

The main task of logic

- to give the laws of valid arguments (inferences, consequence relations)

Valid arguments

- Valid arguments (inferences):
  - an argument (an inference): a relation between premise(s) and conclusion
  - a consequence relation
    - input: premise(s)
    - output: conclusion
  - Valid arguments (inferences, consequence relations): if all premises are true, then the conclusion is true.
  - Logically valid arguments: when the former holds necessarily.

Definition/1

Classical zero–order language is an ordered triple

$$L^{(0)} = \langle LC, Con, Form \rangle$$

where

1. $LC = \{\neg, \supset, \wedge, \vee, \equiv, (,)\}$ (the set of logical constants).
2. $Con \neq \emptyset$ the countable set of non-logical constants (propositional parameters)
3. $LC \cap Con = \emptyset$
4. The set of formulae i.e. the set *Form* is given by the following inductive definition:

Definition/2

- $Con \subseteq Form$
- If $A \in Form$, then $\neg A \in Form$.
- If $A, B \in Form$, then
  - $(A \supset B) \in Form$,
  - $(A \wedge B) \in Form$,
  - $(A \vee B) \in Form$,
  - $(A \equiv B) \in Form$.

Remark

The members of the set *Con* are the atomic formulae (prime formulae).

Definition

- If A is an atomic formula, then it has no direct subformula;
- $\neg A$ has exactly one direct subformula: A;
- Direct subformulae of formulae $(A \supset B)$, $(A \wedge B)$, $(A \vee B)$, $(A \equiv B)$ are formulae A and B, respectively.

Definition

The set of subformulae of formula A [denoting: $SF(A)$] is given by the following inductive definition:

1. $A \in RF(A)$ (i.e. the formula A is a subformula of itself);
2. if $A' \in RF(A)$ and B is a direct subformula of A'-nek, then $B \in RF(A)$
   (i.e., if A' is a subformula of A, then all direct subformulae of A' are subformulae of A).

Definition

The contruction tree of a formula A is a finite ordered tree whose nodes are formulae,

- the root of the tree is the formmula A,
- the node with formula $\neg B$ has one child: he node with the formula B,
- the node with formulae $(B \supset C)$, $(B \wedge C)$, $(B \vee C)$, $(B \equiv C)$ has two children: the nodes with B, and C
- the leaves of the tree are atomic formulae.

Definition

The function ϱ is an interpretation of the language $L^{(0)}$ if

1. $Dom(\varrho) = Con$
2. If $p \in Con$, then $\varrho(p) \in \{0, 1\}$.

Definition

Let ϱ be an interpretation and $|A|_\varrho$ be the semantic value of the formula A formula with respect to ϱ.

1. If $p \in Con$, then $|p|_\varrho = \varrho(p)$
2. If $A \in Form$, then $|\neg A|_\varrho = 1 - |A|_\varrho$.
3. If $A, B \in Form$, then
    - $|(A \supset B)|_\varrho = \begin{cases} 0 & \text{if } |A|_\varrho = 1, \text{ and } |B|_\varrho = 0; \\ 1, & \text{otherwise} \end{cases}$
    - $|(A \wedge B)|_\varrho = \begin{cases} 1 & \text{if } |A|_\varrho = 1, \text{ and } |B|_\varrho = 1; \\ 0, & \text{otherwise} \end{cases}$
    - $|(A \vee B)|_\varrho = \begin{cases} 0 & \text{if } |A|_\varrho = 0, \text{ and } |B|_\varrho = 0; \\ 1, & \text{otherwise.} \end{cases}$
    - $|(A \equiv B)|_\varrho = \begin{cases} 1 & \text{if } |A|_\varrho = |B|_\varrho; \\ 0, & \text{otherwise.} \end{cases}$

Definition (model – a set of formulas)

Let $\Gamma \subseteq Form$ be a set of formulas. An interpretation ϱ is a model of the set of formulas Γ, if $|A|_\varrho = 1$ for all $A \in \Gamma$.

Definition – a model of a formula

A model of a formula A is the model of the singleton $\{A\}$.

Definition – satisfiable a set of formulas

The set of formulas $\Gamma \subseteq Form$ is satisfiable if it has a model.
(If there is an interpretation in which all members of the set Γ are ture.)

Definition – satisfiable a formula

A formula $A \in Form$ is satisfiable, if the singleton $\{A\}$ is satisfiable.

Remark

- A satisfiable set of formulas does not involve a logical contradiction; its formulas may be true together.
- A safisfiable formula may be true.
- If a set of formulas is satisfiable, then its members are satisfiable.
- But: all members of the set $\{p, \neg p\}$ are satisfiable, and the set is not satisfiable.

Theorem

All subsets of a satisfiable set are satisfiable.

Proof

- Let $\Gamma \subseteq Form$ be a set of formulas and $\Delta \subseteq \Gamma$.
- Γ is satisfiable: it has a model. Let ϱ be a model of Γ.
- A property of ϱ: If $A \in \Gamma$, then $|A|_\varrho = 1$
- Since $\Delta \subseteq \Gamma$, if $A \in \Delta$, then $A \in \Gamma$, and so $|A|_\varrho = 1$. That is the interpretation ϱ is a model of Δ, and so Δ is satisfiable.

Definition – unsatisfiable set

The set $\Gamma \subseteq Form$ is unsatisfiable if it is not satisfiable.

Definition – unsatisfiable formula

A formula $A \in Form$ is unsatisfiable if the singleton $\{A\}$ is unsatisfiable.

Remark

A unsatisfiable set of formulas involve a logical contradiction. (Its members cannot be true together.)

Theorem

All expansions of an unsatisfiable set of formulas are unsatisfiable.

Indirect proof

- Suppose that $\Gamma \subseteq Form$ is an unsatisfiable set of formulas and $\Delta \subseteq Form$ is a set of formulas.
- Indirect condition: Γ is unsatisfiable, and $\Gamma \cup \Delta$ satisfiable.
- $\Gamma \subseteq \Gamma \cup \Delta$
- According to the former theorem Γ is satisfiable, and it is a contradiction.

Definition

A formula A is the logical consequence of the set of formulas Γ if the set $\Gamma \cup \{\neg A\}$ is unsatifiable. $(Notation : \Gamma \vDash A)$

Definition

$A \vDash B$, if $\{A\} \vDash B$.

Definition

The formula A is valid if $\emptyset \vDash A$. (Notation: $\vDash A$)

The formulas A and B are logically equivalent if $A \vDash B$ and $B \vDash A$. (Notation: $A \Leftrightarrow B$)

Theorem

Let $\Gamma \subseteq Form$, and $A \in Form$. $\Gamma \models A$ if and only if all models of the set Γ are the models of formula A. (i.e. the singleton $\{A\}$).

Proof

$\rightarrow$ Indirect condition: There is a model of $\Gamma \models A$ such that it is not a model of the formula A.
Let the interpretation ϱ be this model.
The properties of ϱ:

1. $|B|_\varrho = 1$ for all $B \in \Gamma$;
2. $|A|_\varrho = 0$, and so $|\neg A|_\varrho = 1$

In this case all members of the set $\Gamma \cup \{\neg A\}$ are true wrt ϱ-ban, and so $\Gamma \cup \{\neg A\}$ is satisfiable. It means that $\Gamma \not\models A$, and it is a contradiction.

Proof

$\leftarrow$ Indirect condition: All models of the set Γ are the models of formula A, but (and) $\Gamma \not\models A$.
In this case $\Gamma \cup \{\neg A\}$ is satisfiable, i.e. it has a model.
Let the interpretation ϱ be a model.
The properties of ϱ:

1. $|B|_\varrho = 1$ for all $B \in \Gamma$;
2. $|\neg A|_\varrho = 1$, i.e. $|A|_\varrho = 0$

So the set Γ has a model such that it is not a model of formula A, and it is a contradiction.

Corollary

Let $\Gamma \subseteq Form$, and $A \in Form$. $\Gamma \models A$ if and only if for all interpretations in which all members of Γ are true, the formula A is true.

Theorem

If A is a valid formula $((\vDash A))$, then $\Gamma \vDash A$ for all sets of formulas Γ. (A valid formula is a consequence of any set of formulas.)

Proof

- If A is a valid formula, then $\emptyset \vDash A$ (according to its definition).
- $\emptyset \cup \{\neg A\}$ $(= \{\neg A\})$ is unsatisfiable, and so its expansions are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of $\{\neg A\}$, and so it is unsatisfiable, i.e. $\Gamma \vDash A$.

Theorem

If Γ is unsatisfiable, then $\Gamma \vDash A$ for all A. (All formulas are the consequences of an unsatisfiable set of formulas.)

Proof

- According to a proved theorem: If Γ is unsatisfiable, the all expansions of Γ are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of Γ, and so it is unsatisfiable, i.e. $\Gamma \vDash A$.

Theorem

Deduction theorem: If $\Gamma \cup \{A\} \vDash B$, then $\Gamma \vDash (A \supset B)$.

Proof

- Indirect condition: Suppose, that $\Gamma \cup \{A\} \vDash B$, and $\Gamma \nvDash (A \supset B)$.
- $\Gamma \cup \{\neg(A \supset B)\}$ is satisfiable, and so it has a model. Let the interpretation ϱ be a model.
- The properties of ϱ:
  1. All members of Γ are true wrt ϱ.
  2. $|\neg(A \supset B)|_\varrho = 1$
- $|(A \supset B)|_\varrho = 0$, i.e. $|A|_\varrho = 1$ and $|B|_\varrho = 0$. So$|\neg B|_\varrho = 1$.
- All members of $\Gamma \cup \{A\} \cup \{\neg B\}$ are true wrt interpretation ϱ, i.e. $\Gamma \cup \{A\} \nvDash B$, and it is a contradiction.

Theorem

In the opposite direction: If $\Gamma \vDash (A \supset B)$, then $\Gamma \cup \{A\} \vDash B$.

Proof

- Indirect condition: Suppose that $\Gamma \vDash (A \supset B)$, and $\Gamma \cup \{A\} \nvDash B$.
- So $\Gamma \cup \{A\} \cup \{\neg B\}$ is satisfiable, i.e. it has a model. Let the interpretation ϱ a model.
- The properties of ϱ:
  1. All members of Γ are true wrt the interpretation ϱ.
  2. $|A|_\varrho = 1$
  3. $|\neg B|_\varrho = 1$, and so $|B|_\varrho = 0$
- $|(A \supset B)|_\varrho = 0$, $|\neg(A \supset B)|_\varrho = 1$.
- All members of $\Gamma \cup \{\neg(A \supset B)\}$ are true wrt the interpretation ϱ, i.e. $\Gamma \nvDash (A \supset B)$.

Corollary

$A \vDash B$ if and only if $\vDash (A \supset B)$

Proof

Let $\Gamma = \emptyset$ in the former theorems.

Cut elimination theorem

If $\Gamma \cup \{A\} \vDash B$ and $\Delta \vDash A$, then $\Gamma \cup \Delta \vDash B$.

Proof

Indirect.

The truth table of negation

$\neg$	$\neg p$
0	1
1	0

- The law of double negation: $\neg\neg A \Leftrightarrow A$

The truth table of conjunction

$\wedge$	0	1	(q)
0	0	0	
1	0	1	
(p)			

- Commutative: $(A \wedge B) \Leftrightarrow (B \wedge A)$ for all $A, B \in Form$.
- Associative: $(A \wedge (B \wedge C)) \Leftrightarrow ((A \wedge B) \wedge C)$ for all $A, B, C \in Form$.
- Idempotent: $(A \wedge A) \Leftrightarrow A$ for all $A \in Form$.

- $(A \wedge B) \vDash A$, $(A \wedge B) \vDash B$
- The law of contradiction: $\vDash \neg(A \wedge \neg A)$
- The set $\{A_1, A_2, \dots, A_n\}$ $(A_1, A_2, \dots, A_n \in Form)$ is satisfiable iff the formula $A_1 \wedge A_2 \wedge \dots \wedge A_n$ is satisfiable.
- The set $\{A_1, A_2, \dots, A_n\}$ $(A_1, A_2, \dots, A_n \in Form)$ is unsatisfiable iff the formula $A_1 \wedge A_2 \wedge \dots \wedge A_n$ is unsatisfiable.
- $\{A_1, A_2, \dots, A_n\} \vDash A$ $(A_1, A_2, \dots, A_n, A \in Form)$ iff $A_1 \wedge A_2 \wedge \dots \wedge A_n \vDash A$.
- $\{A_1, A_2, \dots, A_n\} \vDash A$ $(A_1, A_2, \dots, A_n, A \in Form)$ iff the formula $((A_1 \wedge A_2 \wedge \dots \wedge A_n) \wedge \neg A)$ is unsatisfiable.

The truth table of disjunction:

$\vee$	0	1
0	0	1
1	1	1

- Commutative: $(A \vee B) \Leftrightarrow (B \vee A)$ for all $A, B \in Form$.
- Associative: $(A \vee (B \vee C)) \Leftrightarrow ((A \vee B) \vee C)$ for all $A, B, C \in Form$.
- Idempotent: $(A \vee A) \Leftrightarrow A$ for all $A \in Form$.
- $A \vDash (A \vee B)$ for all $A, B \in Form$.
- $\{(A \vee B), \neg A\} \vDash B$
- The law of excluded middle: $\vDash (A \vee \neg A)$

- Connection between conjunction and disjunction:

$\wedge$	0	1
0	0	0
1	0	1

	1	0
1	1	1
0	1	0

$\vee$	0	1
0	0	1
1	1	1

- Conjunction and disjunction are dual truth functors.

- Two laws of distributivity:
  - $(A \vee (B \wedge C)) \Leftrightarrow ((A \vee B) \wedge (A \vee C))$
  - $(A \wedge (B \vee C)) \Leftrightarrow ((A \wedge B) \vee (A \wedge C))$

- Properties of absorption
  - $(A \wedge (B \vee A)) \Leftrightarrow A$
  - $(A \vee (B \wedge A)) \Leftrightarrow A$

De Morgan's laws

- What do we say when we deny a conjunction?
- What do we say when we deny a disjunction?
- $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$
- $\neg(A \vee B) \Leftrightarrow (\neg A \wedge \neg B)$

- The proofs of De Morgan's laws.
-

A	B	$\neg A$	$\neg B$	$(\neg A \wedge \neg B)$	$(A \vee B)$	$\neg(A \vee B)$
0	0	1	1	1	0	1
0	1	1	0	0	1	0
1	0	0	1	0	1	0
1	1	0	0	0	1	0

- The truth table of implication:

$\supset$	0	1
0	1	1
1	0	1

- $\models (A \supset A)$
- Modus ponens: $\{(A \supset B), A\} \models B$
- Modus tollens:
  $\{(A \supset B), \neg B\} \models \neg A$
- Chain rule: $\{(A \supset B), (B \supset C)\} \models (A \supset C)$
- Reduction to absurdity: $\{(A \supset B), (A \supset \neg B)\} \models \neg A$

- $\neg A \models (A \supset B)$
- $B \models (A \supset B)$
- $((A \wedge B) \supset C) \Leftrightarrow (A \supset (B \supset C))$
- Contraposition: $(A \supset B) \Leftrightarrow (\neg B \supset \neg A)$
- $(A \supset \neg A) \models \neg A$
- $(\neg A \supset A) \models A$
- $(A \supset (B \supset C)) \Leftrightarrow ((A \supset B) \supset (A \supset C))$
- $\models (A \supset (\neg A \supset B))$
- $((A \vee B) \supset C) \Leftrightarrow ((A \supset C) \wedge (B \supset C))$
- $\{A_1, A_2, \ldots, A_n\} \models A$ $(A_1, A_2, \ldots, A_n, A \in \mathit{Form})$ iff the formula $((A_1 \wedge A_2 \wedge \cdots \wedge A_n) \supset A)$ is valid.

- The truth table of (material) equivalence:

$\equiv$	0	1
0	1	0
1	0	1

- $\models (A \equiv A)$
- $\models \neg(A \equiv \neg A)$

Expressibility

- $(A \supset B) \Leftrightarrow \neg(A \wedge \neg B)$
- $(A \supset B) \Leftrightarrow (\neg A \vee B)$
- $(A \wedge B) \Leftrightarrow \neg(A \supset \neg B)$
- $(A \vee B) \Leftrightarrow (\neg A \supset B)$
- $(A \vee B) \Leftrightarrow \neg(\neg A \wedge \neg B)$
- $(A \wedge B) \Leftrightarrow \neg(\neg A \vee \neg B)$
- $(A \equiv B) \Leftrightarrow ((A \supset B) \wedge (B \supset A))$

Theory of truth functors

Base

- A base is a set of truth functors whose members can express all truth functors.
  - For example: $\{\neg, \supset\}$,$\{\neg, \wedge\}$, $\{\neg, \vee\}$
    1. $(p \wedge q) \Leftrightarrow \neg(p \supset \neg q)$
    2. $(p \vee q) \Leftrightarrow (\neg p \supset q)$
  - Truth functor Sheffer: $(p|q) \Leftrightarrow_{def} \neg(p \wedge q)$
  - Truth functor neither-nor: $(p \parallel q) \Leftrightarrow_{def} (\neg p \wedge \neg q)$
  - Remark: Singleton bases: $(p|q)$, $(p \parallel q)$

Definition

If $p \in Con$, then formulas $p, \neg p$ are literals (p is the base of the literals).

Definition

If the formula A is a literal or a conjunction of literals with different bases, then A is an elementary conjunction.

Definition

If the formula A is a literal or a disjunction of literals with different bases, the A is an elementary disjunction.

Definition

A disjunction of elementary conjunctions is a disjunctive normal form.

Definition

A conjunction of elementary disjunctions is a conjunctive normal form.

Theorem

There is a normal form of any formula of proposition logic, i. e. if $A \in Form$, then there is a formula B such that B is a normal form and $A \Leftrightarrow B$

Definition/1

The language of first–order logic is a

$$L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$$

ordered 5–tuple, where

1. $LC = \{\neg, \supset, \wedge, \vee, \equiv, =, \forall, \exists, (,)\}$: (the set of logical constants).
2. Var $(= \{x_n : n = 0, 1, 2, \ldots\})$: countable infinite set of variables

Definition/2

3. $Con = \bigcup_{n=0}^{\infty}(\mathcal{F}(n) \cup \mathcal{P}(n))$ the set of non–logical constants (at best countable infinite)
    - $\mathcal{F}(0)$: the set of name parameters,
    - $\mathcal{F}(n)$: the set of n argument function parameters,
    - $\mathcal{P}(0)$: the set of prposition parameters,
    - $\mathcal{P}(n)$: the set of predicate parameters.
4. The sets LC, Var, $\mathcal{F}(n)$, $\mathcal{P}(n)$ are pairwise disjoint $(n = 0, 1, 2, \dots)$.

Definition/3

5. The set of terms, i.e. the set $Term$ is given by the following inductive definition:
    (a) $Var \cup \mathcal{F}(0) \subseteq Term$
    (b) If $f \in \mathcal{F}(n)$, $(n = 1, 2, \dots)$, s $t_1, t_2, \dots, t_n \in Term$, then $f(t_1, t_2, \dots, t_n) \in Term$.

Definition/4

6. The set of formulas, i.e. the set *Form* is given by the following inductive definition:
   (a) $\mathcal{P}(0) \subseteq Form$
   (b) If $t_1, t_2 \in Term$, then $(t_1 = t_2) \in Form$
   (c) If $P \in \mathcal{P}(n)$, $(n = 1, 2, \ldots)$, s $t_1, t_2, \ldots, t_n \in Term$, then $P(t_1, t_2, \ldots, t_n) \in Form$.
   (d) If $A \in Form$, then $\neg A \in Form$.
   (e) If $A, B \in Form$, then $(A \supset B)$, $(A \wedge B)$, $(A \vee B)$, $(A \equiv B) \in Form$.
   (f) If $x \in Var$, $A \in Form$, then $\forall x A$, $\exists x A \in Form$.

Megjegyzs:

- Azokat a formulkat, amelyek a 6. (a), (b), (c) szablyok ltal jnnek ltre, atomi formulknak vagy prmformulknak nevezzk.

Definci:

Definition (interpretation)

The ordered pair $\langle U, \varrho\rangle$ is an interpretation of the language $L^{(1)}$ if

- $U \neq \emptyset$ (i.e. U is a nonempty set);
- $Dom(\varrho) = Con$
  - If $a \in \mathcal{F}(0)$, then $\varrho(a) \in U$;
  - If $f \in \mathcal{F}(n)$ $\quad (n \neq 0)$, then $\varrho(f) \in U^{U^{(n)}}$
  - If $p \in \mathcal{P}(0)$, then $\varrho(p) \in \{0, 1\}$;
  - If $P \in \mathcal{P}(n)$ $\quad (n \neq 0)$, then $\varrho(P) \subseteq U^{(n)}$ $(\varrho(P) \in \{0, 1\}^{U^{(n)}})$.

Definition (assignment)

The function v is an assignment relying on the interpretation $\langle U, \varrho\rangle$ if the followings hold:

- $Dom(v) = Var$;
- If $x \in Var$, then $v(x) \in U$.

Definition (modified assignment)

Let v be an assignment relying on the interpretation $\langle U, \varrho\rangle$, $x \in Var$ and $u \in U$.

$$v[x : u](y) = \begin{cases} u, & \text{if } y = x; \\ v(y), & \text{otherwise.} \end{cases}$$

for all $y \in Var$.

Definition (Semantic rules/1)

Let $\langle U, \varrho\rangle$ be a given interpretation and v be an assignment relying on $\langle U, \varrho\rangle$.

- If $a \in \mathcal{F}(0)$, then $|a|_v^{\langle U,\varrho\rangle} = \varrho(a)$.
- If $x \in Var$, then $|x|_v^{\langle U,\varrho\rangle} = v(x)$.
- If $f \in \mathcal{F}(n)$, $(n = 1, 2, \ldots)$, and $t_1, t_2, \ldots, t_n \in Term$, then $|f(t_1)(t_2)\ldots(t_n)|_v^{\langle U,\varrho\rangle} = \varrho(f)(\langle |t_1|_v^{\langle U,\varrho\rangle}, |t_2|_v^{\langle U,\varrho\rangle}, \ldots, |t_n|_v^{\langle U,\varrho\rangle}\rangle)$
- If $p \in \mathcal{P}(0)$, then $|p|_v^{\langle U,\varrho\rangle} = \varrho(p)$
- If $t_1, t_2 \in Term$, then

$$|(t_1 = t_2)|_v^{\langle U,\varrho\rangle} = \begin{cases} 1, & \text{if } |t_1|_v^{\langle U,\varrho\rangle} = |t_2|_v^{\langle U,\varrho\rangle} \\ 0, & \text{otherwise.} \end{cases}$$

Classical first–order logic | Semantics of classical first–order logic

Definition (Semantic rules/2)

- If $P \in \mathcal{P}(n)$ $(n \neq 0)$, $t_1, \ldots, t_n \in Term$, then

$$|P(t_1)\ldots(t_n)|_v^{\langle U,\varrho\rangle} = \begin{cases} 1, & \text{if } \langle |t_1|_v^{\langle U,\varrho\rangle}, \ldots, |t_n|_v^{\langle U,\varrho\rangle}\rangle \in \varrho(P); \\ 0, & \text{otherwise.} \end{cases}$$

Definition (Semantic rules/3)

- If $A \in Form$, then $|\neg A|_v^{\langle U,\varrho\rangle} = 1 - |A|_v^{\langle U,\varrho\rangle}$.
- If $A, B \in Form$, then

$$|(A \supset B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 0 & \text{if } |A|_v^{\langle U,\varrho\rangle} = 1, \text{ and } |B|_v^{\langle U,\varrho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$$

$$|(A \wedge B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 1 & \text{if } |A|_v^{\langle U,\varrho\rangle} = 1, \text{ and } |B|_v^{\langle U,\varrho\rangle} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$|(A \vee B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 0 & \text{if } |A|_v^{\langle U,\varrho\rangle} = 0, \text{ and } |B|_v^{\langle U,\varrho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$$

$$|(A \equiv B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 1 & \text{if } |A|_v^{\langle U,\varrho\rangle} = |B|_v^{\langle U,\varrho\rangle} = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Definition (Semantic rules/4)

- If $A \in Form, x \in Var$, then

$$|\forall x A|_v^{\langle U,\varrho\rangle} = \begin{cases} 0, & \text{if there is an } u \in U \text{ such that } |A|_{v[x:u]}^{\langle U,\varrho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$$

$$|\exists x A|_v^{\langle U,\varrho\rangle} = \begin{cases} 1, & \text{if there is an } u \in U \text{ such that } |A|_{v[x:u]}^{\langle U,\varrho\rangle} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Definition (model – a set of formulas)

Let $L(1) = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $\Gamma \subseteq Form$ be a set of formulas. An ordered triple $\langle U, \varrho, v \rangle$ is a model of the set Γ, if

- $\langle U, \varrho \rangle$ is an interpretation of $L^{(1)}$;
- v is an assignment relying on $\langle U, \varrho \rangle$;
- $|A|_v^{\langle U, \varrho \rangle} = 1$ for all $A \in \Gamma$.

Definition – a model of a formula

A model of a formula A is the model of the singleton $\{A\}$.

Classical first–order logic | Central logical (semantic) notions — FoL

Definition – satisfiable a set of formulas

The set of formulas $\Gamma \subseteq Form$ is satisfiable if it has a model.
(If there is an interpretation and an assignment in which all members of the set Γ are true.)

Definition – satisfiable a formula

A formula $A \in Form$ is satisfiable, if the singleton $\{A\}$ is satisfiable.

Remark

- A satisfiable set of formulas does not involve a logical contradiction; its formulas may be true together.
- A satisfiable formula may be true.
- If a set of formulas is satisfiable, then its members are satisfiable.
- But: all members of the set $\{P(a), \neg P(a)\}$ are satisfiable, and the set is not satisfiable.

Theorem

All subsets of a satisfiable set are satisfiable.

Proof

- Let $\Gamma \subseteq Form$ be a set of formulas and $\Delta \subseteq \Gamma$.
- Γ is satisfiable: it has a model. Let $\langle U, \varrho, v\rangle$ be a model of Γ.
- A property of $\langle U, \varrho, v\rangle$: If $A \in \Gamma$, then $|A|_v^{\langle U,\varrho\rangle} = 1$
- Since $\Delta \subseteq \Gamma$, if $A \in \Delta$, then $A \in \Gamma$, and so $|A|_v^{\langle U,\varrho\rangle} = 1$. That is the ordered triple $\langle U, \varrho, v\rangle$ is a model of Δ, and so Δ is satisfiable.

Definition – unsatisfiable set

The set $\Gamma \subseteq Form$ is unsatisfiable if it is not satisfiable.

Definition – unsatisfiable formula

A formula $A \in Form$ is unsatisfiable if the singleton $\{A\}$ is unsatisfiable.

Remark

A unsatisfiable set of formulas involve a logical contradiction. (Its members cannot be true together.)

Theorem

All expansions of an unsatisfiable set of formulas are unsatisfiable.

Indirect proof

- Suppose that $\Gamma \subseteq Form$ is an unsatisfiable set of formulas and $\Delta \subseteq Form$ is a set of formulas.
- Indirect condition: Γ is unsatisfiable, and $\Gamma \cup \Delta$ satisfiable.
- $\Gamma \subseteq \Gamma \cup \Delta$
- According to the former theorem Γ is satisfiable, and it is a contradiction.

Definition

A formula A is the logical consequence of the set of formulas Γ if the set $\Gamma \cup \{\neg A\}$ is unsatifiable. ($Notation : \Gamma \vDash A$)

Definition

$A \vDash B$, if $\{A\} \vDash B$.

Definition

The formula A is valid if $\emptyset \vDash A$. (Notation: $\vDash A$)

Definition

The formulas A and B are logically equivalent if $A \vDash B$ and $B \vDash A$. (Notation: $A \Leftrightarrow B$)

Theorem

Let $\Gamma \subseteq Form$, and $A \in Form$. $\Gamma \vDash A$ if and only if all models of the set Γ are the models of formula A. (i.e. the singleton $\{A\}$).

Proof

$\rightarrow$ Indirect condition: There is a model of $\Gamma \vDash A$ such that it is not a model of the formula A.
Let the ordered triple $\langle U, \varrho, v\rangle$ be this model.
The properties of $\langle U, \varrho, v\rangle$:

1. $|B|_v^{\langle U,\varrho\rangle} = 1$ for all $B \in \Gamma$;
2. $|A|\langle U, \varrho\rangle_v = 0$, and so $|\neg A|_v^{\langle U,\varrho\rangle} = 1$

In this case all members of the set $\Gamma \cup \{\neg A\}$ are true wrt the interpretation $\langle U, \varrho\rangle$ and assignment v, so $\Gamma \cup \{\neg A\}$ is satisfiable. It means that $\Gamma \nvDash A$, and it is a contradiction.

Proof

$\leftarrow$ Indirect condition: All models of the set Γ are the models of formula A, but (and) $\Gamma \not\models A$.
In this case $\Gamma \cup \{\neg A\}$ is satisfiable, i.e. it has a model.
Let the ordered triple $\langle U, \varrho, v\rangle$ be a model.
The properties of $\langle U, \varrho, v\rangle$:

1. $|B|_v^{\langle U,\varrho\rangle} = 1$ for all $B \in \Gamma$;
2. $|\neg A|_v^{\langle U,\varrho\rangle} = 1$, i.e. $|A|_v^{\langle U,\varrho\rangle} = 0$

So the set Γ has a model such that it is not a model of formula A, and it is a contradiction.

Corollary

Let $\Gamma \subseteq Form$, and $A \in Form$. $\Gamma \models A$ if and only if for all interpretations in which all members of Γ are true, the formula A is true.

Theorem

If A is a valid formula $((\models A))$, then $\Gamma \models A$ for all sets of formulas Γ. (A valid formula is a consequence of any set of formulas.)

Proof

- If A is a valid formula, then $\emptyset \models A$ (according to its definition).
- $\emptyset \cup \{\neg A\}$ $(= \{\neg A\})$ is unsatisfiable, and so its expansions are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of $\{\neg A\}$, and so it is unsatisfiable, i.e. $\Gamma \models A$.

Theorem

If Γ is unsatisfiable, then $\Gamma \vDash A$ for all A. (All formulas are the consequences of an unsatisfiable set of formulas.)

Proof

- According to a proved theorem: If Γ is unsatisfiable, the all expansions of Γ are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of Γ, and so it is unsatisfiable, i.e. $\Gamma \vDash A$.

Theorem

Deduction theorem: If $\Gamma \cup \{A\} \vDash B$, then $\Gamma \vDash (A \supset B)$.

Proof

- Indirect condition: Suppose, that $\Gamma \cup \{A\} \vDash B$, and $\Gamma \nvDash (A \supset B)$.
- $\Gamma \cup \{\neg(A \supset B)\}$ is satisfiable, and so it has a model. Let the ordered triple $\langle U, \varrho, v\rangle$ be a model.
- The properties of $\langle U, \varrho, v\rangle$:
    1. All members of Γ are true wrt $\langle U, \varrho\rangle$ and v.
    2. $|\neg(A \supset B)|_v^{\langle U,\varrho\rangle} = 1$
- $|(A \supset B)|_v^{\langle U,\varrho\rangle} = 0$, i.e. $|A|_v^{\langle U,\varrho\rangle} = 1$ and $|B|_v^{\langle U,\varrho\rangle} = 0$.
  So$|\neg B|_v^{\langle U,\varrho\rangle} = 1$.
- All members of $\Gamma \cup \{A\} \cup \{\neg B\}$ are true wrt $\langle U, \varrho\rangle$ and v, i.e. $\Gamma \cup \{A\} \nvDash B$, and it is a contradiction.

Theorem

In the opposite direction: If $\Gamma \vDash (A \supset B)$, then $\Gamma \cup \{A\} \vDash B$.

Proof

- Indirect condition: Suppose that $\Gamma \vDash (A \supset B)$, and $\Gamma \cup \{A\} \nvDash B$.
- So $\Gamma \cup \{A\} \cup \{\neg B\}$ is satisfiable, i.e. it has a model. Let the ordered triple $\langle U, \varrho, v\rangle$ a model.
- The properties of $\langle U, \varrho, v\rangle$:
  1. All members of Γ are true wrt $\langle U, \varrho\rangle$ and v.
  2. $|A|_v^{\langle U,\varrho\rangle} = 1$
  3. $|\neg B|_v^{\langle U,\varrho\rangle} = 1$, and so $|B|_v^{\langle U,\varrho\rangle} = 0$
- $|(A \supset B)|_v^{\langle U,\varrho\rangle} = 0$, $|\neg(A \supset B)|_v^{\langle U,\varrho\rangle} = 1$.
- All members of $\Gamma \cup \{\neg(A \supset B)\}$ are true wrt $\langle U, \varrho\rangle$ and v, i.e. $\Gamma \nvDash (A \supset B)$.

Corollary

$A \vDash B$ if and only if $\vDash (A \supset B)$

Proof

Let $\Gamma = \emptyset$ in the former theorems.

Cut elimination theorem

If $\Gamma \cup \{A\} \vDash B$ and $\Delta \vDash A$, then $\Gamma \cup \Delta \vDash B$.

Proof

Indirect.

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula. The set of free variables of the formula A (in notation: $FreeVar(A)$) is given by the following inductive definition:

- If A is an atomic formula (i.e. $A \in AtForm$), then the members of the set $FreeVar(A)$ are the variables occuring in A.
- If the formula A is $\neg B$, then $FreeVar(A) = FreeVar(B)$.
- If the formula A is $(B \supset C)$, $(B \wedge C)$, $(B \vee C)$ or $(B \equiv C)$, then $FreeVar(A) = FreeVar(B) \bigcup FreeVar(C)$.
- If the formula A is $\forall x B$ or $\exists x B$, then $FreeVar(A) = FreeVar(B) \setminus \{x\}$.

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form\rangle$ be a first order language and $A \in Form$ be a formula. The set of bound variables of the formula A (in notation: $BoundVar(A)$) is given by the following inductive definition:

- If A is an atomic formula (i.e. $A \in AtForm$), then $BoundVar(A) = \emptyset$.
- If the formula A is $\neg B$, then $BoundVar(A) = FreeVar(B)$.
- If the formula A is $(B \supset C)$, $(B \wedge C)$, $(B \vee C)$ or $(B \equiv C)$, then $BoundVar(A) = BoundVar(B) \bigcup BoundVar(C)$.
- If the formula A is $\forall xB$ or $\exists xB$, then $BoundVar(A) = BoundVar(B) \cup \{x\}$.

Remark

- The bases of inductive definitions of sest of free and bound variables are given by the first requirement of the corresponding definitions.
- The sets of free and bound variables of a formula are not disoint necessarily:
  $FreeVar((P(x) \wedge \exists xR(x))) = \{x\} = BoundVar((P(x) \wedge \exists xR(x)))$

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula, and $x \in Var$ be a variable.

- A fixed occurence of the variable x in the formula A is free if it is not in the subformulas $\forall x B$ or $\exists x B$ of the formula A.
- A fixed occurence of the variable x in the formula A is bound if it is not free.

Remark

- If x is a free variable of the formula A (i.e. $x \in FreeVar(A)$), then it has at least one free occurence in A.
- If x is a bound variable of the formula A (i.e. $x \in BoundVar(A)$), then it has at least one bound occurence in A.
- A fixed occurence of a variable x in the formula A is free if
  - it does not follow a universal or an existential quantifier, or
  - it is not in a scope of a $\forall x$ or a $\exists x$ quantification.
- A variable x may be a free and a bound variable of the formula A: $(P(x) \wedge \exists x R(x))$

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order languuage and $A \in Form$ be a formula.

- If $FreeVar(A) \neq \emptyset$, then the formula A is an open formula.
- If $FreeVar(A) = \emptyset$, then the formula A is a closed formula.

Remark:
The formula A is open if there is at least one variable which has at least one free occurence in A.
The formula A is closed if there is no variable which has a free occurence in A.

De Morgan Laws of quantifications

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula and $x \in Var$ be a variable. Then

- $\neg \exists x A \Leftrightarrow \forall x \neg A$
- $\neg \forall x A \Leftrightarrow \exists x \neg A$

Expressibilty of quantifications

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula and $x \in Var$ be a variable. Then

- $\exists x A \Leftrightarrow \neg \forall x \neg A$
- $\forall x A \Leftrightarrow \neg \exists x \neg A$

Conjunction and quantifications

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language,
$A, B \in Form$ be formulas and $x \in Var$ be a variable.
If $x \notin FreeVar(A)$, then

- $A \wedge \forall x B \Leftrightarrow \forall x (A \wedge B)$
- $A \wedge \exists x B \Leftrightarrow \exists x (A \wedge B)$

Remark:
According to the commutativity of conjunction the followings hold:
If $x \notin FreeVar(A)$, then

- $\forall x B \wedge A \Leftrightarrow \forall x (B \wedge A)$
- $\exists x B \wedge A \Leftrightarrow \exists x (B \wedge A)$

Disjunction and quantifications

Let $L^{(1)} = \langle LC, Var, Con, Term, Form\rangle$ be a first order language, $A, B \in Form$ be formulas and $x \in Var$ be a variable.
If $x \notin FreeVar(A)$, then

- $A \vee \forall x B \Leftrightarrow \forall x(A \vee B)$
- $A \vee \exists x B \Leftrightarrow \exists x(A \vee B)$

Remark:
According to the commutativity of disjunction the followings hold:
If $x \notin FreeVar(A)$, then

- $\forall x B \vee A \Leftrightarrow \forall x(B \vee A)$
- $\exists x B \vee A \Leftrightarrow \exists x(B \vee A)$

Implication with existential quantification

Let $L^{(1)} = \langle LC, Var, Con, Term, Form\rangle$ be a first order language, $A, B \in Form$ be formulas and $x \in Var$ be a variable.
If $x \notin FreeVar(A)$, then

- $A \supset \exists x B \Leftrightarrow \exists x(A \vee B)$
- $\exists x B \supset A \Leftrightarrow \forall x(B \supset A)$

Implication with universal quantification

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A, B \in Form$ be formulas and $x \in Var$ be a variable.
If $x \notin FreeVar(A)$, then

- $A \supset \forall x B \Leftrightarrow \forall x(A \vee B)$
- $\forall x B \supset A \Leftrightarrow \exists x(B \supset A)$

Substitutabily a variable with an other variable

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula and $x, y \in Var$ be variables.
The variable x is subtitutable with the variable y in the formula A if there is no a free occurence of x in A which is in the subformulas $\forall y B$ or $\exists y B$ of A.

Example:

- In the formula $\forall z P(x, z)$ the variable x is substitutable with the variable y, but x is not substitutable with the variable z.

Substitutabily a variable with a term

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula, $x \in Var$ be a variable and $t \in Term$ be a term.
The variable x is subtitutable with the term t in the formula A if in the formula A the variable x is substitutable with all variables occuring in the term t.

Example

- In the formula $\forall z P(x, z)$ the variable x is substitutable with the term $f(y_1, y_2)$, but x is not substitutable with the term $f(y, z)$.

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Result of a substitution

If the variable x is subtitutable with the term t in the formula A, then $[A]_x^t$ denotes the formula which appear when all free occurences of the variable x in A are substituted with the term t.

Renaming

Let $L^{(1)} = \langle LC, Var, Con, Term, Form\rangle$ be a first order language, $A \in Form$ be a formula, and $x, y \in Var$ be variables.
If the variable x is subtitutable with the variable y in the formula A and $y \notin FreeVar(A)$, then

- the formula $\forall y[A]_x^y$ is a regular renaming of the formula $\forall xA$;
- the formula $\exists y[A]_x^y$ is a regular renaming of the formula $\exists xA$.

Congruent formulas

Let $L^{(1)} = \langle LC, Var, Con, Term, Form\rangle$ be a first order language and $A \in Form$ be a formula.
The set $Cong(A)$ (the set ot formulas which are congruent with A) is given by the following inductive definition:

- $A \in Cong(A)$;
- if $\neg B \in Cong(A)$ and $B' \in Cong(B)$, then $\neg B' \in Cong(A)$;
- if $(B \circ C) \in Cong(A)$, $B' \in Cong(B)$ and $C' \in Cong(C)$, then $(B' \circ C') \in Cong(A)$ $(\circ \in \{\supset, \wedge, \vee, \equiv\})$;
- if $\forall xB \in Cong(A)$ and $\forall y[B]_x^y$ is a regular renaming of the formula $\forall xB$, then $\forall y[B]_x^y \in Cong(A)$;
- if $\exists xB \in Cong(A)$ and $\exists y[B]_x^y$ is a regular renaming of the formula $\exists xB$, then $\exists y[B]_x^y \in Cong(A)$.

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A, B \in Form$ be formulas.

- If $B \in Cong(A)$, then the formula A is congruent with the formula B.
- If $B \in Cong(A)$, then the formula B is a syntactical synonym of the formula A.

Theorem

Congruent formulas are logically equivalent, i.e. if $B \in Cong(A)$, then $A \Leftrightarrow B$.

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula.
The formula A is standardized if

- $FreeVar(A) \bigcap BoundVar(A) = \emptyset$;
- all bound variables of the formula A have exactly one occurences next a quantifier.

Theorem

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula.
Then there is a formula $B \in Form$ such that

- the formula B is standardized;
- the formula B is congruent with the formula A, i.e. $B \in Cong(A)$.

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula.
The formula A is prenex if

- there is no quantifier in A or
- the formula A is in the form $Q_1 x_1 Q_2 x_2 \dots Q_n x_n B$ $(n = 1, 2, \dots)$, where
  - there is no quantifier in the formula $B \in Form$;
  - $x_1, x_2 \dots x_n \in Var$ are diffrent variables;
  - $Q_1, Q_2, \dots, Q_n \in \{\forall, \exists\}$ are quantifiers.

Theorem

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula.
Then there is a formula $B \in Form$ such that

- the formula B is prenex;
- $A \Leftrightarrow B$.