1. Introduction to Logic and Computer Science

2. Foundations of Computer Science

3. Logic in Computer Science

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The main task of logic

- to give the laws of valid arguments (inferences, consequence relations)

Valid arguments

- Valid arguments (inferences):
  - an argument (an inference): a relation between premise(s) and conclusion
  - a consequence relation
    - input: premise(s)
    - output: conclusion
  - Valid arguments (inferences, consequence relations): if all premises are true, then the conclusion is true.
  - Logically valid arguments: when the former holds necessarily.
Definition/1
Classical zero–order language is an ordered triple
\[ L^{(0)} = \langle LC, Con, Form \rangle \]
where
1. \( LC = \{ \neg, \supset, \land, \lor, \equiv, (, ) \} \) (the set of logical constants).
2. \( Con \neq \emptyset \) the countable set of non-logical constants (propositional parameters)
3. \( LC \cap Con = \emptyset \)
4. The set of formulae i.e. the set \( Form \) is given by the following inductive definition:

Definition/2
- \( Con \subseteq Form \)
- If \( A \in Form \), then \( \neg A \in Form \).
- If \( A, B \in Form \), then
  - \( (A \supset B) \in Form \),
  - \( (A \land B) \in Form \),
  - \( (A \lor B) \in Form \),
  - \( (A \equiv B) \in Form \).

Remark
The members of the set \( Con \) are the atomic formulae (prime formulae).
**Definition**

If $A$ is an atomic formula, then it has no **direct subformula**;

- $\neg A$ has exactly one **direct subformula**: $A$;
- **Direct subformulae** of formulae $(A \supset B)$, $(A \land B)$, $(A \lor B)$, $(A \equiv B)$ are formulae $A$ and $B$, respectively.

**Definition**

The set of subformulae of formula $A$ [denoting: $SF(A)$] is given by the following inductive definition:

1. $A \in RF(A)$ (i.e. the formula $A$ is a subformula of itself);
2. if $A' \in RF(A)$ and $B$ is a direct subformula of $A'$-nek, then $B \in RF(A)$ (i.e., if $A'$ is a subformula of $A$, then all direct subformulae of $A'$ are subformulae of $A$).

**Definition**

The **contraction tree** of a formula $A$ is a finite ordered tree whose nodes are formulae,

- the root of the tree is the formmula $A$,
- the node with formula $\neg B$ has one child: the node with the formula $B$,
- the node with formulae $(B \supset C)$, $(B \land C)$, $(B \lor C)$, $(B \equiv C)$ has two children: the nodes with $B$, and $C$,
- the leaves of the tree are atomic formulae.
**Definition**

The function $\varrho$ is an interpretation of the language $L^{(0)}$ if

1. $\text{Dom}(\varrho) = \text{Con}$
2. If $p \in \text{Con}$, then $\varrho(p) \in \{0, 1\}$. 

**Definition**

Let $\varrho$ be an interpretation and $|A|_{\varrho}$ be the semantic value of the formula $A$ formula with respect to $\varrho$.

1. If $p \in \text{Con}$, then $|p|_{\varrho} = \varrho(p)$
2. If $A \in \text{Form}$, then $|\neg A|_{\varrho} = 1 - |A|_{\varrho}$.
3. If $A, B \in \text{Form}$, then
   
   \[
   |(A \supset B)|_{\varrho} = \begin{cases} 
   0 & \text{if } |A|_{\varrho} = 1, \text{ and } |B|_{\varrho} = 0; \\
   1 & \text{otherwise}
   \end{cases}
   \]
   \[
   |(A \land B)|_{\varrho} = \begin{cases} 
   1 & \text{if } |A|_{\varrho} = 1, \text{ and } |B|_{\varrho} = 1; \\
   0 & \text{otherwise}
   \end{cases}
   \]
   \[
   |(A \lor B)|_{\varrho} = \begin{cases} 
   0 & \text{if } |A|_{\varrho} = 0, \text{ and } |B|_{\varrho} = 0; \\
   1 & \text{otherwise.}
   \end{cases}
   \]
   \[
   |(A \equiv B)|_{\varrho} = \begin{cases} 
   1 & \text{if } |A|_{\varrho} = |B|_{\varrho}; \\
   0 & \text{otherwise.}
   \end{cases}
   \]
Definition (model – a set of formulas)
Let $\Gamma \subseteq \text{Form}$ be a set of formulas. An interpretation $\varrho$ is a model of the set of formulas $\Gamma$, if $|A|_\varrho = 1$ for all $A \in \Gamma$.

Definition – a model of a formula
A model of a formula $A$ is the model of the singleton $\{A\}$.

Definition – satisfiable a set of formulas
The set of formulas $\Gamma \subseteq \text{Form}$ is satisfiable if it has a model. (If there is an interpretation in which all members of the set $\Gamma$ are true.)

Definition – satisfiable a formula
A formula $A \in \text{Form}$ is satisfiable, if the singleton $\{A\}$ is satisfiable.

Remark
- A satisfiable set of formulas does not involve a logical contradiction; its formulas may be true together.
- A satisfiable formula may be true.
- If a set of formulas is satisfiable, then its members are satisfiable.
- But: all members of the set $\{p, \neg p\}$ are satisfiable, and the set is not satisfiable.
Theorem
All subsets of a satisfiable set are satisfiable.

Proof
- Let $\Gamma \subseteq \text{Form}$ be a set of formulas and $\Delta \subseteq \Gamma$.
- $\Gamma$ is satisfiable: it has a model. Let $\varrho$ be a model of $\Gamma$.
- A property of $\varrho$: If $A \in \Gamma$, then $|A|_\varrho = 1$
- Since $\Delta \subseteq \Gamma$, if $A \in \Delta$, then $A \in \Gamma$, and so $|A|_\varrho = 1$. That is the interpretation $\varrho$ is a model of $\Delta$, and so $\Delta$ is satisfiable.
**Theorem**
All expansions of an unsatisfiable set of formulas are unsatisfiable.

**Indirect proof**
- Suppose that $\Gamma \subseteq Form$ is an unsatisfiable set of formulas and $\Delta \subseteq Form$ is a set of formulas.
- Indirect condition: $\Gamma$ is unsatisfiable, and $\Gamma \cup \Delta$ satisfiable.
- $\Gamma \subseteq \Gamma \cup \Delta$
- According to the former theorem $\Gamma$ is satisfiable, and it is a contradiction.

**Definition**
A formula $A$ is the logical consequence of the set of formulas $\Gamma$ if the set $\Gamma \cup \{\neg A\}$ is unsatisfiable. (*Notation*: $\Gamma \models A$)

**Definition**
$A \models B$, if $\{A\} \models B$.

**Definition**
The formula $A$ is valid if $\emptyset \models A$. (*Notation*: $\models A$)

The formulas $A$ and $B$ are logically equivalent if $A \models B$ and $B \models A$. (*Notation*: $A \iff B$)
Theorem
Let $\Gamma \subseteq \text{Form}$, and $A \in \text{Form}$. $\Gamma \vDash A$ if and only if all models of the set $\Gamma$ are the models of formula $A$. (i.e. the singleton $\{A\}$).

Proof
$\rightarrow$ Indirect condition: There is a model of $\Gamma \vDash A$ such that it is not a model of the formula $A$.
Let the interpretation $\varrho$ be this model.
The properties of $\varrho$:
1. $|B|_\varrho = 1$ for all $B \in \Gamma$;
2. $|A|_\varrho = 0$, and so $|\neg A|_\varrho = 1$

In this case all members of the set $\Gamma \cup \{-A\}$ are true wrt $\varrho$-ban, and so $\Gamma \cup \{-A\}$ is satisfiable. It means that $\Gamma \nvDash A$, and it is a contradiction.

$\leftarrow$ Indirect condition: All models of the set $\Gamma$ are the models of formula $A$, but (and) $\Gamma \nvDash A$.
In this case $\Gamma \cup \{-A\}$ is satisfiable, i.e. it has a model.
Let the interpretation $\varrho$ be a model.
The properties of $\varrho$:
3. $|B|_\varrho = 1$ for all $B \in \Gamma$;
4. $|\neg A|_\varrho = 1$, i.e. $|A|_\varrho = 0$

So the set $\Gamma$ has a model such that it is not a model of formula $A$, and it is a contradiction.

Corollary
Let $\Gamma \subseteq \text{Form}$, and $A \in \text{Form}$. $\Gamma \vDash A$ if and only if for all interpretations in which all members of $\Gamma$ are true, the formula $A$ is true.
Theorem

If $A$ is a valid formula ($\models A$), then $\Gamma \models A$ for all sets of formulas $\Gamma$. (A valid formula is a consequence of any set of formulas.)

Proof

- If $A$ is a valid formula, then $\emptyset \models A$ (according to its definition).
- $\emptyset \cup \{\neg A\} (= \{\neg A\})$ is unsatisfiable, and so its expansions are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of $\{\neg A\}$, and so it is unsatisfiable, i.e. $\Gamma \models A$.

Theorem

If $\Gamma$ is unsatisfiable, then $\Gamma \models A$ for all $A$. (All formulas are the consequences of an unsatisfiable set of formulas.)

Proof

- According to a proved theorem: If $\Gamma$ is unsatisfiable, the all expansions of $\Gamma$ are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of $\Gamma$, and so it is unsatisfiable, i.e. $\Gamma \models A$. 
Theorem

Deduction theorem: If \( \Gamma \cup \{A\} \models B \), then \( \Gamma \models (A \supset B) \).

Proof

- Indirect condition: Suppose, that \( \Gamma \cup \{A\} \models B \), and \( \Gamma \not\models (A \supset B) \).
- \( \Gamma \cup \{\neg(A \supset B)\} \) is satisfiable, and so it has a model. Let the interpretation \( \varrho \) be a model.
- The properties of \( \varrho \):
  1. All members of \( \Gamma \) are true wrt \( \varrho \).
  2. \( |\neg(A \supset B)|_\varrho = 1 \)
  3. \( |(A \supset B)|_\varrho = 0 \), i.e. \( |A|_\varrho = 1 \) and \( |B|_\varrho = 0 \). So \( |\neg B|_\varrho = 1 \).
- All members of \( \Gamma \cup \{A\} \cup \{\neg B\} \) are true wrt interpretation \( \varrho \), i.e. \( \Gamma \cup \{A\} \not\models B \), and it is a contradiction.

Theorem

In the opposite direction: If \( \Gamma \models (A \supset B) \), then \( \Gamma \cup \{A\} \not\models B \).

Proof

- Indirect condition: Suppose that \( \Gamma \models (A \supset B) \), and \( \Gamma \cup \{A\} \not\models B \).
- So \( \Gamma \cup \{A\} \cup \{\neg B\} \) is satisfiable, i.e. it has a model. Let the interpretation \( \varrho \) a model.
- The properties of \( \varrho \):
  1. All members of \( \Gamma \) are true wrt the interpretation \( \varrho \).
  2. \( |A|_\varrho = 1 \)
  3. \( |\neg B|_\varrho = 1 \), and so \( |B|_\varrho = 0 \)
  4. \( |(A \supset B)|_\varrho = 0 \), \( |\neg(A \supset B)|_\varrho = 1 \).
- All members of \( \Gamma \cup \{\neg(A \supset B)\} \) are true wrt the interpretation \( \varrho \), i.e. \( \Gamma \not\models (A \supset B) \).
Corollary
A ⊨ B if and only if ⊨ (A ⊃ B)

Proof
Let Γ = ∅ in the former theorems.

Cut elimination theorem
If Γ ∪ {A} ⊨ B and Δ ⊨ A, then Γ ∪ Δ ⊨ B.

Proof
Indirect.
The truth table of negation

<table>
<thead>
<tr>
<th>¬p</th>
<th>0</th>
<th>1</th>
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<tbody>
<tr>
<td>¬</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- The law of double negation: \( \neg \neg A \Leftrightarrow A \)

The truth table of conjunction

<table>
<thead>
<tr>
<th>( \land )</th>
<th>0</th>
<th>1</th>
<th>(q)</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>(p)</td>
</tr>
</tbody>
</table>

- Commutative: \( (A \land B) \Leftrightarrow (B \land A) \) for all \( A, B \in \text{Form} \).
- Associative: \( (A \land (B \land C)) \Leftrightarrow ((A \land B) \land C) \) for all \( A, B, C \in \text{Form} \).
- Idempotent: \( (A \land A) \Leftrightarrow A \) for all \( A \in \text{Form} \).
Classical propositional logic (classical zero–order logic)

Properties of truth functors

- $(A \land B) \models A$, $(A \land B) \models B$
- The law of contradiction: $\models \neg(A \land \neg A)$
- The set $\{A_1, A_2, \ldots, A_n\}$ ($A_1, A_2, \ldots, A_n \in Form$) is satisfiable iff the formula $A_1 \land A_2 \land \cdots \land A_n$ is satisfiable.
- The set $\{A_1, A_2, \ldots, A_n\}$ ($A_1, A_2, \ldots, A_n \in Form$) is unsatisfiable iff the formula $A_1 \land A_2 \land \cdots \land A_n$ is unsatisfiable.
- $\{A_1, A_2, \ldots, A_n\} \models A$ ($A_1, A_2, \ldots, A_n, A \in Form$) iff $A_1 \land A_2 \land \cdots \land A_n \not\models A$.
- $\{A_1, A_2, \ldots, A_n\} \models A$ ($A_1, A_2, \ldots, A_n, A \in Form$) iff the formula $(A_1 \land A_2 \land \cdots \land A_n) \land \neg A$ is unsatisfiable.

The truth table of disjunction:

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<tr>
<th>$\lor$</th>
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<tr>
<td>1</td>
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</tbody>
</table>

- Commutative: $(A \lor B) \iff (B \lor A)$ for all $A, B \in Form$.
- Associative: $(A \lor (B \lor C)) \iff ((A \lor B) \lor C)$ for all $A, B, C \in Form$.
- Idempotent: $(A \lor A) \iff A$ for all $A \in Form$.
- $A \models (A \lor B)$ for all $A, B \in Form$.
- $\{A \lor B, \neg A\} \not\models B$
- The law of excluded middle: $\models (A \lor \neg A)$
Classical propositional logic (classical zero–order logic)

Properties of truth functors

- Connection between conjunction and disjunction:

<table>
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<tr>
<th>$\land$</th>
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<table>
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- Conjunction and disjunction are dual truth functors.

- Two laws of distributivity:
  - $(A \lor (B \land C)) \Leftrightarrow ((A \lor B) \land (A \lor C))$
  - $(A \land (B \lor C)) \Leftrightarrow ((A \land B) \lor (A \land C))$

- Properties of absorption
  - $(A \land (B \lor A)) \Leftrightarrow A$
  - $(A \lor (B \land A)) \Leftrightarrow A$

De Morgan’s laws

- What do we say when we deny a conjunction?
- What do we say when we deny a disjunction?
- $\neg (A \land B) \Leftrightarrow (\neg A \lor \neg B)$
- $\neg (A \lor B) \Leftrightarrow (\neg A \land \neg B)$

- The proofs of De Morgan’s laws.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$\neg A$</th>
<th>$\neg B$</th>
<th>$\neg (A \land B)$</th>
<th>$A \lor B$</th>
<th>$\neg (A \lor B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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</tbody>
</table>
The truth table of implication:

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<thead>
<tr>
<th>⊃</th>
<th>0</th>
<th>1</th>
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<td>0</td>
<td>1</td>
<td>1</td>
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<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

⊨ \((A \supset A)\)

Modus ponens: \{\((A \supset B), A\)\} ⊨ B

Modus tollens:
\{\((A \supset B), \neg B\)\} ⊨ \neg A

Chain rule: \{\((A \supset B), (B \supset C)\)\} ⊨ (A ⊃ C)

Reduction to absurdity: \{\((A \supset B), (A \supset \neg B)\)\} ⊨ \neg A

−A ⊨ (A ⊃ B)

B ⊨ (A ⊃ B)

\(((A \land B) \supset C) \iff (A \supset (B \supset C))\)

Contraposition: (A ⊃ B) ⇔ (¬B ⊃ ¬A)

(A ⊃ ¬A) ⊨ ¬A

(¬A ⊃ A) ⊨ A

(A ⊃ (B ⊃ C)) ⇔ ((A ⊃ B) ⊃ (A ⊃ C))

⊨ (A ⊃ (¬A ⊃ B))

\(((A \lor B) \supset C) ⇔ ((A \supset C) \land (B \supset C))\)

\{A_1, A_2, \ldots, A_n\} ⊨ A (A_1, A_2, \ldots, A_n, A ∈ Form) iff the formula \(((A_1 \land A_2 \land \cdots \land A_n) \supset A)\) is valid.
Classical propositional logic (classical zero–order logic)

Properties of truth functors

The truth table of (material) equivalence:

<table>
<thead>
<tr>
<th>≡</th>
<th>0</th>
<th>1</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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</tbody>
</table>

- $\models (A \equiv A)$
- $\not\models (A \equiv \neg A)$

Expressibility

- $(A \supset B) \iff \neg(A \land \neg B)$
- $(A \supset B) \iff (\neg A \lor B)$
- $(A \land B) \iff \neg(A \supset \neg B)$
- $(A \lor B) \iff (\neg A \supset B)$
- $(A \lor B) \iff \neg(\neg A \land \neg B)$
- $(A \lor B) \iff \neg(\neg A \lor \neg B)$
- $(A \equiv B) \iff ((A \supset B) \land (B \supset A))$
Theory of truth functors

Base

- A base is a set of truth functors whose members can express all truth functors.
  - For example: \{\neg, \supset\}, \{\neg, \land\}, \{\neg, \lor\}
    \[
    \begin{align*}
    (p \land q) & \iff \neg (p \supset \neg q) \\
    (p \lor q) & \iff (\neg p \supset q)
    \end{align*}
    \]
  - Truth functor Sheffer: \((p|q) \iff_{\text{def}} \neg (p \land q)\)
  - Truth functor neither-nor: \((p \parallel q) \iff_{\text{def}} (\neg p \land \neg q)\)
  - Remark: Singleton bases: \((p|q), (p \parallel q)\)

Normal forms

Definition

If \(p \in Con\), then formulas \(p, \neg p\) are literals (\(p\) is the base of the literals).

Definition

If the formula \(A\) is a literal or a conjunction of literals with different bases, then \(A\) is an elementary conjunction.

Definition

If the formula \(A\) is a literal or a disjunction of literals with different bases, the \(A\) is an elementary disjunction.
**Definition**

A disjunction of elementary conjunctions is a disjunctive normal form.

**Definition**

A conjunction of elementary disjunctions is a conjunctive normal form.

**Theorem**

There is a normal form of any formula of proposition logic, i.e. if $A \in \text{Form}$, then there is a formula $B$ such that $B$ is a normal form and $A \iff B$.

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**Definition/1**

The language of first-order logic is a

$$L^{(1)} = \langle \text{LC}, \text{Var}, \text{Con}, \text{Term}, \text{Form} \rangle$$

ordered 5–tuple, where

1. $\text{LC} = \{ \neg, \lor, \land, \forall, \exists, =, \langle, \rangle \}$: (the set of logical constants).
2. $\text{Var} (= \{ x_n : n = 0, 1, 2, \ldots \})$: countable infinite set of variables.
Definition/2

3. \( \text{Con} = \bigcup_{n=0}^{\infty}(\mathcal{F}(n) \cup \mathcal{P}(n)) \) the set of non–logical constants (at best countable infinite)
   - \( \mathcal{F}(0) \): the set of name parameters,
   - \( \mathcal{F}(n) \): the set of \( n \) argument function parameters,
   - \( \mathcal{P}(0) \): the set of proposition parameters,
   - \( \mathcal{P}(n) \): the set of predicate parameters.

4. The sets \( \text{LC} \), \( \text{Var} \), \( \mathcal{F}(n) \), \( \mathcal{P}(n) \) are pairwise disjoint \((n = 0, 1, 2, \ldots)\).

Definition/3

5. The set of terms, i.e. the set \( \text{Term} \) is given by the following inductive definition:
   - (a) \( \text{Var} \cup \mathcal{F}(0) \subseteq \text{Term} \)
   - (b) If \( f \in \mathcal{F}(n) \), \( (n = 1, 2, \ldots) \), \( s t_1, t_2, \ldots, t_n \in \text{Term} \), then \( f(t_1, t_2, \ldots, t_n) \in \text{Term} \).
Definition/4

6. The set of formulas, i.e. the set $Form$ is given by the following inductive definition:

(a) $P(0) \subseteq Form$
(b) If $t_1, t_2 \in Term$, then $(t_1 = t_2) \in Form$
(c) If $P \in P(n), (n = 1, 2, \ldots)$, s $t_1, t_2, \ldots, t_n \in Term$, then $P(t_1, t_2, \ldots, t_n) \in Form$.
(d) If $A \in Form$, then $\neg A \in Form$.
(e) If $A, B \in Form$, then $(A \supset B), (A \land B), (A \lor B), (A \equiv B) \in Form$.
(f) If $x \in Var, A \in Form$, then $\forall xA, \exists xA \in Form$.

Megjegyzés:

- Azokat a formulákat, amelyek a 6. (a), (b), (c) szabályok Ital jnnek ltre, atomi formulának vagy prmformulának nevezzk.

Definci:
Definition (interpretation)

The ordered pair $\langle U, \varrho \rangle$ is an interpretation of the language $L^{(1)}$ if

- $U \neq \emptyset$ (i.e. $U$ is a nonempty set);
- $\text{Dom}(\varrho) = \text{Con}$
  - If $a \in F(0)$, then $\varrho(a) \in U$;
  - If $f \in F(n)$ ($n \neq 0$), then $\varrho(f) \in U^{(n)}$;
  - If $p \in P(0)$, then $\varrho(p) \in \{0, 1\}$;
  - If $P \in P(n)$ ($n \neq 0$), then $\varrho(P) \subseteq U^{(n)}$ ($\varrho(P) \in \{0, 1\}^{U^{(n)}}$).

Definition (assignment)

The function $\nu$ is an assignment relying on the interpretation $\langle U, \varrho \rangle$ if the followings hold:

- $\text{Dom}(\nu) = \text{Var}$;
- If $x \in \text{Var}$, then $\nu(x) \in U$.

Definition (modified assignment)

Let $\nu$ be an assignment relying on the interpretation $\langle U, \varrho \rangle$, $x \in \text{Var}$ and $u \in U$.

$$\nu[x : u](y) = \begin{cases} u, & \text{if } y = x; \\ \nu(y), & \text{otherwise.} \end{cases}$$

for all $y \in \text{Var}$. 
Definition (Semantic rules/1)

Let $\langle U, \varrho \rangle$ be a given interpretation and $\nu$ be an assignment relying on $\langle U, \varrho \rangle$.

- If $a \in F(0)$, then $|a|^{(U, \varrho)}_{\nu} = \varrho(a)$.
- If $x \in \text{Var}$, then $|x|^{(U, \varrho)}_{\nu} = \nu(x)$.
- If $f \in F(n)$, $(n = 1, 2, \ldots)$, and $t_1, t_2, \ldots, t_n \in \text{Term}$, then $|f(t_1)(t_2) \ldots (t_n)|^{(U, \varrho)}_{\nu} = \varrho(f)(\langle |t_1|^{(U, \varrho)}_{\nu}, |t_2|^{(U, \varrho)}_{\nu}, \ldots, |t_n|^{(U, \varrho)}_{\nu} \rangle)$
- If $p \in P(0)$, then $|p|^{(U, \varrho)}_{\nu} = \varrho(p)$
- If $t_1, t_2 \in \text{Term}$, then $|(t_1 = t_2)|^{(U, \varrho)}_{\nu} = \begin{cases} 1, & \text{if } |t_1|^{(U, \varrho)}_{\nu} = |t_2|^{(U, \varrho)}_{\nu} \\ 0, & \text{otherwise} \end{cases}$

Definition (Semantic rules/2)

- If $P \in P(n)$ ($n \neq 0$), $t_1, \ldots, t_n \in \text{Term}$, then $|P(t_1) \ldots (t_n)|^{(U, \varrho)}_{\nu} = \begin{cases} 1, & \text{if } \langle |t_1|^{(U, \varrho)}_{\nu}, \ldots, |t_n|^{(U, \varrho)}_{\nu} \rangle \in \varrho(P); \\ 0, & \text{otherwise} \end{cases}$
Definition (Semantic rules/3)

- If $A \in \text{Form}$, then $|\neg A|_{v}^{(U, \varrho)} = 1 - |A|_{v}^{(U, \varrho)}$.
- If $A, B \in \text{Form}$, then
  
  \[ |(A \supset B)|_{v}^{(U, \varrho)} = \begin{cases} 
  0 & \text{if } |A|_{v}^{(U, \varrho)} = 1, \text{ and } |B|_{v}^{(U, \varrho)} = 0; \\
  1 & \text{otherwise.} 
  \end{cases} \]

  \[ |(A \land B)|_{v}^{(U, \varrho)} = \begin{cases} 
  1 & \text{if } |A|_{v}^{(U, \varrho)} = 1, \text{ and } |B|_{v}^{(U, \varrho)} = 1; \\
  0 & \text{otherwise.} 
  \end{cases} \]

  \[ |(A \lor B)|_{v}^{(U, \varrho)} = \begin{cases} 
  0 & \text{if } |A|_{v}^{(U, \varrho)} = 0, \text{ and } |B|_{v}^{(U, \varrho)} = 0; \\
  1 & \text{otherwise.} 
  \end{cases} \]

  \[ |(A \equiv B)|_{v}^{(U, \varrho)} = \begin{cases} 
  1 & \text{if } |A|_{v}^{(U, \varrho)} = |B|_{v}^{(U, \varrho)} = 0; \\
  0 & \text{otherwise.} 
  \end{cases} \]

Definition (Semantic rules/4)

- If $A \in \text{Form}, x \in \text{Var}$, then
  
  \[ |\forall x A|_{v}^{(U, \varrho)} = \begin{cases} 
  0, & \text{if there is an } u \in U \text{ such that } |A|_{v[x:u]}^{(U, \varrho)} = 0; \\
  1, & \text{otherwise.} 
  \end{cases} \]

  \[ |\exists x A|_{v}^{(U, \varrho)} = \begin{cases} 
  1, & \text{if there is an } u \in U \text{ such that } |A|_{v[x:u]}^{(U, \varrho)} = 1; \\
  0, & \text{otherwise.} 
  \end{cases} \]
Definition (model – a set of formulas)

Let $L(1) = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $\Gamma \subseteq Form$ be a set of formulas. An ordered triple $\langle U, \varrho, \nu \rangle$ is a model of the set $\Gamma$, if

- $\langle U, \varrho \rangle$ is an interpretation of $L^{(1)}$;
- $\nu$ is an assignment relying on $\langle U, \varrho \rangle$;
- $|A|^{\langle U, \varrho \rangle}_\nu = 1$ for all $A \in \Gamma$.

Definition – a model of a formula

A model of a formula $A$ is the model of the singleton $\{A\}$.

Definition – satisfiable a set of formulas

The set of formulas $\Gamma \subseteq Form$ is satisfiable if it has a model. (If there is an interpretation and an assignment in which all members of the set $\Gamma$ are true.)

Definition – satisfiable a formula

A formula $A \in Form$ is satisfiable, if the singleton $\{A\}$ is satisfiable.
Remark
- A satisfiable set of formulas does not involve a logical contradiction; its formulas may be true together.
- A satisfiable formula may be true.
- If a set of formulas is satisfiable, then its members are satisfiable.
- But: all members of the set \{P(a), \neg P(a)\} are satisfiable, and the set is not satisfiable.

Theorem
All subsets of a satisfiable set are satisfiable.

Proof
- Let \( \Gamma \subseteq Form \) be a set of formulas and \( \Delta \subseteq \Gamma \).
- \( \Gamma \) is satisfiable: it has a model. Let \( \langle U, \varrho, v \rangle \) be a model of \( \Gamma \).
- A property of \( \langle U, \varrho, v \rangle \): If \( A \in \Gamma \), then \(|A|^{(U,\varrho)}_v = 1\)
- Since \( \Delta \subseteq \Gamma \), if \( A \in \Delta \), then \( A \in \Gamma \), and so \(|A|^{(U,\varrho)}_v = 1\). That is the ordered triple \( \langle U, \varrho, v \rangle \) is a model of \( \Delta \), and so \( \Delta \) is satisfiable.
Definition – unsatisfiable set
The set $\Gamma \subseteq \text{Form}$ is unsatisfiable if it is not satisfiable.

Definition – unsatisfiable formula
A formula $A \in \text{Form}$ is unsatisfiable if the singleton $\{A\}$ is unsatisfiable.

Remark
A unsatisfiable set of formulas involve a logical contradiction. (Its members cannot be true together.)

Theorem
All expansions of an unsatisfiable set of formulas are unsatisfiable.

Indirect proof
- Suppose that $\Gamma \subseteq \text{Form}$ is an unsatisfiable set of formulas and $\Delta \subseteq \text{Form}$ is a set of formulas.
- Indirect condition: $\Gamma$ is unsatisfiable, and $\Gamma \cup \Delta$ satisfiable.
- $\Gamma \subseteq \Gamma \cup \Delta$
- According to the former theorem $\Gamma$ is satisfiable, and it is a contradiction.
Definition
A formula $A$ is the logical consequence of the set of formulas $\Gamma$ if the set $\Gamma \cup \{\neg A\}$ is unsatisfiable. (Notation: $\Gamma \models A$)

Definition
$A \models B$, if $\{A\} \models B$.

Definition
The formula $A$ is valid if $\emptyset \models A$. (Notation: $\models A$)

Definition
The formulas $A$ and $B$ are logically equivalent if $A \models B$ and $B \models A$. (Notation: $A \iff B$)

Theorem
Let $\Gamma \subseteq \text{Form}$, and $A \in \text{Form}$. $\Gamma \models A$ if and only if all models of the set $\Gamma$ are the models of formula $A$. (i.e. the singleton $\{A\}$).

Proof
$\rightarrow$ Indirect condition: There is a model of $\Gamma \models A$ such that it is not a model of the formula $A$.
Let the ordered triple $\langle U, \varrho, \nu \rangle$ be this model. The properties of $\langle U, \varrho, \nu \rangle$:
1. $|B|_{\nu}^{\langle U, \varrho \rangle} = 1$ for all $B \in \Gamma$;
2. $|A|_{\nu}^{\langle U, \varrho \rangle} = 0$, and so $|\neg A|_{\nu}^{\langle U, \varrho \rangle} = 1$

In this case all members of the set $\Gamma \cup \{\neg A\}$ are true wrt the interpretation $\langle U, \varrho \rangle$ and assignment $\nu$, so $\Gamma \cup \{\neg A\}$ is satisfiable. It means that $\Gamma \not\models A$, and it is a contradiction.
Proof
← Indirect condition: All models of the set $\Gamma$ are the models of formula $A$, but (and) $\Gamma \not\models A$.
In this case $\Gamma \cup \{\neg A\}$ is satisfiable, i.e. it has a model.
Let the ordered triple $\langle U, \varrho, v \rangle$ be a model.
The properties of $\langle U, \varrho, v \rangle$:
1. $|B|^{\langle U, \varrho \rangle}_{v} = 1$ for all $B \in \Gamma$;
2. $|\neg A|^{\langle U, \varrho \rangle}_{v} = 1$, i.e. $|A|^{\langle U, \varrho \rangle}_{v} = 0$
So the set $\Gamma$ has a model such that it is not a model of formula $A$, and it is a contradiction.

Corollary
Let $\Gamma \subseteq Form$, and $A \in Form$. $\Gamma \models A$ if and only if for all interpretations in which all members of $\Gamma$ are true, the formula $A$ is true.

Theorem
If $A$ is a valid formula ($\models A$), then $\Gamma \models A$ for all sets of formulas $\Gamma$. (A valid formula is a consequence of any set of formulas.)

Proof
- If $A$ is a valid formula, then $\emptyset \models A$ (according to its definition).
- $\emptyset \cup \{\neg A\} (= \{\neg A\})$ is unsatisfiable, and so its expansions are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of $\{\neg A\}$, and so it is unsatisfiable, i.e. $\Gamma \not\models A$. 

Theorem
If $\Gamma$ is unsatisfiable, then $\Gamma \models A$ for all $A$. (All formulas are the consequences of an unsatisfiable set of formulas.)

Proof
- According to a proved theorem: If $\Gamma$ is unsatisfiable, the all expansions of $\Gamma$ are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of $\Gamma$, and so it is unsatisfiable, i.e. $\Gamma \models A$.

Theorem
Deduction theorem: If $\Gamma \cup \{A\} \models B$, then $\Gamma \models (A \supset B)$.

Proof
- Indirect condition: Suppose, that $\Gamma \cup \{A\} \models B$, and $\Gamma \not\models (A \supset B)$.
- $\Gamma \cup \{\neg (A \supset B)\}$ is satisfiable, and so it has a model. Let the ordered triple $\langle U, \varrho, \nu \rangle$ be a model.
- The properties of $\langle U, \varrho, \nu \rangle$:
  1. All members of $\Gamma$ are true wrt $\langle U, \varrho \rangle$ and $\nu$.
  2. $|\neg (A \supset B)|_{\nu}^{\langle U, \varrho \rangle} = 1$
  3. $|(A \supset B)|_{\nu}^{\langle U, \varrho \rangle} = 0$, i.e. $|A|_{\nu}^{\langle U, \varrho \rangle} = 1$ and $|B|_{\nu}^{\langle U, \varrho \rangle} = 0$. So $|\neg B|_{\nu}^{\langle U, \varrho \rangle} = 1$.
- All members of $\Gamma \cup \{A\} \cup \{\neg B\}$ are true wrt $\langle U, \varrho \rangle$ and $\nu$, i.e. $\Gamma \cup \{A\} \not\models B$, and it is a contradiction.
Theorem

In the opposite direction: If $\Gamma \vdash (A \supset B)$, then $\Gamma \cup \{A\} \not\vdash B$.

Proof

- Indirect condition: Suppose that $\Gamma \vdash (A \supset B)$, and $\Gamma \cup \{A\} \not\vdash B$.
- So $\Gamma \cup \{A\} \cup \{\neg B\}$ is satisfiable, i.e. it has a model. Let the ordered triple $\langle U, \varrho, v \rangle$ a model.
- The properties of $\langle U, \varrho, v \rangle$:
  1. All members of $\Gamma$ are true wrt $\langle U, \varrho \rangle$ and $v$.
  2. $|A|_{\langle U, \varrho \rangle} = 1$
  3. $|\neg B|_{\langle U, \varrho \rangle} = 1$, and so $|B|_{\langle U, \varrho \rangle} = 0$
  4. $|(A \supset B)|_{\langle U, \varrho \rangle} = 0$, $|\neg (A \supset B)|_{\langle U, \varrho \rangle} = 1$.
- All members of $\Gamma \cup \{\neg (A \supset B)\}$ are true wrt $\langle U, \varrho \rangle$ and $v$, i.e. $\Gamma \not\vdash (A \supset B)$.

Corollary

$A \vdash B$ if and only if $\not\vdash (A \supset B)$

Proof

Let $\Gamma = \emptyset$ in the former theorems.
### Cut elimination theorem

If $\Gamma \cup \{A\} \models B$ and $\Delta \models A$, then $\Gamma \cup \Delta \models B$.

**Proof**

Indirect.

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### Definition

Let $L^{(1)} = \langle LC, \ Var, \ Con, \ Term, \ Form \rangle$ be a first order language and $A \in Form$ be a formula. The set of free variables of the formula $A$ (in notation: $\text{FreeVar}(A)$) is given by the following inductive definition:

- If $A$ is an atomic formula (i.e. $A \in AtForm$), then the members of the set $\text{FreeVar}(A)$ are the variables occurring in $A$.
- If the formula $A$ is $\lnot B$, then $\text{FreeVar}(A) = \text{FreeVar}(B)$.
- If the formula $A$ is $(B \supset C)$, $(B \land C)$, $(B \lor C)$ or $(B \equiv C)$, then $\text{FreeVar}(A) = \text{FreeVar}(B) \cup \text{FreeVar}(C)$.
- If the formula $A$ is $\forall xB$ or $\exists xB$, then $\text{FreeVar}(A) = \text{FreeVar}(B) \setminus \{x\}$. 
Definition

Let $L^{(1)} = \langle LC, \text{Var}, Con, \text{Term}, Form \rangle$ be a first order language and $A \in Form$ be a formula. The set of bound variables of the formula $A$ (in notation: $BoundVar(A)$) is given by the following inductive definition:

- If $A$ is an atomic formula (i.e. $A \in \text{AtForm}$), then $BoundVar(A) = \emptyset$.
- If the formula $A$ is $\neg B$, then $BoundVar(A) = FreeVar(B)$.
- If the formula $A$ is $(B \supset C)$, $(B \land C)$, $(B \lor C)$ or $(B \equiv C)$, then $BoundVar(A) = BoundVar(B) \cup BoundVar(C)$.
- If the formula $A$ is $\forall x B$ or $\exists x B$, then $BoundVar(A) = BoundVar(B) \cup \{x\}$.

Remark

- The bases of inductive definitions of set of free and bound variables are given by the first requirement of the corresponding definitions.
- The sets of free and bound variables of a formula are not disjoint necessarily:
  $$FreeVar((P(x) \land \exists x R(x))) = \{x\} = BoundVar((P(x) \land \exists x R(x)))$$
Definition

Let \( L^{(1)} = \langle LC, Var, Con, Term, Form \rangle \) be a first order language, \( A \in Form \) be a formula, and \( x \in Var \) be a variable.

- A fixed occurrence of the variable \( x \) in the formula \( A \) is free if it is not in the subformulas \( \forall x B \) or \( \exists x B \) of the formula \( A \).
- A fixed occurrence of the variable \( x \) in the formula \( A \) is bound if it is not free.

Remark

- If \( x \) is a free variable of the formula \( A \) (i.e. \( x \in FreeVar(A) \)), then it has at least one free occurrence in \( A \).
- If \( x \) is a bound variable of the formula \( A \) (i.e. \( x \in BoundVar(A) \)), then it has at least one bound occurrence in \( A \).
- A fixed occurrence of a variable \( x \) in the formula \( A \) is free if
  - it does not follow a universal or an existential quantifier, or
  - it is not in a scope of a \( \forall x \) or a \( \exists x \) quantification.
- A variable \( x \) may be a free and a bound variable of the formula \( A \):
  \( (P(x) \land \exists x R(x)) \)
Definition

Let $L^{(1)} = \langle LC, \text{Var}, \text{Con}, \text{Term}, \text{Form} \rangle$ be a first order language and $A \in \text{Form}$ be a formula.

- If $\text{FreeVar}(A) \neq \emptyset$, then the formula $A$ is an open formula.
- If $\text{FreeVar}(A) = \emptyset$, then the formula $A$ is a closed formula.

Remark:
The formula $A$ is open if there is at least one variable which has at least one free occurrence in $A$.
The formula $A$ is closed if there is no variable which has a free occurrence in $A$.

De Morgan Laws of quantifications

Let $L^{(1)} = \langle LC, \text{Var}, \text{Con}, \text{Term}, \text{Form} \rangle$ be a first order language, $A \in \text{Form}$ be a formula and $x \in \text{Var}$ be a variable. Then

- $\neg \exists x A \iff \forall x \neg A$
- $\neg \forall x A \iff \exists x \neg A$
Expressibilty of quantifications

Let \( L^{(1)} = \langle LC, Var, Con, Term, Form \rangle \) be a first order language, \( A \in Form \) be a formula and \( x \in Var \) be a variable. Then

- \( \exists xA \iff \neg \forall x \neg A \)
- \( \forall xA \iff \neg \exists x \neg A \)

Conjunction and quantifications

Let \( L^{(1)} = \langle LC, Var, Con, Term, Form \rangle \) be a first order language, \( A, B \in Form \) be formulas and \( x \in Var \) be a variable.

If \( x \notin \text{FreeVar}(A) \), then

- \( A \land \forall xB \iff \forall x(A \land B) \)
- \( A \land \exists xB \iff \exists x(A \land B) \)

Remark:

According to the commutativity of conjunction the followings hold:

If \( x \notin \text{FreeVar}(A) \), then

- \( \forall xB \land A \iff \forall x(B \land A) \)
- \( \exists xB \land A \iff \exists x(B \land A) \)
Disjunction and quantifications

Let $L^{(1)} = \langle LC, \text{Var}, \text{Con}, \text{Term}, \text{Form} \rangle$ be a first order language, $A, B \in \text{Form}$ be formulas and $x \in \text{Var}$ be a variable.

If $x \notin \text{FreeVar}(A)$, then

- $A \lor \forall x B \iff \forall x (A \lor B)$
- $A \lor \exists x B \iff \exists x (A \lor B)$

Remark:

According to the commutativity of disjunction the followings hold:

If $x \notin \text{FreeVar}(A)$, then

- $\forall x B \lor A \iff \forall x (B \lor A)$
- $\exists x B \lor A \iff \exists x (B \lor A)$

Implication with existential quantification

Let $L^{(1)} = \langle LC, \text{Var}, \text{Con}, \text{Term}, \text{Form} \rangle$ be a first order language, $A, B \in \text{Form}$ be formulas and $x \in \text{Var}$ be a variable.

If $x \notin \text{FreeVar}(A)$, then

- $A \supset \exists x B \iff \exists x (A \lor B)$
- $\exists x B \supset A \iff \forall x (B \supset A)$
Implication with universal quantification

Let $L^{(1)} = \langle L, \text{Var}, \text{Con}, \text{Term}, \text{Form} \rangle$ be a first order language, $A, B \in \text{Form}$ be formulas and $x \in \text{Var}$ be a variable. If $x \notin \text{FreeVar}(A)$, then

- $A \supset \forall x B \iff \forall x (A \lor B)$
- $\forall x B \supset A \iff \exists x (B \supset A)$

Substitutabily a variable with an other variable

Let $L^{(1)} = \langle L, \text{Var}, \text{Con}, \text{Term}, \text{Form} \rangle$ be a first order language, $A \in \text{Form}$ be a formula and $x, y \in \text{Var}$ be variables. The variable $x$ is subtitutable with the variable $y$ in the formula $A$ if there is no a free occurence of $x$ in $A$ which is in the subformulas $\forall y B$ or $\exists y B$ of $A$.

Example:

- In the formula $\forall z P(x, z)$ the variable $x$ is substitutable with the variable $y$, but $x$ is not substitutable with the variable $z$. 
Substitutabily a variable with a term

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula, $x \in Var$ be a variable and $t \in Term$ be a term. The variable $x$ is substitutable with the term $t$ in the formula $A$ if in the formula $A$ the variable $x$ is substitutable with all variables occurring in the term $t$.

Example

- In the formula $\forall z P(x, z)$ the variable $x$ is substitutable with the term $f(y_1, y_2)$, but $x$ is not substitutable with the term $f(y, z)$.

Result of a substitution

If the variable $x$ is substitutable with the term $t$ in the formula $A$, then $[A]_x^t$ denotes the formula which appear when all free occurrences of the variable $x$ in $A$ are substituted with the term $t$. 
### Renaming

Let $\mathcal{L}^{(1)} = \langle \mathcal{L}C, \mathcal{V}ar, \mathcal{C}on, \mathcal{T}erm, \mathcal{F}orm \rangle$ be a first order language, $A \in \mathcal{F}orm$ be a formula, and $x, y \in \mathcal{V}ar$ be variables. If the variable $x$ is substitutable with the variable $y$ in the formula $A$ and $y \not\in \mathcal{F}ree\mathcal{V}ar(A)$, then

- the formula $\forall y[A]_x$ is a regular renaming of the formula $\forall xA$;
- the formula $\exists y[A]_x$ is a regular renaming of the formula $\exists xA$.

### Congruent formulas

Let $\mathcal{L}^{(1)} = \langle \mathcal{L}C, \mathcal{V}ar, \mathcal{C}on, \mathcal{T}erm, \mathcal{F}orm \rangle$ be a first order language and $A \in \mathcal{F}orm$ be a formula. The set $\mathcal{C}ong(A)$ (the set of formulas which are congruent with $A$) is given by the following inductive definition:

- $A \in \mathcal{C}ong(A)$;
- if $\neg B \in \mathcal{C}ong(A)$ and $B' \in \mathcal{C}ong(B)$, then $\neg B' \in \mathcal{C}ong(A)$;
- if $(B \circ C) \in \mathcal{C}ong(A)$, $B' \in \mathcal{C}ong(B)$ and $C' \in \mathcal{C}ong(C)$, then $(B' \circ C') \in \mathcal{C}ong(A)$ ($\circ \in \{\rightarrow, \land, \lor, \equiv\}$);
- if $\forall xB \in \mathcal{C}ong(A)$ and $\forall y[B]_x \in \mathcal{C}ong(A)$, then $\forall y[B]_x \in \mathcal{C}ong(A)$;
- if $\exists xB \in \mathcal{C}ong(A)$ and $\exists y[B]_x \in \mathcal{C}ong(A)$, then $\exists y[B]_x \in \mathcal{C}ong(A)$.
Definition
Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A, B \in Form$ be formulas.

- If $B \in Cong(A)$, then the formula $A$ is congruent with the formula $B$.
- If $B \in Cong(A)$, then the formula $B$ is a syntactical synonym of the formula $A$.

Theorem
Congruent formulas are logically equivalent, i.e. if $B \in Cong(A)$, then $A \leftrightarrow B$.

Definition
Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula.
The formula $A$ is standardized if

- $FreeVar(A) \cap BoundVar(A) = \emptyset$;
- all bound variables of the formula $A$ have exactly one occurrence next a quantifier.

Theorem
Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula.
Then there is a formula $B \in Form$ such that

- the formula $B$ is standardized;
- the formula $B$ is congruent with the formula $A$, i.e. $B \in Cong(A)$. 
Definition
Let $L^{(1)} = \langle LC, \text{Var}, \text{Con}, \text{Term}, \text{Form} \rangle$ be a first order language and $A \in \text{Form}$ be a formula.
The formula $A$ is prenex if
- there is no quantifier in $A$ or
- the formula $A$ is in the form $Q_1x_1Q_2x_2 \ldots Q_nx_nB$ ($n = 1, 2, \ldots$),
where
  - there is no quantifier in the formula $B \in \text{Form}$;
  - $x_1, x_2 \ldots x_n \in \text{Var}$ are different variables;
  - $Q_1, Q_2, \ldots, Q_n \in \{\forall, \exists\}$ are quantifiers.

Theorem
Let $L^{(1)} = \langle LC, \text{Var}, \text{Con}, \text{Term}, \text{Form} \rangle$ be a first order language and $A \in \text{Form}$ be a formula.
Then there is a formula $B \in \text{Form}$ such that
- the formula $B$ is prenex;
- $A \iff B$. 