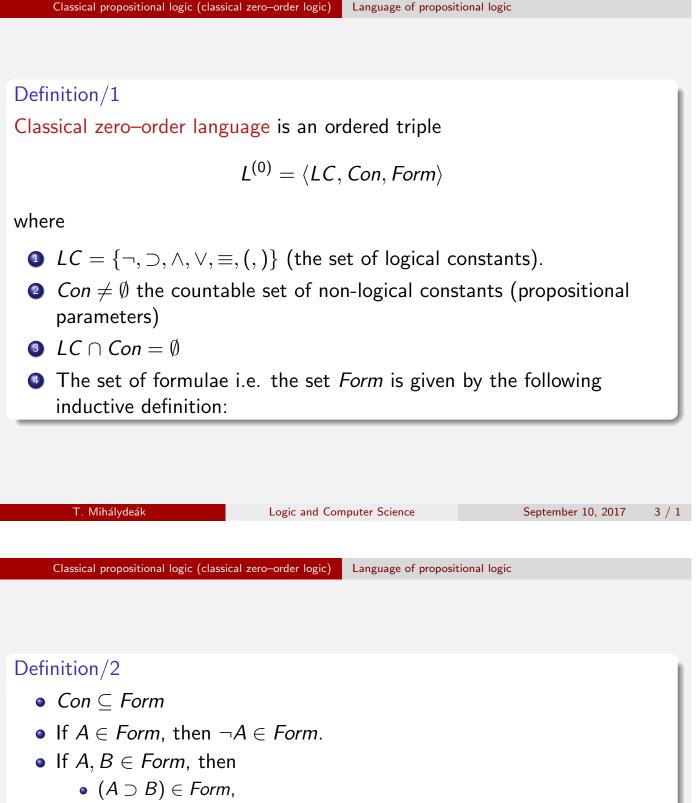


#### The main task of logic

• to give the laws of valid arguments (inferences, consequence relations)

#### Valid arguments

- Valid arguments (inferences):
  - an argument (an inference): a relation between premise(s) and conclusion
  - a consequence relation
    - input: premise(s)
    - output: conclusion
  - Valid arguments (inferences, consequence relations): if all premises are true, then the conclusion is true.
  - Logically valid arguments: when the former holds necessarily.



- $(A \land B) \in Form$ ,
- $(A \lor B) \in Form$ ,
- $(A \equiv B) \in Form.$

#### Remark

The members of the set *Con* are the atomic formulae (prime formulae).

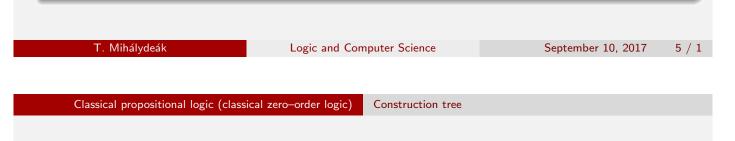
- If A is an atomic formula, then it has no direct subformula;
- $\neg A$  has exactly one direct subformula: A;
- Direct subformulae of formulae (A ⊃ B), (A ∧ B), (A ∨ B), (A ≡ B) are formulae A and B, respectively.

#### Definition

The set of subformulae of formula A [denoting: SF(A)] is given by the following inductive definition:

- $A \in RF(A)$  (i.e. the formula A is a subformula of itself);
- if A' ∈ RF(A) and B is a direct subformula of A'-nek, then B ∈ RF(A)

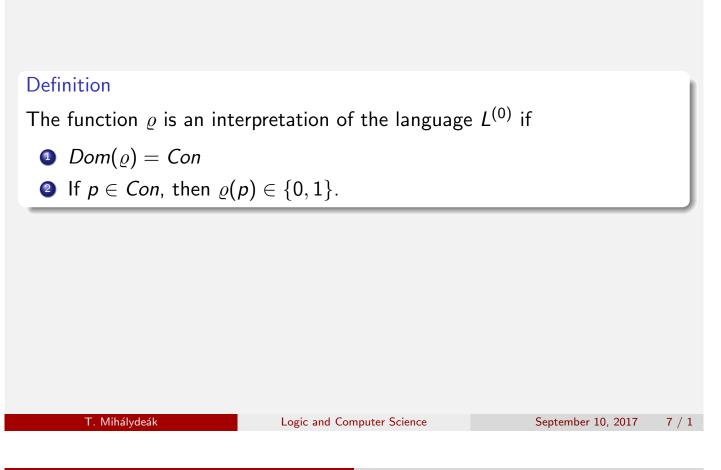
(i.e., if A' is a subformula of A, then all direct subformulae of A' are subformulae of A).



#### Definition

The contruction tree of a formula *A* is a finite ordered tree whose nodes are formulae,

- the root of the tree is the formmula A,
- the node with formula  $\neg B$  has one child: he node with the formula B,
- the node with formulae (B ⊃ C), (B ∧ C), (B ∨ C), (B ≡ C) has two children: the nodes with B, and C
- the leaves of the tree are atomic formulae.

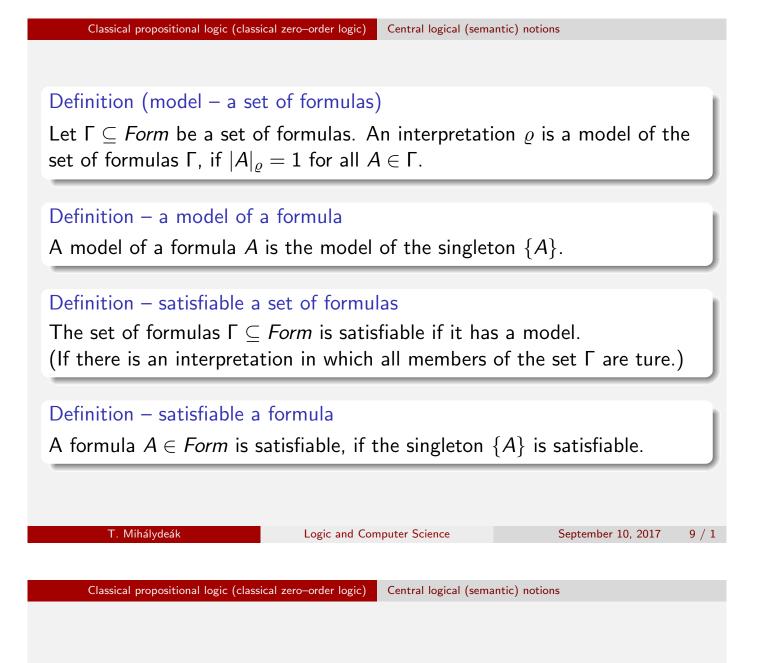


Classical propositional logic (classical zero-order logic) The semantic rules of propositional logic

#### Definition

Let  $\rho$  be an interpretation and  $|A|_{\rho}$  be the semantic value of the formula A formula with respect to  $\rho$ .

If  $p \in Con$ , then  $|p|_{\varrho} = \varrho(p)$ If  $A \in Form$ , then  $|\neg A|_{\varrho} = 1 - |A|_{\varrho}$ .
If  $A, B \in Form$ , then  $|(A \supset B)|_{\varrho} = \begin{cases} 0 & \text{if } |A|_{\varrho} = 1, \text{ and } |B|_{\varrho} = 0; \\ 1, & \text{otherwise} \end{cases}$   $|(A \land B)|_{\varrho} = \begin{cases} 1 & \text{if } |A|_{\varrho} = 1, \text{ and } |B|_{\varrho} = 1; \\ 0, & \text{otherwise} \end{cases}$   $|(A \lor B)|_{\varrho} = \begin{cases} 0 & \text{if } |A|_{\varrho} = 0, \text{ and } |B|_{\varrho} = 0; \\ 1, & \text{otherwise} \end{cases}$   $|(A \lor B)|_{\varrho} = \begin{cases} 1 & \text{if } |A|_{\varrho} = 0, \text{ and } |B|_{\varrho} = 0; \\ 1, & \text{otherwise} \end{cases}$   $|(A \equiv B)|_{\varrho} = \begin{cases} 1 & \text{if } |A|_{\varrho} = |B|_{\varrho}; \\ 0, & \text{otherwise}. \end{cases}$ 



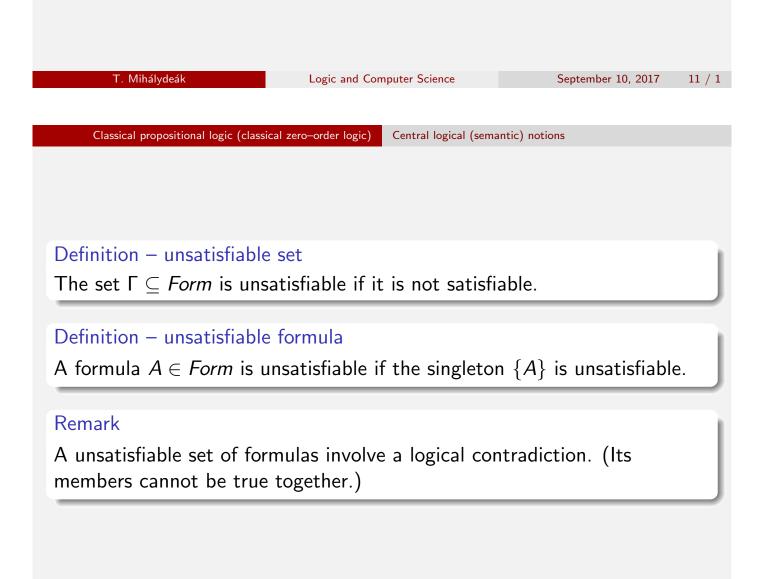
#### Remark

- A satisfiable set of formulas does not involve a logical contradiction; its formulas may be true together.
- A safisfiable formula may be true.
- If a set of formulas is satisfiable, then its members are satisfiable.
- But: all members of the set {p, ¬p} are satisfiable, and the set is not satisfiable.

All subsets of a satisfiable set are satisfiable.

#### Proof

- Let  $\Gamma \subseteq$  *Form* be a set of formulas and  $\Delta \subseteq \Gamma$ .
- $\Gamma$  is satisfiable: it has a model. Let  $\varrho$  be a model of  $\Gamma$ .
- A property of  $\varrho$ : If  $A \in \Gamma$ , then  $|A|_{\rho} = 1$
- Since Δ ⊆ Γ, if A ∈ Δ, then A ∈ Γ, and so |A|<sub>ρ</sub> = 1. That is the interpretation ρ is a model of Δ, and so Δ is satisfiable.



All expansions of an unsatisfiable set of formulas are unsatisfiable.

#### Indirect proof

- Suppose that  $\Gamma \subseteq Form$  is an unsatisfiable set of formulas and  $\Delta \subseteq Form$  is a set of formulas.
- Indirect condition:  $\Gamma$  is unsatisfiable, and  $\Gamma \cup \Delta$  satisfiable.
- $\Gamma \subseteq \Gamma \cup \Delta$
- According to the former theorem Γ is satisfiable, and it is a contradiction.

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Classical propositional logic (classical zero-order logic) Central logical (semantic) notions

## Definition

```
A formula A is the logical consequence of the set of formulas \Gamma if the set \Gamma \cup \{\neg A\} is unsatifiable. (Notation : \Gamma \vDash A)
```

```
Definition A \vDash B, if \{A\} \vDash B.
```

#### Definition

The formula A is valid if  $\emptyset \vDash A$ . (Notation:  $\vDash A$ )

```
The formulas A and B are logically equivalent if A \vDash B and B \vDash A.
(Notation: A \Leftrightarrow B)
```

Let  $\Gamma \subseteq Form$ , and  $A \in Form$ .  $\Gamma \vDash A$  if and only if all models of the set  $\Gamma$  are the models of formula A. (i.e. the singleton  $\{A\}$ ).

#### Proof

 $\rightarrow$  Indirect condition: There is a model of  $\Gamma \vDash A$  such that it is not a model of the formula A.

Let the interpretation  $\rho$  be this model.

The properties of  $\varrho$ :

• 
$$|B|_{\rho} = 1$$
 for all  $B \in \Gamma$ ;

2 
$$|A|_{\varrho} = 0$$
, and so  $|\neg A|_{\varrho} = 1$ 

In this case all members of the set  $\Gamma \cup \{\neg A\}$  are true wrt  $\rho$ -ban, and so  $\Gamma \cup \{\neg A\}$  is satisfiable. It means that  $\Gamma \nvDash A$ , and it is a contradiction.

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Classical propositional logic (class	ical zero, order logic) Control logical (con	mantic) nations	

#### Proof

← Indirect condition: All models of the set  $\Gamma$  are the models of formula A, but (and)  $\Gamma \nvDash A$ .

In this case  $\Gamma \cup \{\neg A\}$  is satisfiable, i.e. it has a model.

Let the interpretation  $\rho$  be a model.

The properties of  $\varrho$ :

•  $|B|_{\rho} = 1$  for all  $B \in \Gamma$ ;

2 
$$|\neg A|_{\varrho} = 1$$
, i.e.  $|A|_{\varrho} = 0$ 

So the set  $\Gamma$  has a model such that it is not a model of formula A, and it is a contradiction.

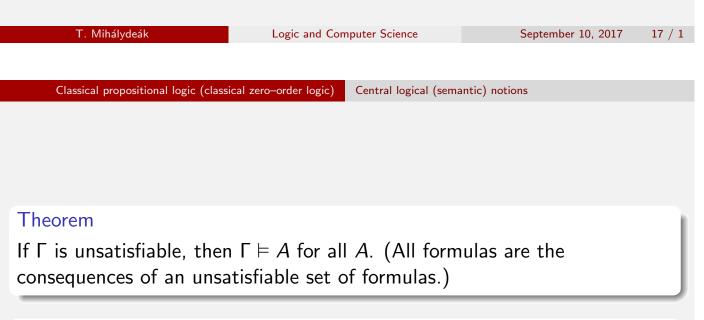
#### Corollary

Let  $\Gamma \subseteq Form$ , and  $A \in Form$ .  $\Gamma \vDash A$  if and only if for all interpretations in which all members of  $\Gamma$  are true, the formula A is true.

If A is a valid formula  $((\models A))$ , then  $\Gamma \models A$  for all sets of formulas  $\Gamma$ . (A valid formula is a consequence of any set of formulas.)

#### Proof

- If A is a valid formula, then  $\emptyset \vDash A$  (according to its definition).
- Ø ∪ {¬A} (= {¬A}) is unsatisfiable, and so its expansions are unsatisfiable.
- $\Gamma \cup \{\neg A\}$  is an expansion of  $\{\neg A\}$ , and so it is unsatisfiable, i.e.  $\Gamma \vDash A$ .



## Proof

- According to a proved theorem: If Γ is unsatisfiable, the all expansions of Γ are unsatisfiable.
- $\Gamma \cup \{\neg A\}$  is an expansion of  $\Gamma$ , and so it is unsatisfiable, i.e.  $\Gamma \vDash A$ .

**Deduction theorem**: If  $\Gamma \cup \{A\} \vDash B$ , then  $\Gamma \vDash (A \supset B)$ .

#### Proof

- Indirect condition: Suppose, that  $\Gamma \cup \{A\} \vDash B$ , and  $\Gamma \nvDash (A \supset B)$ .
- Γ ∪ {¬(A ⊃ B)} is satisfiable, and so it has a model. Let the interpretation *ρ* be a model.
- The properties of  $\varrho$ :
  - **1** All members of  $\Gamma$  are true wrt  $\varrho$ .

$$2 |\neg (A \supset B)|_{\varrho} = 1$$

- $|(A \supset B)|_{\varrho} = 0$ , i.e.  $|A|_{\varrho} = 1$  and  $|B|_{\varrho} = 0$ . So $|\neg B|_{\varrho} = 1$ .
- All members of Γ ∪ {A} ∪ {¬B} are true wrt interpretation *ρ*, i.e.
   Γ ∪ {A} ⊭ B, and it is a contradiction.

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#### Theorem

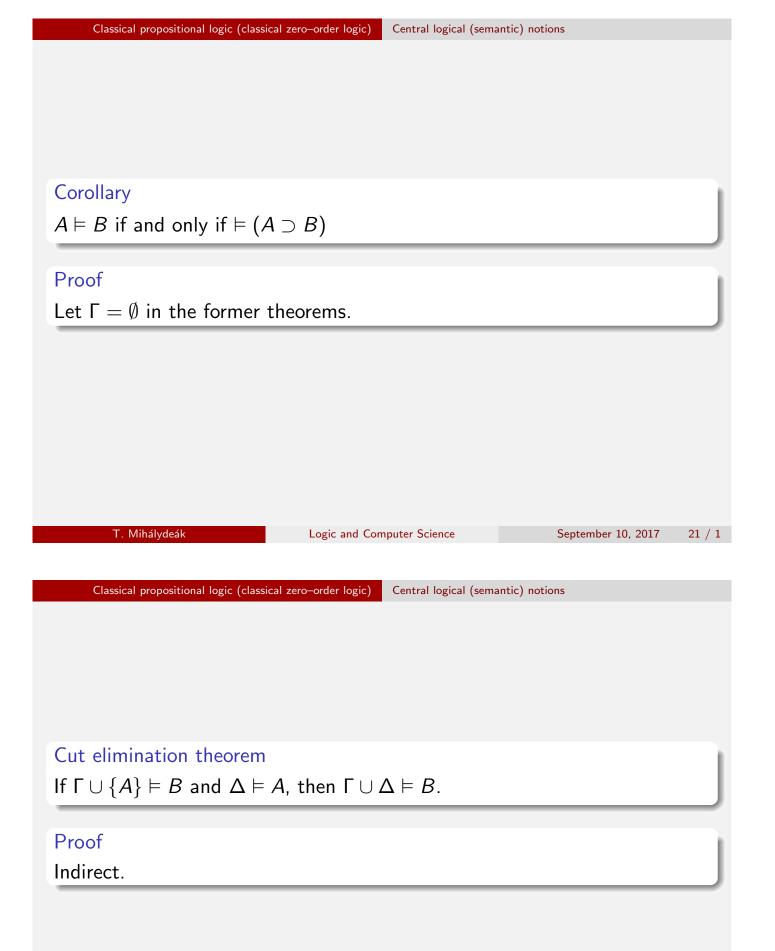
In the opposite direction: If  $\Gamma \vDash (A \supset B)$ , then  $\Gamma \cup \{A\} \vDash B$ .

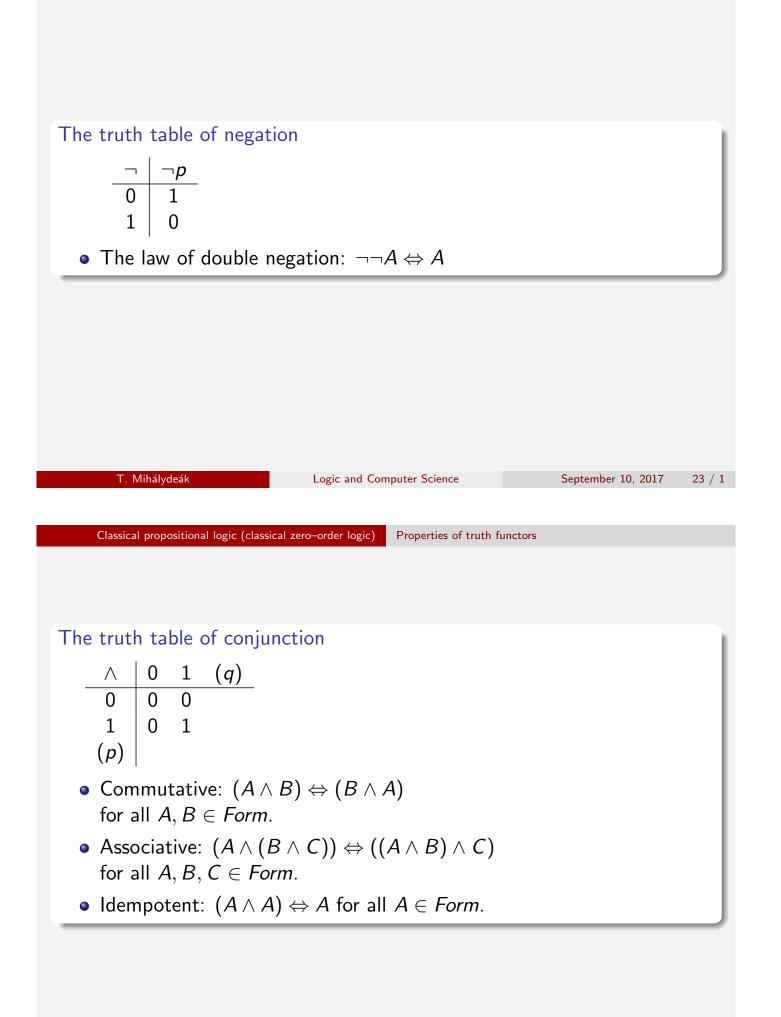
#### Proof

- Indirect condition: Suppose that  $\Gamma \vDash (A \supset B)$ , and  $\Gamma \cup \{A\} \nvDash B$ .
- So Γ ∪ {A} ∪ {¬B} is satisfiable, i.e. it has a model. Let the interpretation *ρ* a model.
- The properties of  $\varrho$ :
  - **1** All members of  $\Gamma$  are true wrt the interpretation  $\varrho$ .

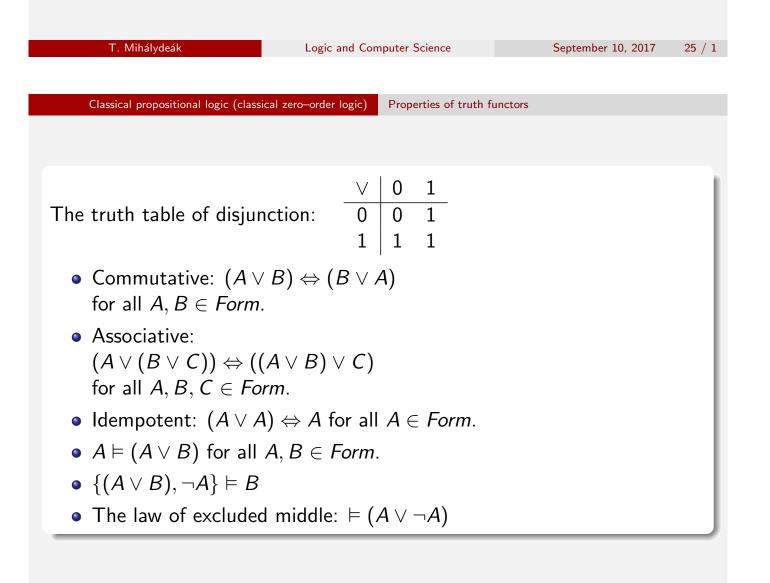
**2** 
$$|A|_o = 1$$

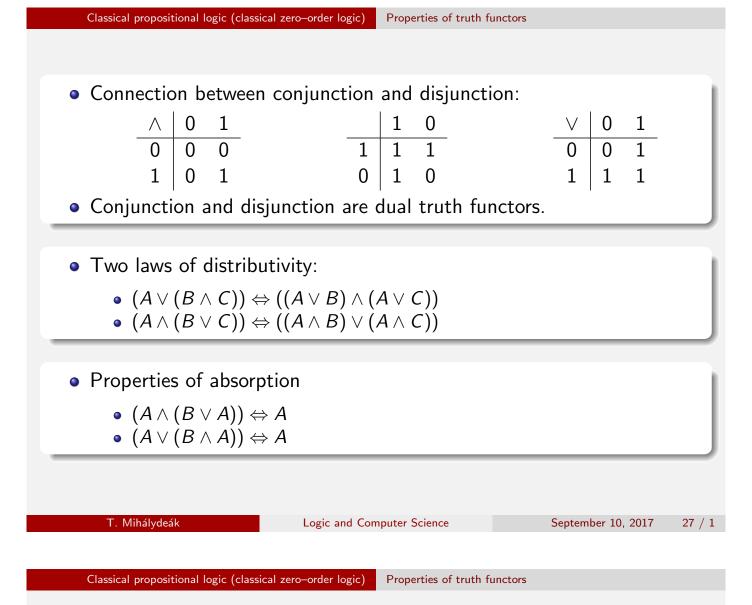
- (3)  $|\neg B|_{\varrho} = 1$ , and so  $|B|_{\varrho} = 0$
- $|(A \supset B)|_{\varrho} = 0$ ,  $|\neg (A \supset B)|_{\varrho} = 1$ .
- All members of  $\Gamma \cup \{\neg(A \supset B)\}$  are true wrt the interpretation  $\varrho$ , i.e.  $\Gamma \nvDash (A \supset B)$ .





- $(A \land B) \vDash A$ ,  $(A \land B) \vDash B$
- The law of contradiction:  $\vDash \neg (A \land \neg A)$
- The set {A<sub>1</sub>, A<sub>2</sub>,..., A<sub>n</sub>} (A<sub>1</sub>, A<sub>2</sub>,..., A<sub>n</sub> ∈ Form) is satisfiable iff the formula A<sub>1</sub> ∧ A<sub>2</sub> ∧ · · · ∧ A<sub>n</sub> is satisfiable.
- The set {A<sub>1</sub>, A<sub>2</sub>, ..., A<sub>n</sub>} (A<sub>1</sub>, A<sub>2</sub>, ..., A<sub>n</sub> ∈ Form) is unsatisfiable iff the formula A<sub>1</sub> ∧ A<sub>2</sub> ∧ ··· ∧ A<sub>n</sub> is unsatisfiable.
- $\{A_1, A_2, \ldots, A_n\} \vDash A (A_1, A_2, \ldots, A_n, A \in Form)$  iff  $A_1 \land A_2 \land \cdots \land A_n \vDash A$ .
- $\{A_1, A_2, \ldots, A_n\} \vDash A (A_1, A_2, \ldots, A_n, A \in Form)$  iff the formula  $((A_1 \land A_2 \land \cdots \land A_n) \land \neg A)$  is unsatisfiable.



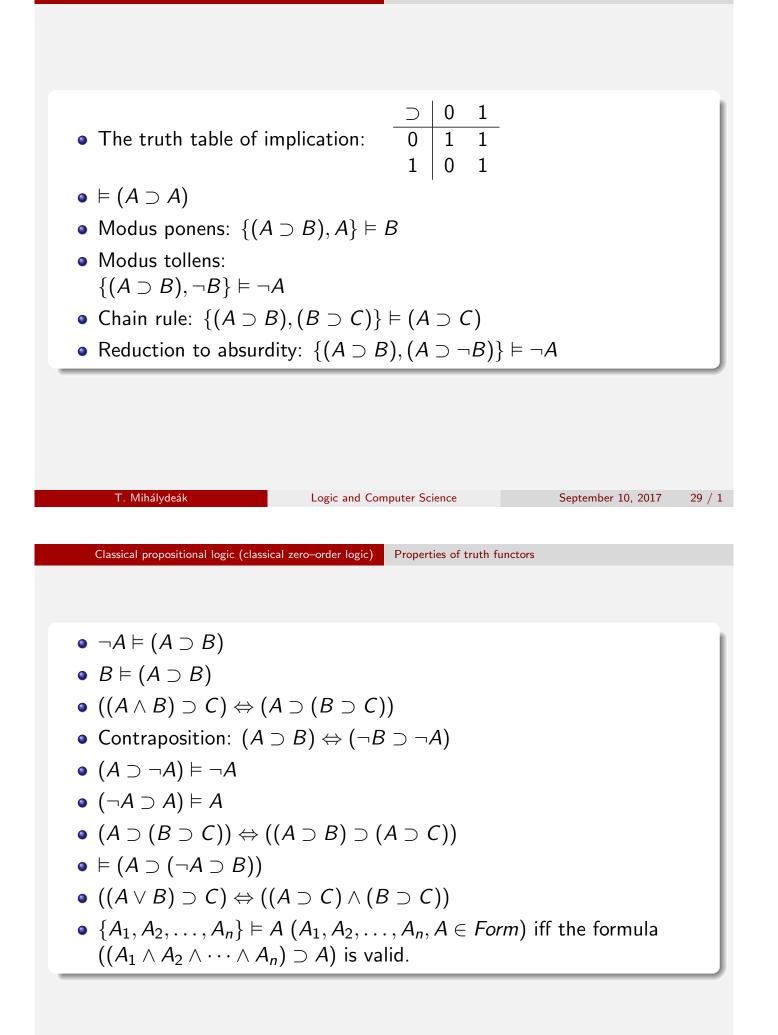


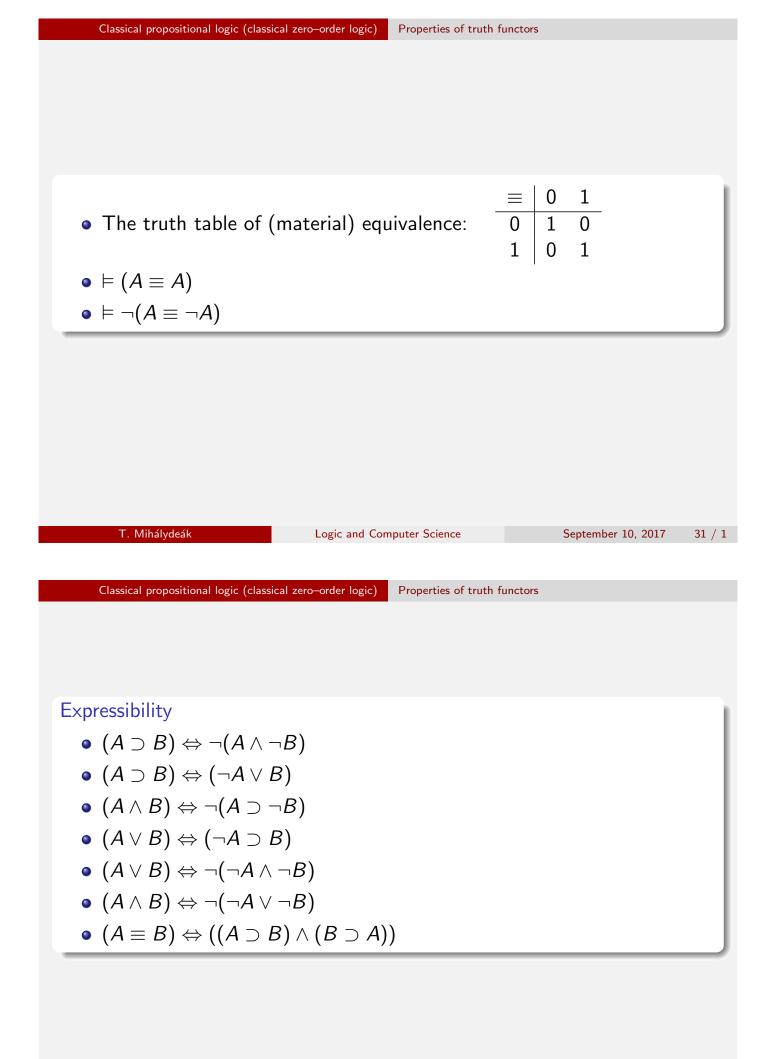
# De Morgan's laws

- What do we say when we deny a conjunction?
- What do we say when we deny a disjunction?
- $\neg (A \land B) \Leftrightarrow (\neg A \lor \neg B)$
- $\neg (A \lor B) \Leftrightarrow (\neg A \land \neg B)$

#### • The proofs of De Morgan's laws.

	Α	В	$\neg A$	$\neg B$	$(\neg A \land \neg B)$	$(A \lor B)$	$\neg (A \lor B)$
	0	0	1	1	1	0	1
۲	0	1	1	0	0	1	0
	1	0	0	1	0	1	0
	1	1	0	0	0	1	0





# Theory of truth functors

#### Base

- A base is a set of truth functors whose members can express all truth functors.
  - For example:  $\{\neg, \supset\}, \{\neg, \wedge\}, \{\neg, \vee\}$ 
    - $\begin{array}{c} \textcircled{1} \quad (p \land q) \Leftrightarrow \neg (p \supset \neg q) \\ \textcircled{2} \quad (p \lor q) \Leftrightarrow (\neg p \supset q) \end{array}$
  - Truth functor Sheffer:  $(p|q) \Leftrightarrow_{def} 
    eg(p \wedge q)$
  - Truth functor neither-nor:  $(p \parallel q) \Leftrightarrow_{def} (\neg p \land \neg q)$
  - Remark: Singleton bases: (p|q),  $(p \parallel q)$



#### Definition

If  $p \in Con$ , then formulas  $p, \neg p$  are literals (p is the base of the literals).

#### Definition

If the formula A is a literal or a conjunction of literals with different bases, then A is an elementary conjunction.

#### Definition

If the formula A is a literal or a disjunction of literals with different bases, the A is an elementary disjunction.

A disjunction of elementary conjunctions is a disjunctive normal form.

## Definition

A conjunction of elementary disjunctions is a conjunctive normal form.

#### Theorem

There is a normal form of any formula of proposition logic, i. e. if  $A \in Form$ , then there is a formula B such that B is a normal form and  $A \Leftrightarrow B$ 

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 Classical first-order logic

 Language of classical first-order logic

 Definition/1

 The language of first-order logic is a

 
$$L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$$

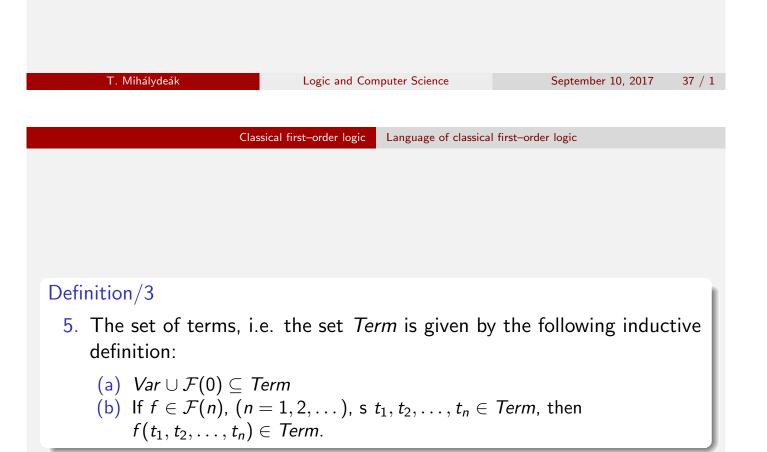
 ordered 5-tuple, where

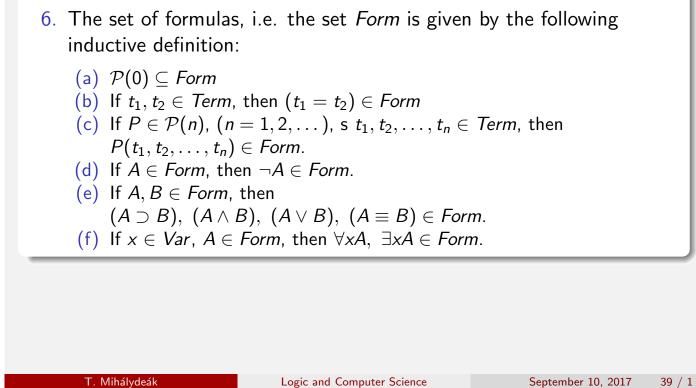
 1.
  $LC = \{\neg, \supset, \land, \lor, \equiv, =, \forall, \exists, (, )\}$ : (the set of logical constants).

 2.

 Var (= { $x_n : n = 0, 1, 2, ... \}$ ): countable infinite set of variables

- 3.  $Con = \bigcup_{n=0}^{\infty} (\mathcal{F}(n) \cup \mathcal{P}(n))$  the set of non-logical constants (at best countable infinite)
  - $\mathcal{F}(0)$ : the set of name parameters,
  - $\mathcal{F}(n)$ : the set of *n* argument function parameters,
  - $\mathcal{P}(0)$ : the set of prposition parameters,
  - $\mathcal{P}(n)$ : the set of predicate parameters.
- 4. The sets LC, Var,  $\mathcal{F}(n)$ ,  $\mathcal{P}(n)$  are pairwise disjoint (n = 0, 1, 2, ...).





Classical first-order logic

Megjegyzs:

• Azokat a formulkat, amelyek a 6. (a), (b), (c) szablyok Ital jnnek Itre, atomi formulknak vagy prmformulknak nevezzk.

Syntactical definitions

## Definci:

#### Definition (interpretation)

The ordered pair  $\langle U, \varrho \rangle$  is an interpretation of the language  $L^{(1)}$  if

•  $U \neq \emptyset$  (i.e. U is a nonempty set);

• 
$$Dom(\varrho) = Con$$

- If  $a \in \mathcal{F}(0)$ , then  $\varrho(a) \in U$ ;
- If  $f \in \mathcal{F}(n)$   $(n \neq 0)$ , then  $\varrho(f) \in U^{U^{(n)}}$
- If  $p \in \mathcal{P}(0)$ , then  $\varrho(p) \in \{0,1\}$ ;
- If  $P \in \mathcal{P}(n)$   $(n \neq 0)$ , then  $\varrho(P) \subseteq U^{(n)}$   $(\varrho(P) \in \{0,1\}^{U^{(n)}})$ .



#### Classical first-order logic Semantics of classical first-order logic

#### Definition (assignment)

The function v is an assignment relying on the interpretation  $\langle U, \varrho \rangle$  if the followings hold:

- Dom(v) = Var;
- If  $x \in Var$ , then  $v(x) \in U$ .

#### Definition (modified assignment)

Let v be an assignment relying on the interpretation  $\langle U, \varrho \rangle$ ,  $x \in Var$  and  $u \in U$ .

$$v[x:u](y) = \begin{cases} u, & \text{if } y = x; \\ v(y), & \text{otherwise.} \end{cases}$$

for all  $y \in Var$ .

## Definition (Semantic rules/1)

Let  $\langle U, \varrho \rangle$  be a given interpretation and v be an assignment relying on  $\langle U, \varrho \rangle$ .

• If 
$$a \in \mathcal{F}(0)$$
, then  $|a|_{v}^{\langle U,\varrho \rangle} = \varrho(a)$ .  
• If  $x \in Var$ , then  $|x|_{v}^{\langle U,\varrho \rangle} = v(x)$ .  
• If  $f \in \mathcal{F}(n)$ ,  $(n = 1, 2, ...)$ , and  $t_{1}, t_{2}, ..., t_{n} \in Term$ , then  
 $|f(t_{1})(t_{2})...(t_{n})|_{v}^{\langle U,\varrho \rangle} = \varrho(f)(\langle |t_{1}|_{v}^{\langle U,\varrho \rangle}, |t_{2}|_{v}^{\langle U,\varrho \rangle}, ..., |t_{n}|_{v}^{\langle U,\varrho \rangle}\rangle)$   
• If  $p \in \mathcal{P}(0)$ , then  $|p|_{v}^{\langle U,\varrho \rangle} = \varrho(p)$   
• If  $t_{1}, t_{2} \in Term$ , then  
 $|(t_{1} = t_{2})|_{v}^{\langle U,\varrho \rangle} = \begin{cases} 1, & \text{if } |t_{1}|_{v}^{\langle U,\varrho \rangle} = |t_{2}|_{v}^{\langle U,\varrho \rangle} \\ 0, & \text{otherwise.} \end{cases}$   
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Definition (Semantic rules/2)

• If 
$$P \in \mathcal{P}(n)$$
  $(n \neq 0), t_1, \ldots, t_n \in Term$ , then

$$|P(t_1)\dots(t_n)|_{v}^{\langle U,\varrho\rangle} = \begin{cases} 1, & \text{if } \langle |t_1|_{v}^{\langle U,\varrho\rangle},\dots,|t_n|_{v}^{\langle U,\varrho\rangle}\rangle \in \varrho(P); \\ 0, & \text{otherwise.} \end{cases}$$

## Definition (Semantic rules/3)

- If  $A \in Form$ , then  $|\neg A|_{v}^{\langle U, \varrho \rangle} = 1 |A|_{v}^{\langle U, \varrho \rangle}$ .
- If  $A, B \in Form$ , then

$$\begin{split} |(A \supset B)|_{\nu}^{\langle U,\varrho\rangle} &= \begin{cases} 0 & \text{if } |A|_{\nu}^{\langle U,\varrho\rangle} = 1, \text{ and } |B|_{\nu}^{\langle U,\varrho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases} \\ |(A \land B)|_{\nu}^{\langle U,\varrho\rangle} &= \begin{cases} 1 & \text{if } |A|_{\nu}^{\langle U,\varrho\rangle} = 1, \text{ and } |B|_{\nu}^{\langle U,\varrho\rangle} = 1; \\ 0, & \text{otherwise.} \end{cases} \\ |(A \lor B)|_{\nu}^{\langle U,\varrho\rangle} &= \begin{cases} 0 & \text{if } |A|_{\nu}^{\langle U,\varrho\rangle} = 0, \text{ and } |B|_{\nu}^{\langle U,\varrho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases} \\ |(A \equiv B)|_{\nu}^{\langle U,\varrho\rangle} &= \begin{cases} 1 & \text{if } |A|_{\nu}^{\langle U,\varrho\rangle} = |B|_{\nu}^{\langle U,\varrho\rangle} = 0; \\ 0, & \text{otherwise.} \end{cases} \end{split}$$

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Classical first-order logic Semantics of classical first-order logic

Definition (Semantic rules/4)

- If  $A \in Form, x \in Var$ , then
  - $|\forall x A|_{\nu}^{\langle U, \varrho \rangle} = \begin{cases} 0, & \text{if there is an } u \in U \text{ such that } |A|_{\nu[x:u]}^{\langle U, \varrho \rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$

$$|\exists x A|_{v}^{\langle U, \varrho \rangle} = \begin{cases} 1, & \text{if there is an } u \in U \text{ such that } |A|_{v[x:u]}^{\langle U, \varrho \rangle} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

#### Definition (model – a set of formulas)

Let  $L(1) = \langle LC, Var, Con, Term, Form \rangle$  be a first order language and  $\Gamma \subseteq Form$  be a set of formulas. An ordered triple  $\langle U, \varrho, v \rangle$  is a model of the set  $\Gamma$ , if

- $\langle U, \varrho \rangle$  is an interpretation of  $L^{(1)}$ ;
- v is an assignment relying on  $\langle U, \varrho \rangle$ ;

• 
$$|A|_{v}^{\langle U,\varrho\rangle} = 1$$
 for all  $A \in \Gamma$ .

## Definition – a model of a formula

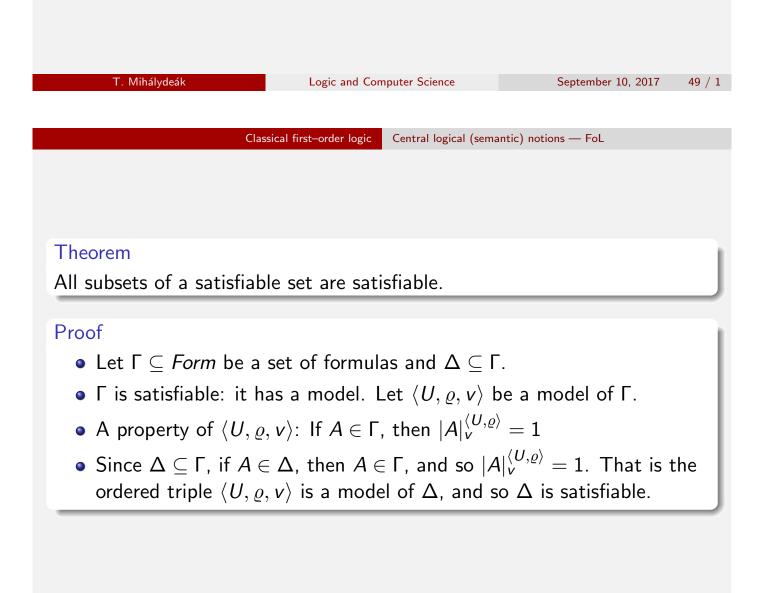
A model of a formula A is the model of the singleton  $\{A\}$ .

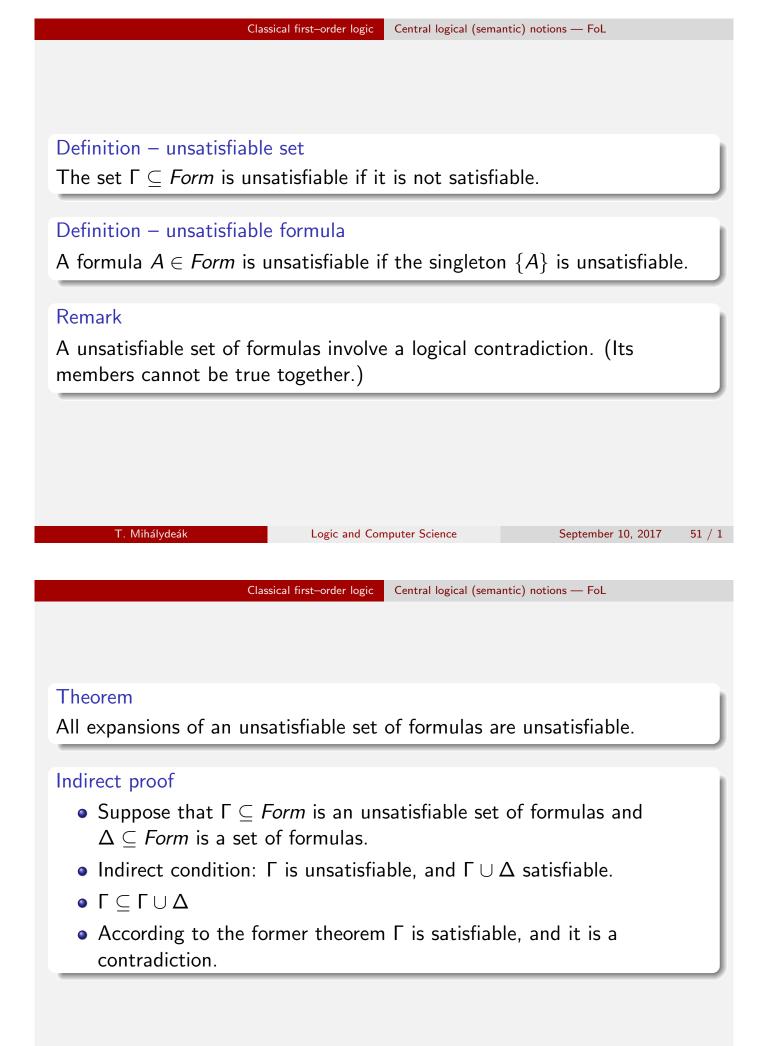
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Clas	sical first–order logic	Central logical (sema	ntic) notions — FoL
Definition – satisfiable a	set of formula	S	
The set of formulas $\Gamma \subseteq$	Form is satisfi	able if it has	s a model.
It there is an interpreta		signmont in	Which all members of
If there is an interpreta he set Γ are true.)	tion and an as	signment in	which all members of

A formula  $A \in Form$  is satisfiable, if the singleton  $\{A\}$  is satisfiable.

#### Remark

- A satisfiable set of formulas does not involve a logical contradiction; its formulas may be true together.
- A satisfiable formula may be true.
- If a set of formulas is satisfiable, then its members are satisfiable.
- But: all members of the set {P(a), ¬P(a)} are satisfiable, and the set is not satisfiable.





A formula A is the logical consequence of the set of formulas  $\Gamma$  if the set  $\Gamma \cup \{\neg A\}$  is unsatifiable. (*Notation* :  $\Gamma \vDash A$ )

#### Definition

 $A \vDash B$ , if  $\{A\} \vDash B$ .

#### Definition

The formula A is valid if  $\emptyset \vDash A$ . (Notation:  $\vDash A$ )

#### Definition

```
The formulas A and B are logically equivalent if A \vDash B and B \vDash A.
(Notation: A \Leftrightarrow B)
```

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Classical first-order logic Properties of first order central logical notions

#### Theorem

Let  $\Gamma \subseteq Form$ , and  $A \in Form$ .  $\Gamma \vDash A$  if and only if all models of the set  $\Gamma$  are the models of formula A. (i.e. the singleton  $\{A\}$ ).

#### Proof

 $\rightarrow$  Indirect condition: There is a model of  $\Gamma \vDash A$  such that it is not a model of the formula A.

Let the ordered triple  $\langle U, \varrho, v \rangle$  be this model. The properties of  $\langle U, \varrho, v \rangle$ :

$$|B|_{\nu}^{\langle U,\varrho\rangle} = 1 \text{ for all } B \in \Gamma;$$

() 
$$|A|\langle U, arrho 
angle_{m{v}} = 0$$
, and so  $|
eg A|_{m{v}}^{\langle U, arrho 
angle} = 1$ 

In this case all members of the set  $\Gamma \cup \{\neg A\}$  are true wrt the interpretation  $\langle U, \varrho \rangle$  and assignment v, so  $\Gamma \cup \{\neg A\}$  is satisfiable. It means that  $\Gamma \nvDash A$ , and it is a contradiction.

### Proof

← Indirect condition: All models of the set  $\Gamma$  are the models of formula A, but (and)  $\Gamma \nvDash A$ .

In this case  $\Gamma \cup \{\neg A\}$  is satisfiable, i.e. it has a model.

Let the ordered triple  $\langle U, \varrho, v \rangle$  be a model.

The properties of  $\langle U, \varrho, v \rangle$ :

$$\begin{array}{l} \bullet \quad |B|_{v}^{\langle U,\varrho\rangle} = 1 \text{ for all } B \in \Gamma; \\ \bullet \quad |\neg A|_{v}^{\langle U,\varrho\rangle} = 1, \text{ i.e. } |A|_{v}^{\langle U,\varrho\rangle} = 0 \end{array}$$

So the set  $\Gamma$  has a model such that it is not a model of formula A, and it is a contradiction.

## Corollary

Let  $\Gamma \subseteq Form$ , and  $A \in Form$ .  $\Gamma \vDash A$  if and only if for all interpretations in which all members of  $\Gamma$  are true, the formula A is true.

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#### Theorem

If A is a valid formula  $((\models A))$ , then  $\Gamma \models A$  for all sets of formulas  $\Gamma$ . (A valid formula is a consequence of any set of formulas.)

## Proof

- If A is a valid formula, then  $\emptyset \vDash A$  (according to its definition).
- Ø ∪ {¬A} (= {¬A}) is unsatisfiable, and so its expansions are unsatisfiable.
- $\Gamma \cup \{\neg A\}$  is an expansion of  $\{\neg A\}$ , and so it is unsatisfiable, i.e.  $\Gamma \vDash A$ .

If  $\Gamma$  is unsatisfiable, then  $\Gamma \vDash A$  for all A. (All formulas are the consequences of an unsatisfiable set of formulas.)

#### Proof

- According to a proved theorem: If Γ is unsatisfiable, the all expansions of Γ are unsatisfiable.
- $\Gamma \cup \{\neg A\}$  is an expansion of  $\Gamma$ , and so it is unsatisfiable, i.e.  $\Gamma \vDash A$ .

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#### Theorem

**Deduction theorem**: If  $\Gamma \cup \{A\} \vDash B$ , then  $\Gamma \vDash (A \supset B)$ .

#### Proof

- Indirect condition: Suppose, that  $\Gamma \cup \{A\} \vDash B$ , and  $\Gamma \nvDash (A \supset B)$ .
- Γ ∪ {¬(A ⊃ B)} is satisfiable, and so it has a model. Let the ordered triple ⟨U, ρ, ν⟩ be a model.
- The properties of  $\langle U, \varrho, \nu \rangle$ :
  - **1** All members of  $\Gamma$  are true wrt  $\langle U, \varrho \rangle$  and v.

$$( |\neg (A \supset B)|_{v}^{\langle U, \varrho \rangle} = 1$$

- $|(A \supset B)|_{\nu}^{\langle U, \varrho \rangle} = 0$ , i.e.  $|A|_{\nu}^{\langle U, \varrho \rangle} = 1$  and  $|B|_{\nu}^{\langle U, \varrho \rangle} = 0$ . So $|\neg B|_{\nu}^{\langle U, \varrho \rangle} = 1$ .
- All members of Γ ∪ {A} ∪ {¬B} are true wrt ⟨U, ρ⟩ and v, i.e.
   Γ ∪ {A} ⊭ B, and it is a contradiction.

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In the opposite direction: If  $\Gamma \vDash (A \supset B)$ , then  $\Gamma \cup \{A\} \vDash B$ .

## Proof

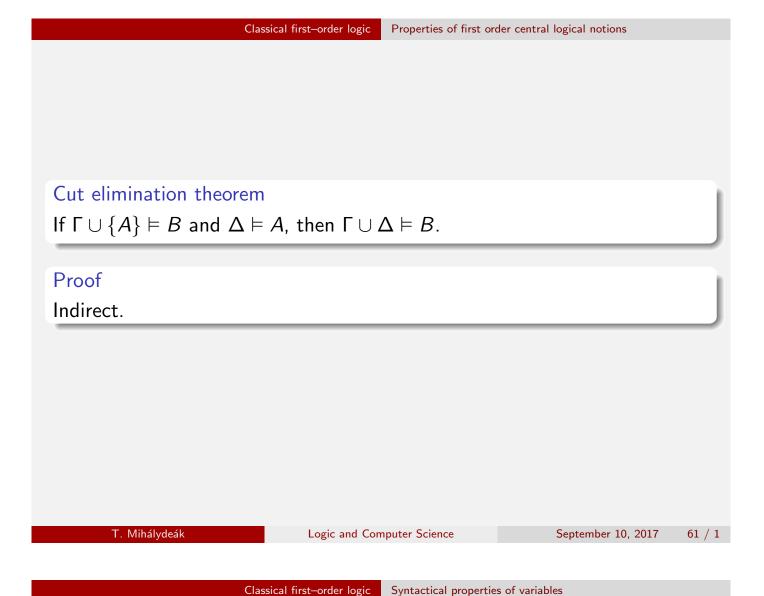
- Indirect condition: Suppose that  $\Gamma \vDash (A \supset B)$ , and  $\Gamma \cup \{A\} \nvDash B$ .
- So Γ ∪ {A} ∪ {¬B} is satisfiable, i.e. it has a model. Let the ordered triple ⟨U, ρ, ν⟩ a model.
- The properties of  $\langle U, \varrho, v \rangle$ :
  - **1** All members of  $\Gamma$  are true wrt  $\langle U, \varrho \rangle$  and v.

2 
$$|A|_{v}^{\langle U,\varrho\rangle} = 1$$
  
3  $|\neg B|_{v}^{\langle U,\varrho\rangle} = 1$ , and so  $|B|_{v}^{\langle U,\varrho\rangle} = 0$ 

• 
$$|(A \supset B)|_{v}^{\langle U, \varrho \rangle} = 0, \ |\neg(A \supset B)|_{v}^{\langle U, \varrho \rangle} = 1.$$

• All members of  $\Gamma \cup \{\neg (A \supset B)\}$  are true wrt  $\langle U, \varrho \rangle$  and v, i.e.  $\Gamma \nvDash (A \supset B)$ .

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Corollony				
Corollary				- 1
$A \vDash B$ if and only if $\vDash (A)$	$A \supset B)$			- 11
Proof				
Let $\Gamma = \emptyset$ in the former	theorems.			

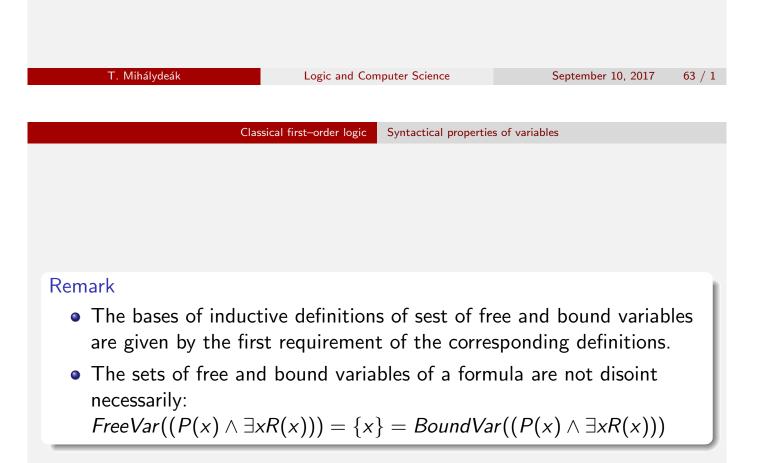


Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language and  $A \in Form$  be a formula. The set of free variables of the formula A (in notation: FreeVar(A)) is given by the following inductive definition:

- If A is an atomic formula (i.e. A ∈ AtForm), then the members of the set FreeVar(A) are the variables occuring in A.
- If the formula A is  $\neg B$ , then FreeVar(A) = FreeVar(B).
- If the formula A is (B ⊃ C), (B ∧ C), (B ∨ C) or (B ≡ C), then FreeVar(A) = FreeVar(B) ∪ FreeVar(C).
- If the formula A is  $\forall xB$  or  $\exists xB$ , then  $FreeVar(A) = FreeVar(B) \setminus \{x\}$ .

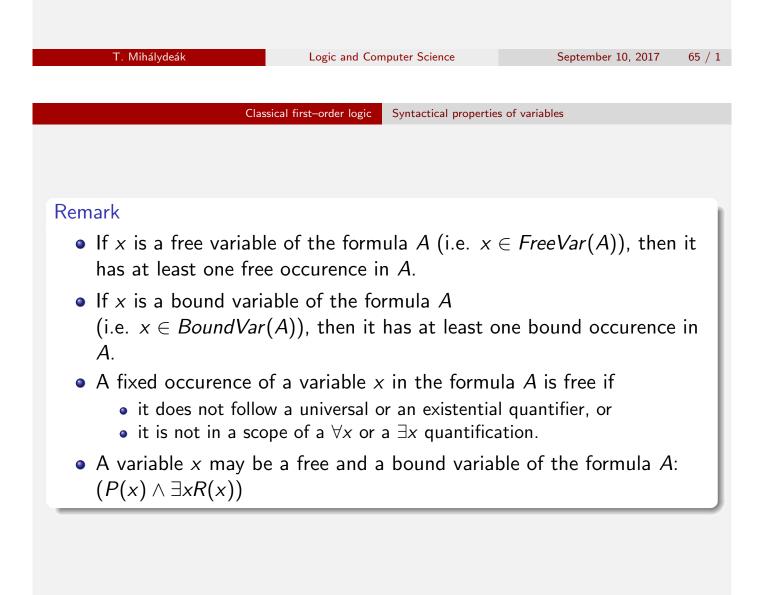
Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language and  $A \in Form$  be a formula. The set of bound variables of the formula A (in notation: BoundVar(A)) is given by the following inductive definition:

- If A is an atomic formula (i.e.  $A \in AtForm$ ), then  $BoundVar(A) = \emptyset$ .
- If the formula A is  $\neg B$ , then BoundVar(A) = FreeVar(B).
- If the formula A is (B ⊃ C), (B ∧ C), (B ∨ C) or (B ≡ C), then BoundVar(A) = BoundVar(B) ∪ BoundVar(C).
- If the formula A is ∀xB or ∃xB, then BoundVar(A) = BoundVar(B) ∪ {x}.



Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language,  $A \in Form$  be a formula, and  $x \in Var$  be a variable.

- A fixed occurrence of the variable x in the formula A is free if it is not in the subformulas ∀xB or ∃xB of the formula A.
- A fixed occurrence of the variable x in the formula A is bound if it is not free.



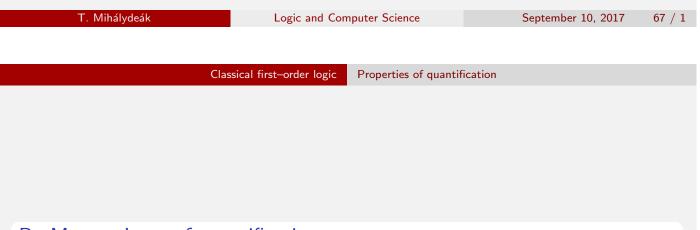
Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language and  $A \in Form$  be a formula.

- If  $FreeVar(A) \neq \emptyset$ , then the formula A is an open formula.
- If  $FreeVar(A) = \emptyset$ , then the formula A is a closed formula.

#### Remark:

The formula A is open if there is at least one variable which has at least one free occurrence in A.

The formula A is closed if there is no variable which has a free occurence in A.



## De Morgan Laws of quantifications

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language,  $A \in Form$  be a formula and  $x \in Var$  be a variable. Then

- $\neg \exists x A \Leftrightarrow \forall x \neg A$
- $\neg \forall x A \Leftrightarrow \exists x \neg A$

### Expressibilty of quantifications

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language,  $A \in Form$  be a formula and  $x \in Var$  be a variable. Then

• 
$$\exists x A \Leftrightarrow \neg \forall x \neg A$$

•  $\forall x A \Leftrightarrow \neg \exists x \neg A$ 



#### Conjunction and quantifications

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language,  $A, B \in Form$  be formulas and  $x \in Var$  be a variable. If  $x \notin FreeVar(A)$ , then

- $A \land \forall xB \Leftrightarrow \forall x(A \land B)$
- $A \land \exists x B \Leftrightarrow \exists x (A \land B)$

#### Remark:

According to the commutativity of conjunction the followings hold: If  $x \notin FreeVar(A)$ , then

- $\forall xB \land A \Leftrightarrow \forall x(B \land A)$
- $\exists x B \land A \Leftrightarrow \exists x (B \land A)$

#### Disjunction and quantifications

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language,  $A, B \in Form$  be formulas and  $x \in Var$  be a variable. If  $x \notin FreeVar(A)$ , then

- $A \lor \forall x B \Leftrightarrow \forall x (A \lor B)$
- $A \lor \exists x B \Leftrightarrow \exists x (A \lor B)$

#### Remark:

According to the commutativity of disjunction the followings hold: If  $x \notin FreeVar(A)$ , then

- $\forall x B \lor A \Leftrightarrow \forall x (B \lor A)$
- $\exists x B \lor A \Leftrightarrow \exists x (B \lor A)$

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Implication with existential quantification

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language,  $A, B \in Form$  be formulas and  $x \in Var$  be a variable. If  $x \notin FreeVar(A)$ , then

- $A \supset \exists x B \Leftrightarrow \exists x (A \lor B)$
- $\exists x B \supset A \Leftrightarrow \forall x (B \supset A)$

#### Implication with universal quantification

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language,  $A, B \in Form$  be formulas and  $x \in Var$  be a variable. If  $x \notin FreeVar(A)$ , then

- $A \supset \forall xB \Leftrightarrow \forall x(A \lor B)$
- $\forall xB \supset A \Leftrightarrow \exists x(B \supset A)$

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#### Substitutabily a variable with an other variable

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language,  $A \in Form$  be a formula and  $x, y \in Var$  be variables.

The variable x is subtitutable with the variable y in the formula A if there is no a free occurrence of x in A which is in the subformulas  $\forall yB$  or  $\exists yB$  of A.

#### Example:

In the formula ∀zP(x, z) the variable x is substitutable with the variable y, but x is not substitutable with the variable z.

#### Substitutabily a variable with a term

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language,  $A \in Form$  be a formula,  $x \in Var$  be a variable and  $t \in Term$  be a term. The variable x is subtitutable with the term t in the formula A if in the formula A the variable x is substitutable with all variables occuring in the term t.

#### Example

• In the formula  $\forall z P(x, z)$  the variable x is substitutable with the term  $f(y_1, y_2)$ , but x is not substitutable with the term f(y, z).

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## Result of a substitution

If the variable x is subtitutable with the term t in the formula A, then  $[A]_x^t$  denotes the formula which appear when all free occurences of the variable x in A are substituted with the term t.

### Renaming

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language,  $A \in Form$  be a formula, and  $x, y \in Var$  be variables.

If the variable x is subtitutable with the variable y in the formula A and  $y \notin FreeVar(A)$ , then

- the formula  $\forall y[A]_x^y$  is a regular renaming of the formula  $\forall xA$ ;
- the formula  $\exists y[A]_x^y$  is a regular renaming of the formula  $\exists xA$ .



## Congruent formulas

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language and  $A \in Form$  be a formula.

The set Cong(A) (the set of formulas which are congruent with A) is given by the following inductive definition:

- $A \in Cong(A)$ ;
- if  $\neg B \in Cong(A)$  and  $B' \in Cong(B)$ , then  $\neg B' \in Cong(A)$ ;
- if  $(B \circ C) \in Cong(A)$ ,  $B' \in Cong(B)$  and  $C' \in Cong(C)$ , then  $(B' \circ C') \in Cong(A)$  ( $\circ \in \{\supset, \land, \lor, \equiv\}$ );
- if  $\forall x B \in Cong(A)$  and  $\forall y [B]_x^y$  is a regular renaming of the formula  $\forall x B$ , then  $\forall y [B]_x^y \in Cong(A)$ ;
- if  $\exists x B \in Cong(A)$  and  $\exists y [B]_x^y$  is a regular renaming of the formula  $\exists x B$ , then  $\exists y [B]_x^y \in Cong(A)$ .

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language and  $A, B \in Form$  be formulas.

- If  $B \in Cong(A)$ , then the formula A is congruent with the formula B.
- If B ∈ Cong(A), then the formula B is a syntactical synonym of the formula A.

#### Theorem

Congruent formulas are logically equivalent, i.e. if  $B \in Cong(A)$ , then  $A \Leftrightarrow B$ .

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#### Classical first-order logic Properties of quantification

## Definition

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language and  $A \in Form$  be a formula.

The formula A is standardized if

- FreeVar(A)  $\bigcap$  BoundVar(A) =  $\emptyset$ ;
- all bound variables of the formula A have exactly one occurences next a quantifier.

#### Theorem

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language and  $A \in Form$  be a formula.

Then there is a formula  $B \in Form$  such that

- the formula *B* is standardized;
- the formula B is congruent with the formula A, i.e.  $B \in Cong(A)$ .

Let  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  be a first order language and  $A \in Form$  be a formula. The formula A is prenex if

- there is no quantifier in A or
- the formula A is in the form  $Q_1x_1Q_2x_2...Q_nx_nB$  (n = 1, 2, ...), where
  - there is no quantifier in the formula  $B \in Form$ ;
  - $x_1, x_2 \dots x_n \in Var$  are diffrent variables;
  - $Q_1, Q_2, \ldots, Q_n \in \{\forall, \exists\}$  are quantifiers.

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Theorem				- II
Let $L^{(1)} = \langle LC, Var, Cor \rangle$	n, Term, Form	angle be a first or	rder language and	- 1
$A \in Form$ be a formula.				- 1
Then there is a formula	$B \in Form such$	ch that		- 1
• the formula <i>B</i> is pr	enex;			- 1
• $A \Leftrightarrow B$ .	·			- 1