Definitions

1. Define the first-order term!

The set of terms, i.e. the set Term is given by the following inductive definition:

1. $Var \cup F(0) \subset Term$
2. If $f \in F(n)$, $(n = 1, 2, \ldots)$, and $t_1 \in Term$, $t_2 \in Term$ and $\ldots$ and $t_n \in Term$ then $f(t_1, t_2, \ldots, t_n) \in Term$.

2. Define the first-order atomic formula!

The set of atomic formulas,(a subset of the $Form$ set) contains the followings:

1. $P(0) \in Form$
2. If $t_1 \in Term$ and $t_2 \in Term$, then $(t1 = t2) \in Form$
3. If $P \in P(n)$, $(n = 1, 2, \ldots)$, and $t_1 \in Term$, $t_2 \in Term$ and $\ldots$ and $t_n \in Term$ then $P(t1, t2, \ldots, tn) \in Form$

Practical exercises

Let $L^{(1)}$ be a first-order language:

$$L^{(1)} = \langle LC, \{x_1, x_2, \ldots\}, \{p, q, f, g, c\}, Term, Form\rangle$$

- $p \in P(1)$ and $q \in P(2)$ are predicate parameters,
- $f \in F(1)$ and $g \in F(2)$ are function parameters,
- $c \in F(0)$ is a name parameter.

Let $\langle U, \varrho\rangle$ be an interpretation of $L^{(1)}$:

- $U = \{1, 2, 3, 4\}$
- $\varrho(p) = p'$ $\quad p'(x) = \begin{cases} 1 & \text{if } x = 1 \text{ or } x = 4 \\ 0 & otherwise \end{cases}$
- $\varrho(q) = q'$ $\quad q'(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & otherwise \end{cases}$
- $\varrho(f) = f'$ $\quad f'(x) = 5 - x,$
- $\varrho(g) = g'$ $\quad g'(x, y) = \text{abs}(x - y) + 1$
- $\varrho(c) = 2$

First-order evaluation

3. $|c|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|c|_v^{\langle U,\varrho\rangle} = 2$$

4. $|x_2|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|x_2|_v^{\langle U,\varrho\rangle} = 3$$

5. $|f(c)|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|f(c)|_v^{\langle U,\varrho\rangle} = 3$$

6. $|f(g(c,c))|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|f(g(c,c))|_v^{\langle U,\varrho\rangle} = 4$$

7. $|g(f(x_1), f(x_2))|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|g(f(x_1), f(x_2))|_v^{\langle U,\varrho\rangle} = 3$$

8. $|f(g(x_1, g(x_2, c)))|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|f(g(x_1, g(x_2, c)))|_v^{\langle U,\varrho\rangle} = 3$$

9. $|x_2|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 2, v(x_2) = 4$

$$|x_2|_v^{\langle U,\varrho\rangle} = 4$$

10. $|g(f(x_1), f(x_2))|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 2, v(x_2) = 4$

$$|g(f(x_1), f(x_2))|_v^{\langle U,\varrho\rangle} = 3$$

11. $|f(g(x_1, g(x_2, c)))|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 2, v(x_2) = 4$

$$|f(g(x_1, g(x_2, c)))|_v^{\langle U,\varrho\rangle} = 3$$

12. $|p(x_2)|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|p(x_2)|_v^{\langle U,\varrho\rangle} = 0$$

13. $|\forall x_1 p(x_1)|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|\forall x_1 p(x_1)|_v^{\langle U,\varrho\rangle} = 0$$

14. $|q(x_1, x_2)|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|q(x_1, x_2)|_v^{\langle U,\varrho\rangle} = 0$$

15. $|q(x_1, x_2) \supset \neg q(f(x_1), f(x_2))|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|q(x_1, x_2) \supset \neg q(f(x_1), f(x_2))|_v^{\langle U,\varrho\rangle} = 1$$

16. $|\forall x_1 (q(f(x_1), x_2) \supset \neg q(x_1, f(x_2)))|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|\forall x_1 (q(f(x_1), x_2) \supset \neg q(x_1, f(x_2)))|_v^{\langle U,\varrho\rangle} = 1$$

17. $|\exists x_2 (q(f(c), x_2) \supset \neg q(c, f(x_2)))|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|\exists x_2 (q(f(c), x_2) \supset \neg q(c, f(x_2)))|_v^{\langle U,\varrho\rangle} = 1$$

18. $|\exists x_1 p(x_1) \supset \forall x_1 \neg q(x_1, f(x_1))|_v^{\langle U,\varrho\rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|\exists x_1 p(x_1) \supset \forall x_1 \neg q(x_1, f(x_1))|_v^{\langle U,\varrho\rangle} = 0$$

19. $|\forall x_1 \exists x_2 \neg q(x_1, f(x_2))|_v^{\langle U, \varrho \rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|\forall x_1 \exists x_2 \neg q(x_1, f(x_2))|_v^{\langle U, \varrho \rangle} = 1$$

20. $|\exists x_1 \forall x_2 \neg q(x_1, f(x_2))|_v^{\langle U, \varrho \rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|\exists x_1 \forall x_2 \neg q(x_1, f(x_2))|_v^{\langle U, \varrho \rangle} = 1$$

21. $|\forall x_1 \forall x_2 (q(x_1, f(x_2)) \vee q(f(x_2), x_1))|_v^{\langle U, \varrho \rangle}$ if $v(x_1) = 1, v(x_2) = 3$

$$|\forall x_1 \forall x_2 (q(x_1, f(x_2)) \vee q(f(x_2), x_1))|_v^{\langle U, \varrho \rangle} = 0$$

22. Prenex form of: $\forall x_1 q(f(x_1), x_2) \supset \neg q(x_1, f(x_2))$

$\forall x_1 q(f(x_1), x_2) \supset \neg q(x_1, f(x_2)) \Leftrightarrow$
$\forall x_3 q(f(x_3), x_2) \supset \neg q(x_1, f(x_2)) \Leftrightarrow$
$\exists x_3 (q(f(x_3), x_2) \supset \neg q(x_1, f(x_2)))$

23. Prenex form of: $\exists x_2 q(f(c), x_2) \supset \neg q(c, f(x_2))$

$\exists x_2 q(f(c), x_2) \supset \neg q(c, f(x_2)) \Leftrightarrow$
$\exists x_1 q(f(c), x_1) \supset \neg q(c, f(x_2)) \Leftrightarrow$
$\forall x_1 (q(f(c), x_1) \supset \neg q(c, f(x_2)))$

24. Prenex form of: $\exists x_1 p(x_1) \supset \forall x_1 \neg q(x_1, f(x_1))$

$\exists x_1 p(x_1) \supset \forall x_1 \neg q(x_1, f(x_1)) \Leftrightarrow$
$\exists x_1 p(x_1) \supset \forall x_2 \neg q(x_2, f(x_2)) \Leftrightarrow$
$\forall x_1 (p(x_1) \supset \forall x_2 \neg q(x_2, f(x_2))) \Leftrightarrow$
$\forall x_1 \forall x_2 (p(x_1) \supset \neg q(x_2, f(x_2)))$

25. Prenex form of: $\neg\forall x_1 \exists x_2 q(x_1, f(x_2))$

$\neg\forall x_1 \exists x_2 q(x_1, f(x_2)) \Leftrightarrow$
$\exists x_1 \neg\exists x_2 q(x_1, f(x_2)) \Leftrightarrow$
$\exists x_1 \forall x_2 \neg q(x_1, f(x_2))$

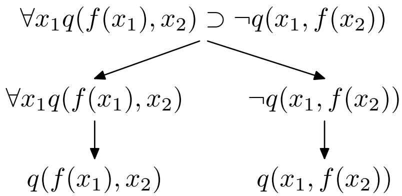
26. Prenex form of: $\neg\exists x_1 \forall x_2 q(x_1, f(x_2))$

$\neg\exists x_1 \forall x_2 q(x_1, f(x_2)) \Leftrightarrow$
$\forall x_1 \neg\forall x_2 q(x_1, f(x_2)) \Leftrightarrow$
$\forall x_1 \exists x_2 \neg q(x_1, f(x_2))$

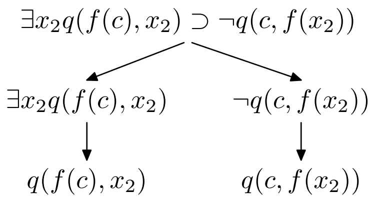
27. Prenex form of: $\forall x_1 \forall x_2 (q(x_1, f(x_2)) \vee q(f(x_2), x_1))$

$\forall x_1 \forall x_2 (q(x_1, f(x_2)) \vee q(f(x_2), x_1))$ is in prenex form.

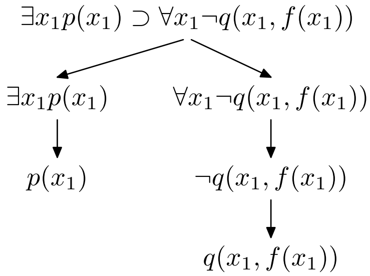
28. Construction tree of: $\forall x_1 q(f(x_1), x_2) \supset \neg q(x_1, f(x_2))$



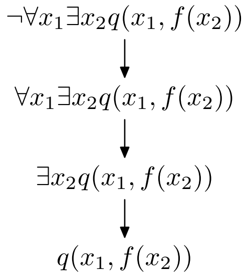
29. Construction tree of: $\exists x_2 q(f(c), x_2) \supset \neg q(c, f(x_2))$



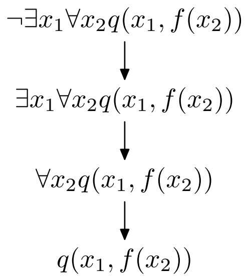
30. Construction tree of: $\exists x_1 p(x_1) \supset \forall x_1 \neg q(x_1, f(x_1))$



31. Construction tree of: $\neg\forall x_1 \exists x_2 q(x_1, f(x_2))$



32. Construction tree of: $\neg\exists x_1 \forall x_2 q(x_1, f(x_2))$



33. Construction tree of: $\forall x_1 \forall x_2 (q(x_1, f(x_2)) \vee q(f(x_2), x_1))$

$$\forall x_1 \forall x_2 (q(x_1, f(x_2)) \vee q(f(x_2), x_1))$$

$$\downarrow$$

$$\forall x_2 (q(x_1, f(x_2)) \vee q(f(x_2), x_1))$$

$$\downarrow$$

$$q(x_1, f(x_2)) \vee q(f(x_2), x_1)$$

$$\swarrow \qquad \searrow$$

$$q(x_1, f(x_2)) \qquad\qquad q(f(x_2), x_1)$$