Definitions

1. Complete the following rule:

$$\frac{\Gamma \vdash (A \supset B); \quad ?}{\Gamma \vdash B}$$

$$\frac{\Gamma \vdash (A \supset B); \quad \Gamma \vdash A}{\Gamma \vdash B}$$

2. Complete the following rule:

$$\frac{\Gamma, A \vdash B; \quad ?}{\Gamma \vdash \neg A}$$

$$\frac{\Gamma, A \vdash B; \quad \Gamma, A \vdash \neg B}{\Gamma \vdash \neg A}$$

3. Complete the following rule:

$$\frac{\Gamma \vdash A; \quad ?}{\Gamma \vdash (A \wedge B)}$$

$$\frac{\Gamma \vdash A; \quad \Gamma \vdash B}{\Gamma \vdash (A \wedge B)}$$

4. Complete the following rule:

$$\frac{\Gamma, B \vdash A; \quad ?}{\Gamma \vdash (A \equiv B)}$$

$$\frac{\Gamma, B \vdash A; \quad \Gamma, A \vdash B}{\Gamma \vdash (A \equiv B)}$$

5. Complete the following rule:

$$\frac{\Gamma \vdash (A \equiv B); \quad ?}{\Gamma \vdash A}$$

$$\frac{\Gamma \vdash (A \equiv B); \quad \Gamma \vdash B}{\Gamma \vdash A}$$

6. Complete the following rule:

$$\frac{?}{\Gamma, A \vdash A}$$

$$\frac{\emptyset}{\Gamma, A \vdash A}$$

7. Complete the following rule:

$$\frac{\Gamma, A \vdash C; \quad ?}{\Gamma, (A \vee B) \vdash C}$$

$$\frac{\Gamma, A \vdash C; \quad \Gamma, B \vdash C;}{\Gamma, (A \vee B) \vdash C}$$

8. Complete the following rule:

$$\frac{\Delta, A \vdash B; \quad ?}{\Gamma, \Delta \vdash B}$$

$$\frac{\Delta, A \vdash B; \quad \Gamma \vdash A}{\Gamma, \Delta \vdash B}$$

Practical exercises

9. Prove the correctness of

$$\frac{\Gamma \vdash A}{\Gamma \vdash (A \vee B)}$$

We indirectly suppose, $\Gamma \models A$ and $\Gamma \not\models (A \vee B)$:
Because of $\Gamma \not\models (A \vee B)$ the $\Gamma \cup \{\neg(A \vee B)\}$ set is satisfiable, so it has a ϱ model.
$|\neg(A \vee B)|_\varrho = 1$ so $|A|_\varrho = 0$ and $|B|_\varrho = 0$.
Furthermore $|\neg A|_\varrho = 1$, so ϱ is a model of $\Gamma \cup \{\neg A\}$.
But in this case $\Gamma \not\models A$.

10. Prove the correctness of

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash (A \supset B)}$$

We indirectly suppose, $\Gamma, A \models B$ and $\Gamma \not\models (A \supset B)$:
Because of $\Gamma \not\models (A \supset B)$ the $\Gamma \cup \{\neg(A \supset B)\}$ set is satisfiable, so it has a ϱ model.
$|\neg(A \supset B)|_\varrho = 1$ so $|A|_\varrho = 1$ and $|B|_\varrho = 0$.
Furthermore $|\neg B|_\varrho = 1$, so ϱ is a model of $\Gamma \cup \{\neg B, A\}$.
But in this case $\Gamma, A \not\models B$.

11. Prove the correctness of

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

We indirectly suppose, $\Gamma \models \neg\neg A$ and $\Gamma \not\models A$:
Because of $\Gamma \not\models A$ the $\Gamma \cup \{\neg A\}$ set is satisfiable, so it has a ϱ model.
$|\neg A|_\varrho = 1$ so $|\neg\neg A|_\varrho = 0$.
Furthermore $|\neg\neg\neg A|_\varrho = 1$, so ϱ is a model of $\Gamma \cup \{\neg\neg\neg A\}$.
But in this case $\Gamma \not\models \neg\neg A$.

12. Prove the validity of $p \wedge q \supset \neg\neg p$ using the natural deduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{p, q, \neg p \vdash p} \qquad \dfrac{\emptyset}{p, q, \neg p \vdash \neg p}}{p, q \vdash \neg\neg p}}{p \wedge q \vdash \neg\neg p}}{\vdash p \wedge q \supset \neg\neg p}$$

13. Prove the validity of $\neg\neg p \supset (q \supset p)$ using the natural deduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{\neg\neg p, q \vdash \neg\neg p}}{\neg\neg p, q \vdash p}}{\neg\neg p \vdash (q \supset p)}}{\vdash \neg\neg p \supset (q \supset p)}$$

14. Prove the validity of $\neg(p \wedge (p \supset q) \wedge (p \supset \neg q))$ using the natural deduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{p, p \supset q, p \supset \neg q \vdash p} \quad \dfrac{\emptyset}{p, p \supset q, p \supset \neg q \vdash p \supset q}}{p, p \supset q, p \supset \neg q \vdash q}}{p, (p \supset q) \wedge (p \supset \neg q) \vdash q}}{p \wedge (p \supset q) \wedge (p \supset \neg q) \vdash q} \quad \dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{p, p \supset q, p \supset \neg q \vdash p} \quad \dfrac{\emptyset}{p, p \supset q, p \supset \neg q \vdash p \supset \neg q}}{p, p \supset q, p \supset \neg q \vdash \neg q}}{p, (p \supset q) \wedge (p \supset \neg q) \vdash \neg q}}{p \wedge (p \supset q) \wedge (p \supset \neg q) \vdash \neg q}}{\vdash \neg(p \wedge (p \supset q) \wedge (p \supset \neg q))}$$

15. Prove the validity of $\neg((q \supset \neg p) \wedge (p \wedge q))$ using the natural deduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{q \supset \neg p, p, q \vdash p}}{q \supset \neg p, p \wedge q \vdash p}}{(q \supset \neg p) \wedge (p \wedge q) \vdash p} \qquad \dfrac{\dfrac{\dfrac{q \supset \neg p, p, q, \vdash q \qquad q \supset \neg p, p, q, \vdash q \supset \neg p}{q \supset \neg p, p, q, \vdash \neg p}}{q \supset \neg p, p \wedge q, \vdash \neg p}}{(q \supset \neg p) \wedge (p \wedge q) \vdash \neg p}}{\vdash \neg((q \supset \neg p) \wedge (p \wedge q))}$$

16. Prove the validity of $\neg\neg p \land q \supset p$ using the natural deduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{\neg\neg p, q \vdash \neg\neg p}}{\neg\neg p, q \vdash p}}{\neg\neg p \land q \vdash p}}{\vdash \neg\neg p \land q \supset p}$$

17. Prove the validity of $p \supset (q \supset \neg\neg p)$ using the natural deduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{p, q, \neg p \vdash p} \qquad \dfrac{\emptyset}{p, q, \neg p \vdash \neg p}}{p, q \vdash \neg\neg p}}{p \vdash q \supset \neg\neg p}}{\vdash p \supset (q \supset \neg\neg p)}$$

18. Prove the validity of $\neg((\neg q \supset p) \land (\neg p \land \neg q))$ using the natural deduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{\neg q \supset p, \neg p, \neg q \vdash \neg q \supset p} \qquad \dfrac{\emptyset}{\neg q \supset p, \neg p, \neg q \vdash \neg q}}{\neg q \supset p, \neg p, \neg q \vdash p}}{\neg q \supset p, \neg p \land \neg q \vdash p}}{(\neg q \supset p) \land (\neg p \land \neg q) \vdash p} \qquad \dfrac{\dfrac{\dfrac{\emptyset}{\neg q \supset p, \neg p, \neg q \vdash \neg p}}{\neg q \supset p, \neg p \land \neg q \vdash \neg p}}{(\neg q \supset p) \land (\neg p \land \neg q) \vdash \neg p}}{\vdash \neg((\neg q \supset p) \land (\neg p \land \neg q))}$$