Exercises for the final exam

The practical part of the final exam contains exercises only from this document.
The theoretical part of the final exam descibed in the lecture slides.
Table of content:

- Conjunctive and disjunctive normal forms, and prenex form,
- first-order semantics: tautology, contradiction, equivalence,
- conclusion checking: formalization, logical consequence,
- deductions in Gentzen style sequent calculus.

Conjunctive and disjunctive normal forms

Put the formulas below into the following categories: literal, elementary conjunction, elementary disjunction, conjunctive normal form, disjunctive normal form.

1. $\neg\neg X$
2. $\neg X \vee \neg Y$
3. $X \supset \neg Y$
4. $\neg Y$
5. $(X \vee Y) \vee Z$
6. $(X \wedge Y) \wedge \neg Z$
7. $\neg(X \wedge Y) \wedge Z$
8. $(X \supset \neg Y) \supset Z$
9. X
10. $(X \vee Y) \wedge Z$
11. $(X \supset Y) \wedge Z$
12. $\neg(X \vee Y) \vee \neg(Y \vee Z)$
13. $(\neg X \vee Y) \wedge (\neg Y \vee Z)$

Conjunctive and disjunctive normal forms

Simplify the following expressions!

1. $\neg\neg\neg X$
2. $(X \vee \neg Y \vee \neg X) \supset Y$
3. $(X \wedge \neg Y \wedge \neg X) \supset Y$
4. $(X \vee Y) \wedge (Z \wedge Y)$
5. $(X \wedge Y) \vee (Z \vee Y)$
6. $\neg(\neg X \vee \neg Y)$
7. $\neg(\neg X \wedge \neg Y)$
8. $\neg X \supset \neg Y$
9. $\neg X \supset \neg Y$
10. $\neg(X \wedge Y) \supset (X \wedge Y)$

Conjunctive and disjunctive normal forms

Construct the disjunctive normal form of the followings!

1. $X \wedge (Y \vee Z)$
2. $(X \vee \neg Y \vee \neg Z) \wedge (U \vee \neg W)$
3. $(X \vee \neg Y) \wedge (\neg Z \vee \neg U \vee W) \wedge (S \vee \neg T)$

Construct the conjunctive normal form of the followings!

1. $X \vee (\neg Y \wedge \neg Z)$
2. $(X \wedge (\neg Y \vee \neg Z)) \vee (U \wedge \neg W)$
3. $(X \vee \neg Y) \vee ((\neg Z \vee \neg U \vee W) \wedge (S \vee \neg T))$

Conjunctive and disjunctive normal forms

Construct the disjunctive normal form and conjunctive normal form of the followings!

1. $X \supset (Y \supset (Z \supset U))$
2. $((X \supset Y) \supset Z) \supset U$
3. $\neg(X \vee Y) \supset \neg(X \wedge \neg Z)$
4. $\neg(X \vee \neg Y \vee \neg Z) \wedge (U \vee \neg W)$
5. $(X \wedge (\neg Y \vee \neg Z)) \supset \neg(U \supset \neg W)$
6. $(X \vee Z) \wedge (Z \supset \neg X)$
7. $\neg((X \vee Z) \supset (Z \supset \neg X))$
8. $(X \wedge \neg Y) \vee \neg((\neg Z \supset \neg U \vee W) \supset (S \vee \neg T))$

Prenex form

Construct the prenex form of the followings!

1. $\neg\exists x P(x) \wedge \neg\forall x Q(x)$
2. $\neg\exists x P(x) \supset \neg\forall x Q(x)$
3. $\exists x P(x) \supset \neg\forall x \neg Q(x)$
4. $\forall x(P(x) \wedge \neg\exists x Q(x)) \vee (\forall x Q(x) \supset R(x))$
5. $\neg\forall x(P(x) \wedge \neg\exists x Q(x)) \vee (\forall x Q(x) \supset R(x))$
6. $\forall x P(x) \supset (\forall x R(x) \supset \exists x Q(x))$
7. $\neg\forall x(\exists y T(x,y) \supset \forall x R(x) \vee P(x))$
8. $\forall x \exists y T(x,y) \supset Q(x) \vee R(y)$
9. $P(x) \supset (\forall y Q(y) \supset \neg\exists x T(x,y))$

First-order semantics

Prove – using the semantical definitions –, that the followings are tautologies.

1. $(\neg\forall x P(x) \supset \exists x \neg P(x)) \wedge (\exists x \neg P(x) \supset \neg\forall x P(x))$
2. $(\neg\exists x P(x) \supset \forall x \neg P(x)) \wedge (\forall x \neg P(x) \supset \neg\exists x P(x))$
3. $\exists x P(x) \vee \exists x Q(x) \supset \exists x (P(x) \vee Q(x))$
4. $\exists x (P(x) \vee Q(x)) \supset \exists x P(x) \vee \exists x Q(x)$
5. $\forall x P(x) \wedge \forall x Q(x) \supset \forall x (P(x) \wedge Q(x))$
6. $\forall x (P(x) \wedge Q(x)) \supset \forall x P(x) \wedge \forall x Q(x)$

Equivalences

Prove – using the semantical definitions –, the followings:

$$\neg\forall xA \sim \exists x\neg A \qquad \neg\exists xA \sim \forall x\neg A$$

$$\forall xA \vee B \sim \forall x(A \vee B) \qquad A \vee \forall xB \sim \forall x(A \vee B)$$

$$\forall xA \wedge B \sim \forall x(A \wedge B) \qquad A \wedge \forall xB \sim \forall x(A \wedge B)$$

$$\exists xA \vee B \sim \exists x(A \vee B) \qquad A \vee \exists xB \sim \exists x(A \vee B)$$

$$\exists xA \wedge B \sim \exists x(A \wedge B) \qquad A \wedge \exists xB \sim \exists x(A \wedge B)$$

$$\forall xA \supset B \sim \exists x(A \supset B) \qquad A \supset \forall xB \sim \forall x(A \supset B)$$

$$\exists xA \supset B \sim \forall x(A \supset B) \qquad A \supset \exists xB \sim \exists x(A \supset B)$$

We suppose, that x has no free occurrence.

Conclusion checking

Premises:

(P_1) *If Peter told us the truth or James is a liar, then Zoltan is a liar too.*

(P_2) *Peter didn't tell us the truth, only if Eve did.*

(P_3) *If James is not a liar, then Eve didn't tell us the truth.*

Are the following conclusions correct?

(a) *If Eve didn't tell us the truth then Zoltan is not a liar.*

(b) *Peter didn't tell us the truth or Zoltan is a liar.*

Conclusion checking

Premises:

(P_1) *It is not true, that if bad weather predicted but the sun is shining, then it is raining.*

(P_2) *If the wind is not blowing then it is not raining.*

(P_3) *The sun is shining, only if the wind is not blowing.*

Are the following conclusions correct?

(a) *If bad weather predicted the wind is not blowing and it is not raining.*

(b) *The wind is not blowing, only if it is not true, that bad weather predicted.*

Conclusion checking

Premises:

(P_1) *If the result was wrong, then the metering device failed, or the sample damaged.*

(P_2) *It is not true, that the metering device failed or the result wasn't wrong.*

(P_3) *The sample damaged, only if the calculation wasn't incorrect or the metering device failed.*

Are the following conclusions correct?

(a) *The sample wasn't damaged, only if the metering device failed.*

(b) *When the result was wrong then the calculation was incorrect.*

Gentzen style sequent calculus

Create an equivalent formula for each sequent above!

1. $\rightarrow (X \supset Y)$
2. $X \rightarrow Y$
3. $X \wedge Y \rightarrow Y \wedge Z$
4. $(X \supset Y) \rightarrow$
5. $X \vee Y \rightarrow Y \vee Z$
6. $X, Y \rightarrow Y, Z$

Gentzen style sequent calculus

Prove – using Gentzen style sequent calculus –, that the followings are tautologies.

1. Axiom schema of the predicate calculus

$$(X \supset (Y \supset Z)) \supset ((X \supset Y) \supset (X \supset Z))$$

2. De'Morgan's laws

   (a) $\neg X \wedge \neg Y \supset \neg(X \vee Y)$
   (b) $\neg(X \vee Y) \supset \neg X \wedge \neg Y$
   (c) $\neg X \vee \neg Y \supset \neg(X \wedge Y)$
   (d) $\neg(X \wedge Y) \supset \neg X \vee \neg Y$

3. Implication transformation

   (a) $(X \supset Y) \supset \neg X \vee Y$
   (b) $\neg X \vee Y \supset (X \supset Y)$
   (c) $\neg(X \supset Y) \supset X \wedge \neg Y$
   (d) $X \wedge \neg Y \supset \neg(X \supset Y)$

4. Pierce's law

$$((X \supset Y) \supset X) \supset X$$

5. Reduktio ad absurdum

$$(X \supset Y) \wedge (X \supset \neg Y) \supset \neg X$$

6. Prefix swapping rule

$$(X \supset (Y \supset Z)) \supset (Y \supset (X \supset Z))$$

7. Self distributivity law for implication

   (a) $(X \supset (Y \supset Z)) \supset ((X \supset Y) \supset (X \supset Z))$
   (b) $(X \supset Y) \supset (X \supset Z)) \supset (X \supset (Y \supset Z))$

8. Transitivity of the implication

$$((A \supset B) \wedge (B \supset C)) \supset (A \supset C)$$

9. Contraposition
   (a) $(X \supset Y) \supset (\neg Y \supset \neg X)$
   (b) $(\neg Y \supset \neg X) \supset (X \supset Y)$
10. Distributivity of conjunction
    (a) $(X \vee Y) \wedge Z \supset (X \wedge Z) \vee (Y \wedge Z)$
    (b) $(X \wedge Z) \vee (Y \wedge Z) \supset (X \vee Y) \wedge Z$
11. Distributivity od disjunction
    (a) $(X \wedge Y) \vee Z \supset (X \vee Z) \wedge (Y \vee Z)$
    (b) $(X \vee Z) \wedge (Y \vee Z) \supset (X \wedge Y) \vee Z$

Gentzen style sequent calculus

Prove – using Gentzen style sequent calculus –, that the followings are tautologies.

1. $(\neg\forall x P(x) \supset \exists x \neg P(x)) \wedge (\exists x \neg P(x) \supset \neg\forall x P(x))$
2. $(\neg\exists x P(x) \supset \forall x \neg P(x)) \wedge (\forall x \neg P(x) \supset \neg\exists x P(x))$
3. $\exists x P(x) \vee \exists x Q(x) \supset \exists x (P(x) \vee Q(x))$
4. $\exists x (P(x) \vee Q(x)) \supset \exists x P(x) \vee \exists x Q(x)$
5. $\forall x P(x) \wedge \forall x Q(x) \supset \forall x (P(x) \wedge Q(x))$
6. $\forall x (P(x) \wedge Q(x)) \supset \forall x P(x) \wedge \forall x Q(x)$