PART III.

Gentzen calculus

sequent calculus

Logic in computer science

Seminar: INGK401-K5; INHK401; INJK401-K4
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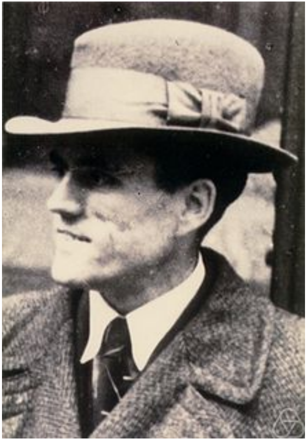

Gerhard Karl Erich Gentzen
November 24, 1909

Sequent

Let $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m$ $(n, m \geqslant 0)$ formulas in a propositional logic language. The formula in the form

$$\top \wedge A_1 \wedge A_2 \wedge \cdots \wedge A_n \supset B_1 \vee B_2 \vee \cdots \vee B_m \vee \bot$$

is called a **sequent**, and it is shortly written as

$$A_1, A_2, \ldots, A_n \to B_1, B_2, \ldots, B_m \qquad \text{or} \qquad \Gamma \to \Delta,$$

where $\Gamma \rightleftharpoons \{A_1, A_2, \ldots, A_n\}$ and $\Delta \rightleftharpoons \{B_1, B_2, \ldots, B_m\}$.

Axiom scheme of Gentzen calculus

$$A, \Gamma \rightarrow \Delta, A$$

Deduction rules of Gentzen calculus

$$(\to\supset) \quad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, (A \supset B)} \qquad (\supset\to) \quad \frac{\Gamma \to \Delta, A; \quad B, \Gamma \to \Delta}{(A \supset B), \Gamma \to \Delta}$$

$$(\to \wedge) \quad \frac{\Gamma \to \Delta, A; \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, (A \wedge B)} \qquad (\wedge \to) \quad \frac{A, B, \Gamma \to \Delta}{(A \wedge B), \Gamma \to \Delta}$$

$$(\to \vee) \quad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, (A \vee B)} \qquad (\vee \to) \quad \frac{A, \Gamma \to \Delta; \quad B, \Gamma \to \Delta}{(A \vee B), \Gamma \to \Delta}$$

$$(\to \neg) \quad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A} \qquad (\neg \to) \quad \frac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta}$$

Deduction rules of Gentzen calculus

$$(\to \exists) \quad \frac{\Gamma \to \Delta, \exists x A, A(x||t)}{\Gamma \to \Delta, \exists x A} \qquad (\exists \to) \quad \frac{[A^x_y], \Gamma \to \Delta}{\exists x A, \Gamma \to \Delta}$$

$$(\to \forall) \quad \frac{\Gamma \to \Delta, [A^x_y],}{\Gamma \to \Delta, \forall x A} \qquad (\forall \to) \quad \frac{\forall x A, A(x||t), \Gamma \to \Delta}{\forall x A, \Gamma \to \Delta}$$

Where y has no free occurrence in Γ or Δ, and t is an arbitrary term.

The $A(x||t)$ expression denotes, that we replace all of the free occurrences of the x variable in the A formula with a t term.

Deductions

A sequent is **provable in the Gentzen calculus**, if

- the sequent is an axiom,
- or exists a deduction rule, where the sequent in question is written under the line, and the sequents on the top of the line are provable.

In other words, if we want to prove a sequent in the Gentzen calculus which is not an axiom, it is enough to find a deduction rule, where the sequent in question is written below the line, and prove the sequents above the line.
(If everything above the line is true, so is everything under the line.)
In the predicate logic, each sequent above the line of a deduction rule have lower degree, than the sequent under the line.

Soundness and completeness

(a) The Gentzen sequent calculus is **soundness**, because when the sequent

$$A_1, A_2, \ldots, A_n \to B_1, B_2, \ldots, B_m$$

is provable in the Gentzen calculus, then the formula

$$\top \wedge A_1 \wedge A_2 \wedge \cdots \wedge A_n \supset B_1 \vee B2 \vee \cdots \vee B_m \vee \bot$$

is a tautology.

(b) The Gentzen sequent calculus is **completeness**, because if the formula A is a tautology, then the sequent $\to A$ is provable in the Gentzen calculus.