PART I.

PROPOSITIONAL LOGIC

ZERO ORDER LOGIC

Logic in computer science

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Statement

Statement:

- is a declarative sentence,
- with clear meaning.

Aristotle's basic rules:

- conflict-free: all of our statements have exactly one value (not able to be true and false in the same time),
- there is not third logical value (only true and false).

PART I. Propositional logic (zero order logic)

Alphabet of the propositional logic:

- propositional letters: X, Y, Z, . . . ,
- logical symbols:
  - $\neg$ negation sign,
  - $\wedge$ conjunction sign,
  - $\vee$ disjunction sign,
  - $\supset$ implication sign,
- delimeter symbols: normal brackets

Propositional formulas

The inductive definition of the propositional formula:

1. A propositional letter is a formula. (These are atomic formulas.)
2. If A is a formula, then $\neg A$ is a formula too.
3. If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$ are formulas.
4. A string of symbols is a formula if and only if it can be obtained by starting with atomic formulas and repeatedly applying the inductive steps (2 and 3), and it must terminates after a finite number of steps.

We denote the set of propositional (or zero order) formulas as $\mathbf{L}_0$.

Syntactical properties

Immediate subformulas:

1. An atomic formula has no immediate subformulas.
2. The only immediate subformula of $\neg A$ is A.
3. The immediate subformulas of $(A \circ B)$, where $\circ \in \{\wedge, \vee, \supset\}$, are A and B.

Let A be a formula. The **set of subformulas** of A is the

- smallest set
- that contains A,
- and contains, with each member, the immediate subformulas of that member.

Syntactical properties

The **logical degree** of a formula: $d : \mathbf{L}_0 \to N_0$

1. $d(A) = 0$, if A is an atomic formula,
2. $d(\neg A) = d(A) + 1$, where A is an arbitrary formula,
3. $d(A \circ B) = d(A) + d(B) + 1$, where A and B are arbitrary formulas, and $\circ \in \{\wedge, \vee, \supset\}$.

Syntactical properties

The **scope of a logical connective (logical symbol)** in a formula A, is a subformula of A,

- which contains the logical symbol, and
- which has the smallest logical degree.

The **main logical connective** of a non atomic formula, is the logical connective in the formula, which scope is the whole formula itself.

Abbrevations

The **precedence** of the logical connectives:

$$\{\neg\},\ \{\wedge, \vee\},\ \{\supset\}$$

$$\text{higher precedence} \ \longrightarrow \ \text{lower precedence}$$

1. The outmost parentheses can be omitted.
2. The parentheses can be omitted, if the logical symbol in the inner expression have higher precedence, than the outer logical symbol.

Semantics

Logical operations

a	b	$\overset{\bullet}{\neg}\, a$	$a \overset{\bullet}{\wedge} b$	$a \overset{\bullet}{\vee} b$	$a \overset{\bullet}{\supset} b$
true	*true*	*false*	*true*	*true*	*true*
true	*false*		*false*	*true*	*false*
false	*true*	*true*	*false*	*true*	*true*
false	*false*		*false*	*false*	*true*

An **interpretation** of a propositional (zero-order) logic language is a function that assigns to each propositional letter a unique truth value.

$$\mathcal{I} : Pr \to \{true, false\} \quad \text{(where } Pr \text{ denotes the set of propositional letters)}$$

Truth valuation of formulas

We denote the **truth valuation** of propositional formula A by a given interpretation $\mathcal{I}$ as $|A|^{\mathcal{I}}$. It is defined as a recursive function:

- $|A|^{\mathcal{I}} = \mathcal{I}(A)$ if A is an atomic formula,
- $|\neg A|^{\mathcal{I}} = \overset{\bullet}{\neg}\ |A|^{\mathcal{I}}$ where A is an arbitrary formula,
- $|(A \wedge B)|^{\mathcal{I}} = |A|^{\mathcal{I}} \overset{\bullet}{\wedge} |B|^{\mathcal{I}}$ where A and B are arbitrary formulas,
- $|(A \vee B)|^{\mathcal{I}} = |A|^{\mathcal{I}} \overset{\bullet}{\vee} |B|^{\mathcal{I}}$ where A and B are arbitrary formulas,
- $|(A \supset B)|^{\mathcal{I}} = |A|^{\mathcal{I}} \overset{\bullet}{\supset} |B|^{\mathcal{I}}$ where A and B are arbitrary formulas.

Semantical properties

An $\mathcal{I}$ interpretaion is called the **model** of an A formula, if $|A|^{\mathcal{I}} = true$. We shortly denote this by $\mathcal{I} \models A$. Otherwise we will use $\mathcal{I} \not\models A$ when $\mathcal{I}$ is not a model of A.

An A formula is called **tautology**, when $\mathcal{I} \models A$ satisfied for all possible $\mathcal{I}$ interpretation. (In this case, the A formula is true for all possible interpretations) We shortly denote this by $\models A$.

An A formula is called **contradiction**, when A has no model. (In this case, the A formula is false for all possible interpretations.) We shortly denote this by $\Dashv A$.

Semantical properties

Let A and B formulas. We say that A and B are **logically equivalent**, when A and B have the same truth value for every possible interpretation. We shortly denote this by $A \sim_0 B$.

Let $A_1, A_2, \ldots, A_n$ (where $n \geqslant 1$) and B formulas. We say that B is a **propositional consequence** of $A_1, A_2, \ldots, A_n$ or$B$ follows from $A_1, A_2, \ldots, A_n$, denoted $A_1, A_2, \ldots, A_n \vDash B$, if B is true in every interpretation, where every formula of $A_1, A_2, \ldots, A_n$ is true.

A conclusion of some premises said to be correct, if and only if the conclusion is the logical consequence of the premises.

Normal forms

A **literal** is an atomic formula, or a negation of an atomic formula.
An **elementary conjunction** is

- a literal, or
- a conjunction of a literal and an elementary conjunction.

$(L_1 \wedge L_2 \wedge \ldots \wedge L_n \quad \text{where } n \geqslant 1 \text{ and } L_1, L_2, \ldots, L_n \text{ are literals})$

An **elementary disjunction** is

- a literal, or
- a disjunction of a literal and an elementary disjunction.

$(L_1 \vee L_2 \vee \ldots \vee L_n \quad \text{where } n \geqslant 1 \text{ and } L_1, L_2, \ldots, L_n \text{ are literals})$

Normal forms

A **conjunctive normal form** is

- an elementary disjunction, or
- a conjunction of an elementary disjunction and a conjunctive normal form.

A **disjunctive normal form** is

- an elementary conjunction, or
- a disjunction of an elementary conjunction and a disjunctive normal form.

Algorithm to create normal forms

1. Until the formula contains implication, remove it using one of the following equivalences:

$$A \supset B \sim_0 \neg A \vee B \quad \text{and} \quad \neg(A \supset B) \sim_0 A \wedge \neg B$$

2. Until the formula contains nagation but not inside the literals, remove the negations using DeMorgan's laws and omitting the double negations:

$$\neg(A \vee B) \sim_0 \neg A \wedge \neg B \quad \text{and} \quad \neg(A \wedge B) \sim_0 \neg A \vee \neg B \quad \text{and} \quad \neg\neg A \sim_0 A$$

3. If necessary, change the order of the conjunctions and disjunctions using the distributivity laws:

$$(A \vee B) \wedge C \sim_0 (A \wedge C) \vee (B \wedge C)$$

$$(A \wedge B) \vee C \sim_0 (A \vee C) \wedge (B \vee C)$$