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Logic in Computer Science

Basic concepts of set theory

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Sets

The terms *set* and *members* are given; they are considered basic concepts without any further official definition. However, the concept of a set is identified as a collection usually.

A specific set can be defined in many different ways, e.g.:

1. By enumerating its members,
   - {spring, summer, autumn, winter} is the set of seasons,
   - $\{0, 1, 2, 3, 4, 5, 6, 7\}$ or shortly $\{0, 1, \ldots, 7\}$ is the set of octal digits, but
   - the empty set is denoted by: $\varnothing$.
2. We can define a set using some common property of its members, constructing a condition that is satisfied by the members (and by the members only). E.g.:
   - $\{\ a \mid a$ is a nonnegative integer and $a < 8\ \}$
3. We can define a set inductively.

Sets

The fact that an object $\mathbf{a}$ is membered in a set $\mathbf{A}$ is written as

$$\mathbf{a} \in \mathbf{A},$$

where the symbol $\in$ represents the *membership relation*.
To indicate that an object $\mathbf{a}$ is not membered in a set $\mathbf{A}$, we can use the notation:

$$\mathbf{a} \notin \mathbf{A}.$$

For example, let we denote the set of natural numbers by $\mathbb{N}$, which can be written in the form:

$$\mathbb{N} \rightleftharpoons \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}$$

- $\mathbf{3}$ is a natural number, which can be expressed as $\mathbf{3} \in \mathbb{N}$,
- $\mathbf{0}$ is not a natural number, which can be written as $\mathbf{0} \notin \mathbb{N}$.

Sets

An inductive definition of a set consists of declaration steps and inductive steps. By using this technique, the inductive definition of the set of natural numbers $\mathbb{N}$ can be the following:

The set of natural numbers $\mathbb{N}$ *is defined inductively as follows:*

- $\mathbf{1} \in \mathbb{N}$, *and*
- *if* $\mathbf{a} \in \mathbb{N}$ *then* $\mathbf{a} + \mathbf{1} \in \mathbb{N}$.

Sets

Let **A** and **B** be two arbitrary sets.

- The set **A** is the *subset* of the set **B**, if all members of **A** are membered in **B**. We denote is as: $\mathbf{A} \subseteq \mathbf{B}$.
- If $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$ then **A** and **B** are equal: $\mathbf{A} = \mathbf{B}$.
- **A** and **B** are *disjoint* if they have no element in common.
- The *power set* of **A**, denoted by $\wp(\mathbf{A})$ or $2^{\mathbf{A}}$, is a set containing all subset of **A**. E.g.:

$$\wp(\{\mathbf{0},\mathbf{1}\}) = 2^{\{0,1\}} = \{\varnothing, \{\mathbf{0}\}, \{\mathbf{1}\}, \{\mathbf{0},\mathbf{1}\}\}$$

Basic operations

Let $\mathbf{A}$ and $\mathbf{B}$ be two arbitrary sets.

- *union*: $\mathbf{A} \cup \mathbf{B} \rightleftharpoons \{\, \mathbf{a} \mid \mathbf{a} \in \mathbf{A} \text{ or } \mathbf{a} \in \mathbf{B} \,\}$,
- *intersection*: $\mathbf{A} \cap \mathbf{B} \rightleftharpoons \{\, \mathbf{a} \mid \mathbf{a} \in \mathbf{A} \text{ and } \mathbf{a} \in \mathbf{B} \,\}$,
- *difference*: $\mathbf{A} \setminus \mathbf{B} \rightleftharpoons \{\, \mathbf{a} \mid \mathbf{a} \in \mathbf{A} \text{ and } \mathbf{a} \notin \mathbf{B} \,\}$.
- *symmetrical difference*: $\mathbf{A} \triangle \mathbf{B} \rightleftharpoons (\mathbf{A} \setminus \mathbf{B}) \cup (\mathbf{B} \setminus \mathbf{A})$.

Let U be the universal set: the set of all objects under our consideration. (A universal set is a set that contains all the elements or objects of other sets.)

- *complement*: $\overline{\mathbf{A}} \rightleftharpoons \mathbf{U} \setminus \mathbf{A}$

Cartesian product

Let $\mathbf{A}$, $\mathbf{B}$ be two arbitrary sets. The *Cartesian product* of the sets $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \times \mathbf{B}$ and it is defined as a set of all ordered pairs $\langle \mathbf{a}, \mathbf{b} \rangle$ where $\mathbf{a} \in \mathbf{A}$ and $\mathbf{b} \in \mathbf{B}$. Formally:

$$\mathbf{A} \times \mathbf{B} \;\rightleftharpoons\; \{\, \langle \mathbf{a}, \mathbf{b} \rangle \mid \mathbf{a} \in \mathbf{A} \text{ és } \mathbf{b} \in \mathbf{B} \,\}.$$

Note that $\langle a, b \rangle = \langle c, d \rangle$ if and only if $a = c$ and $b = d$.

The *Cartesian product* of the sequence of sets $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ ($\mathbf{n} > \mathbf{0}$) can be defined by using n-tuples:

$$\mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_n \;\rightleftharpoons\; \{\, \langle \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \rangle \mid \mathbf{a}_i \in \mathbf{A}_i \text{ for all } \mathbf{i} \in \{\mathbf{1}, \mathbf{2}, \ldots \mathbf{n}\} \,\}.$$

The $\mathbf{A}^n$ is an abbreviation for $\underbrace{\mathbf{A} \times \mathbf{A} \times \cdots \times \mathbf{A}}_{n \text{ times}}$.

Binary relations

Let $\mathbf{A}$, $\mathbf{B}$ be two arbitrary sets. $\mathbf{R}$ is a *binary relation* between the sets $\mathbf{A}$ and $\mathbf{B}$ if

$$\mathbf{R} \subseteq \mathbf{A} \times \mathbf{B}.$$

If $\mathbf{A} = \mathbf{B}$ and so $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$, we can simply say that $\mathbf{R}$ is a binary relation on $\mathbf{A}$.

Let $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{B}$ a binary relation.

- The *domain* of $\mathbf{R}$:

$$\mathrm{dom}(\mathbf{R}) = \{\, \mathbf{a} \mid \mathbf{a} \in \mathbf{A} \text{ and there exists} \mathbf{b} \in \mathbf{B} \text{ such that} (\mathbf{a}, \mathbf{b}) \in \mathbf{R} \,\},$$

- The *range* of $\mathbf{R}$:

$$\mathrm{rng}(\mathbf{R}) = \{\, \mathbf{b} \mid \mathbf{b} \in \mathbf{B} \text{ and there exists } \mathbf{a} \in \mathbf{A} \text{ such that } (\mathbf{a}, \mathbf{b}) \in \mathbf{R} \,\}.$$

Binary relations

Let $\mathbf{R}$ be a binary relation on $\mathbf{A}$, and so $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$.
$\mathbf{R}$ is called

- *reflexive*, if $a\,R\,a$ for all $a \in A$,
- *antireflexive*, if there is no such $a \in A$ where $a\,R\,a$,
- *symmetric*, if it is true that when $a\,R\,b$ then $b\,R\,a$,
- *antisymmetric*, if no $a\,R\,b$ and $b\,R\,a$ unless $a = b$,
- *asymmetric*, if there is no $a \in A$ and $b \in A$ such that $a\,R\,b$ and $b\,R\,a$,
- *transitive*, if it is true that when $a\,R\,b$ and $b\,R\,c$ then $a\,R\,c$.

Function

Let $\mathbf{R}$ be a binary relation so that $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{B}$.
$\mathbf{R}$ is called

- *injective* (or *left unique*), if it is true that

$$\text{when } \mathbf{a}\,\mathbf{R}\,\mathbf{c} \text{ and } \mathbf{b}\,\mathbf{R}\,\mathbf{c} \text{ then } \mathbf{a} = \mathbf{b}$$

- *surjective* (*right total*), if $\mathrm{rng}(\mathbf{R}) = \mathbf{B}$,
- *left total*, if $\mathrm{dom}(\mathbf{R}) = \mathbf{A}$,
- *bijective*, if it is *injective* and *surjective*.
- *functional* (or *right unique*), if it is true that

$$\text{when } \mathbf{a}\,\mathbf{R}\,\mathbf{b} \text{ and } \mathbf{a}\,\mathbf{R}\,\mathbf{c} \text{ then } \mathbf{b} = \mathbf{c}$$

- *function*, if it is *functional* and *left total*.