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# On metrical homogeneous connections of a Finsler point space

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**Abstract.** Finsler point spaces mean that vectors are defined at points of the base manifold and not at line elements. We show a process how to obtain in such a Finsler space a metrical homogeneous connection from an arbitrary linear (or nonlinear) connection. The autoparallel curves of the given and the constructed connections coincide as point sets. It is shown that the curvature tensors are coupled by the angular operator.

## Introduction

In the history of Finsler geometry the search of an appropriate connection — in other aspect, a parallelism structure — was the first challenge. Several solution were obtained (see [2], [3]). Namely, in Finsler spaces it is not possible, in general, to introduce a metrical (i.e. length preserving) linear connection. There are, however, two ways for solving this defect. One of these is lifting all notions and investigations to the tangent manifold, traditionally saying, to the line element bundle (see [2], [3]). Another outway proposes to find metrical, but not linear, only homogeneous connections on the manifold itself. It was found by W. Barthel [1] implicitely already by L. Berwald and E. Cartan, however, such a connection is not unique at all. This second line of approach has not been completely exploited.

Here we show a general construction for obtaining a metrical homogeneous connection from an arbitrary linear connection given in a Finsler space. The basic idea, due to L. Tamássy [5] is the following. Take a

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unit vector, translate it along a curve according to the given linear connection and normalize the translated vectors. Then we obtain a metrical, i.e. length preserving parallel translation along curves. This connection is, however, not linear, only homogeneous in general. We remark that this program can be realized even if the starting connection is nonlinear, moreover the Finsler space can be replaced with a Finsler structure in a vector bundle.

Now we describe this construction in a Finsler vector bundle with the aid of horizontal maps and give the relationships between the corresponding covariant derivations, parallelism structures, geodesics and curvatures.

## §1. Construction of a metrical homogeneous connection

We work in the category of smooth vector bundles. Let $\xi=(E,\pi,B,F)$ be a vector bundle with the total space $E$, the base space $B$, the projection $\pi$ and the typical fibre $F$. $\dot{\xi}$ denotes the splitted bundle with the total space $\dot{E}=E\setminus\{0\}$. A continuous function $L:E\to\mathbb{R}^+$ which is $C^1$ on $\dot{E}$ is called a *Finsler fundamental function* if it is positively homogeneous, i.e. $L(tz)=tL(z)$ for all $t\in\mathbb{R}^+$ and $z\in E$, and, considering a local vector bundle coordinate system $(x^i,y^\alpha,i=1,\dots,\dim B=n,\ \alpha=1,\dots,\dim F=m)$, the fundamental metric tensor defined by

$$g_{\alpha\beta}(x,y)=\frac{\partial^2(L^2/2)}{\partial y^\alpha\partial y^\beta}(x,y)$$

gives a positive definite quadratic form for all $(x,y)=z\in\dot{E}$.

The last assumption implies that the indicatrix

$$I_x=\{z\in E_x=\pi^{-1}(x)\quad|\quad L(z)=1\}$$

at each point $x\in B$ is convex (see [3]). The indicatrix bundle $\mathcal{I}\xi=(IE,\overline{\pi},B,\mathbb{S}^{m-1})$ of a Finsler vector bundle $(\xi,L)$ is formed with the indicatrices $I_x$ as fibres. In the following construction there will be a crucial role of the *normalizing operator* $\eta:\dot{\xi}\to\mathcal{I}\xi$ defined by

$$\eta(z)=\frac{z}{L(z)}.$$

Consider now a connection in the vector bundle $\xi$ given by a horizontal map $\mathcal{H}:\pi^*(\tau_B)\to\tau_E$ (i.e. a vector bundle map satisfying $d\pi(\mathcal{H}(z,\underline{v}))=\underline{v}$ for all $z\in E$, $\underline{v}\in TB$ with $\pi(z)=\pi_B(\underline{v})$, where $\pi^*(\tau_B)$ denotes the pullback bundle $(E\times_B TB,E,pr_1,\mathbb{R}^n)$). A curve $\psi:I\to E$ is called horizontal iff the tangents $\dot{\psi}$ are horizontal vectors. Then we say also that $\psi$ is parallel along the base curve $\pi \circ \psi$. Suppose that $\mathcal{H}$ satisfies the homogeneity condition

$$\mathcal{H}(\mu_t(z), \underline{v}) = d\mu_t(\mathcal{H}(z, \underline{v}))$$

for all $t \in \mathbb{R}^+$, where $\mu_t : E \to E$, $z \mapsto tz$ is the multiplication by $t$ in the fibres. If $\mathcal{H}$ is differentiable on the whole $E$ then we obtain the notion of the linear connection. (Homogeneity and differentiability at $z = 0$ yield linearity.) In this case the parallel translation along curves determines linear maps and the local connection coefficients are linear: $\Gamma_i^\alpha(x, y) = \Gamma_{i\beta}^\alpha(x) y^\beta$. If $\mathcal{H}$ is not supposed to be differentiable on the zero section of $\xi$ then we have a homogeneous connection. Then the parallel translation means only homogeneous maps and $\Gamma_i^\alpha(x, ty) = t\Gamma_i^\alpha(x, y)$ holds for all $t \in \mathbb{R}^+$.

Our construction has two steps. Starting with a linear connection in $\xi$, first a connection in the indicatrix bundle $\mathcal{I}\xi$ is defined, then it is extended to $\xi$ in order to obtain a metrical homogeneous connection in the whole $\xi$.

**Proposition 1.** *If $\mathcal{H}$ is a horizontal map of a linear connection in the Finsler vector bundle $(\xi, L)$ then the map $\mathcal{H}^{(i)} : \overline{\pi}^*(\tau_B) \to \tau_{IE}$ defined by*

$$\mathcal{H}^{(i)} = d\eta \circ \mathcal{H}_{|IE \times_B TB} \tag{1}$$

*is a horizontal map in the indicatrix bundle $\mathcal{I}\xi$.*

Proof. Clearly $\overline{\pi} \circ \eta = \pi$. Using this relation we obtain

$$\begin{aligned} d\overline{\pi} \circ \mathcal{H}^{(i)} = d\overline{\pi} \circ d\eta \circ \mathcal{H}|_{IE \times_B TB} &= d(\overline{\pi} \circ \eta) \circ \mathcal{H}|_{IE \times_B TB} \\ &= d\pi \circ \mathcal{H}|_{IE \times_B TB} = id_{TB}. \end{aligned}$$

□

*Remark.* Having the homogeneity of the horizontal map $\mathcal{H}$ it is not necessary to restrict $\mathcal{H}$ in (1) onto $IE$. In fact, if for $z_1, z_2 \in E$ holds $\eta(z_1) = \eta(z_2)$ then $d\eta(\mathcal{H}(z_1, \underline{v})) = d\eta(\mathcal{H}(z_2, \underline{v}))$. Namely, in this case $z_1 = tz_2 = \mu_t(z_2)$ for some $t \in \mathbb{R}^+$, supposed $z_2 \neq 0$, therefore

$$\begin{aligned} d\eta(\mathcal{H}(z_1, \underline{v})) = d\eta(\mathcal{H}(\mu_t(z_2), \underline{v})) &= d\eta(d\mu_t(\mathcal{H}(z_2, \underline{v}))) \\ &= d(\eta \circ \mu_t)(\mathcal{H}(z_2, \underline{v})) = d\eta(\mathcal{H}(z_2, \underline{v})). \end{aligned}$$

In the second step we extend the connection of the indicatrix bundle $\mathcal{I}\xi$ homogeneously to the whole vector bundle $\xi$. In this way we obtain a metrical homogeneous connection in the Finsler vector bundle.

**Proposition 2.** *The map $\mathcal{H}^{(m)} : \dot{\pi}^*(\tau_B) \to \tau_{\dot{E}}$ constructed from $\mathcal{H}$ and defined by*

$$\mathcal{H}^{(m)}(z, \underline{v}) = d\mu_{L(z)}(\mathcal{H}^{(i)}(\eta(z), \underline{v})) \quad \forall (z, \underline{v}) \in E \times_B TB \tag{2}$$

*is a horizontal map for the vector bundle $\xi$ which satisfies the homogeneity property and means a metrical connection in the Finsler vector bundle $(\xi, L)$.*

Proof. a) Using $\pi \circ \mu_{L(z)} = \pi$ we show that $\mathcal{H}^{(m)}$ is a horizontal map in $\xi$, for

$$\begin{aligned} d\pi(d\mu_{L(z)}(\mathcal{H}^{(i)}(\eta(z), \underline{v}))) &= d(\pi \circ \mu_{L(z)})(\mathcal{H}^{(i)}(\eta(z), \underline{v})) \\ &= d\pi(\mathcal{H}^{(i)}(\eta(z), \underline{v})) = \underline{v}. \end{aligned}$$

b) To check homogeneneity we observe that $\mu_t \circ \mu_{L(z)} = \mu_{tL(z)} = \mu_{L(tz)}$ and $\eta(z) = \eta(tz)$ for $t \in \mathbb{R}^+$. Therefore

$$\begin{aligned} d\mu_t(\mathcal{H}^{(m)}(z, \underline{v})) &= d\mu_t(d\mu_{L(z)}(\mathcal{H}^{(i)}(\eta(z), \underline{v}))) \\ &= d\mu_{L(tz)}(\mathcal{H}^{(i)}(\eta(tz), \underline{v})) = \mathcal{H}^{(m)}(tz, \underline{v}) = \mathcal{H}^{(m)}(\mu_t(z), \underline{v}). \end{aligned}$$

c) $\mathcal{H}^{(m)}$ is metrical by definition iff $dL \circ \mathcal{H}^{(m)} = 0$. To verify this we apply that $L \circ \mu_{L(z)} = L(z)L$ holds for a fix $z \in E$ and $L \circ \eta = 1$, which imply $dL \circ d\mu_{L(z)} = L(z)dL$, $dL \circ d\eta = 0$. Therefore using these and the definition of $\mathcal{H}^{(m)}$ and $\mathcal{H}^{(i)}$, we obtain

$$\begin{aligned} dL(\mathcal{H}^{(m)}(z, \underline{v})) &= dL(d\mu_{L(z)}(\mathcal{H}^{(i)}(\eta(z), \underline{v}))) \\ = dL(d\mu_{L(z)}(d\eta(\mathcal{H}(z, \underline{v})))) &= L(z)dL(d\eta(\mathcal{H}(z, \underline{v})) = 0. \end{aligned}$$ □

The following corollary formulates that this process produces essentially the descriptive construction of L. Tamássy [5].

**Corollary.** *Let $\varphi : I \to B$ be a curve in the base space $B$, $\psi : I \to E$ a horizontal curve along $\varphi$ with respect to $\mathcal{H}$ and $z_0 := \psi(0) \in \pi^{-1}(\varphi(0))$. Thus the curve $\psi^* : I \to E$ defined by $\psi^*(t) = \mu_{L(z_0)}(\eta(\psi(t))\ \forall t \in I$ is a horizontal curve with respect to the constructed homogeneous metrical connection $\mathcal{H}^{(m)}$.*

The corollary follows immediately from the definition: $\psi$ is a horizontal curve iff $\dot{\psi} = \mathcal{H}(\psi, d\pi \circ \dot{\psi})$ where $\dot{\psi}$ denotes the tangents of $\psi$.

Now we give the local description of our construction. Considering a local coordinate system $(x^i, y^\alpha)$ for $\xi$, the map $\eta$ is given locally by

$$\eta : (x^i, y^\alpha) \mapsto \left( x^i, \frac{y^\alpha}{L(x,y)} \right).$$

A short calculation shows that the action of $d\eta : TE \to TE$ is as follows:

$$d\eta \left( \frac{\partial}{\partial y^\alpha} \right) = (\delta^\beta_\alpha L(x,y) - \frac{\partial L(x,y)}{\partial y^\alpha} y^\beta) \frac{1}{L^2(x,y)} \frac{\partial}{\partial y^\beta},$$
$$d\eta \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} - \frac{\partial L(x,y)}{\partial x^i} \frac{y^\alpha}{L^2(x,y)} \frac{\partial}{\partial y^\alpha}.$$

If $\mathcal{H}$ is characterized by the local connection parameters $\Gamma^\alpha_i(x,y)$ which satisfies

$$\mathcal{H} \left( (x,y), \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} - \Gamma^\alpha_i(x,y) \frac{\partial}{\partial y^\alpha}$$

then using the above action of $d\eta$ and (1),(2) we can obtain that the new metrical homogeneous connection $\mathcal{H}^{(m)}$ has the following local connection parameters:

$$\overset{(m)}{\Gamma}{}^\alpha_i(x,y) = \Gamma^\alpha_i(x,y) - \frac{y^\alpha}{L(x,y)} \left( \frac{\partial L(x,y)}{\partial y^\beta} \Gamma^\beta_i(x,y) - \frac{\partial L(x,y)}{\partial x^i} \right).$$

Moreover, in the case of a linear connection $\Gamma^\alpha_i(x,y) = \Gamma^\alpha_{i\gamma}(x) y^\gamma$ we have

$$\overset{(m)}{\Gamma}{}^\alpha_i(x,y) = \Gamma^\alpha_{i\gamma}(x) y^\gamma - \frac{y^\alpha}{L(x,y)} \left( \frac{\partial L(x,y)}{\partial y^\beta} \Gamma^\beta_{i\gamma}(x) y^\gamma - \frac{\partial L(x,y)}{\partial x^i} \right). \tag{3}$$

## §2. The relationships between the original linear and the constructed metrical connections

First we derive a relationship between the corresponding covariant derivations. The covariant derivation $\nabla : \mathcal{X}(B) \times \operatorname{Sec} \xi \to \operatorname{Sec} \xi$ of $\mathcal{H}$ is given by

$$\nabla_X \sigma = \alpha(d\sigma(X) - \mathcal{H}(\sigma, X))$$

where $\alpha : VE \to E$ is given locally as $\alpha(y^\beta \frac{\partial}{\partial y^\beta}) = (y^\beta)$.

**Theorem 1.** *Denoting the covariant derivations $\nabla$ and $\nabla^{(m)}$ belonging to $\mathcal{H}$ and $\mathcal{H}^{(m)}$ resp., we have*

$$\nabla^{(m)}_{\underline{v}} \sigma - \nabla_{\underline{v}} \sigma = \eta(\sigma) dL(\mathcal{H}(\sigma, \underline{v})) \quad \forall \underline{v} \in TB, \ \sigma \in \operatorname{Sec} \xi.$$

As we see, this theorem expresses that the deviation of the two covariant derivations just measures the 'non-metric' property of $\mathcal{H}$.

Proof. We check first that for any fixed $z \in \dot{E}$

$$(id - d\mu_{L(z)} \circ d\eta)(\underline{u}) = \frac{1}{L(z)} dL(\underline{u}) C(z) \quad \forall \underline{u} \in T_z E$$

where $C : E \to VE$ denotes the Liouville vector field given locally as $C(z) = y^\alpha(z)(\frac{\partial}{\partial y^\alpha})_z$. In fact, applying the left side on the coordinate functions

$$\begin{aligned}
(id - d\mu_{L(z)} \circ d\eta)(\underline{u})(x^i) &= \underline{u}(x^i) - \underline{u}(x^i \circ \mu_{L(z)} \circ \eta) = 0. \\
(id - d\mu_{L(z)} \circ d\eta)(\underline{u})(y^\alpha) &= \underline{u}(y^\alpha) - \underline{u}(y^\alpha \circ \mu_{L(z)} \circ \eta) \\
&= \underline{u}(y^\alpha) - \underline{u}\left(\frac{y^\alpha}{L} L(z)\right) \\
&= \underline{u}(y^\alpha) - L(z) \frac{\underline{u}(y^\alpha) L(z) - y^\alpha \underline{u}(L)}{L^2(z)} \\
&= \frac{y^\alpha}{L(z)} \underline{u}(L) = \frac{1}{L(z)} dL(\underline{u}) y^\alpha.
\end{aligned}$$

We calculate now the deviation of $\nabla^{(m)}$ and $\nabla$ with respect to (1) and (2)

$$\begin{aligned}
\nabla^{(m)}_{\underline{v}} \sigma - \nabla_{\underline{v}} \sigma &= \alpha\left(d\sigma(\underline{v}) - \mathcal{H}^{(m)}(\sigma, \underline{v})\right) - \alpha\left(d\sigma(\underline{v}) - \mathcal{H}(\sigma, \underline{v})\right) \\
&= \alpha\left(\mathcal{H}(\sigma, \underline{v}) - \mathcal{H}^{(m)}(\sigma, \underline{v})\right) \\
&= \alpha\left(\mathcal{H}(\sigma, \underline{v}) - d\mu_{L(\sigma)}(d\eta(\mathcal{H}(\sigma, \underline{v})))\right) \\
&= \alpha\left((id - d\mu_{L(\sigma)} \circ d\eta)(\mathcal{H}(\sigma, \underline{v}))\right).
\end{aligned}$$

Using the relation above and $\alpha \circ C = id$ we have further

$$\begin{aligned}
\nabla^{(m)}_{\underline{v}} \sigma - \nabla_{\underline{v}} \sigma &= \alpha\left(\frac{1}{L(\sigma)} dL(\mathcal{H}(\sigma, \underline{v})) C(\sigma)\right) \\
&= \frac{\sigma}{L(\sigma)} dL(\mathcal{H}(\sigma, \underline{v})) = \eta(\sigma) dL(\mathcal{H}(\sigma, \underline{v})). \qquad \square
\end{aligned}$$

The curvature tensor $R$ of a linear connection $\nabla$ is defined by

$$R(X, Y)\sigma := \nabla_X(\nabla_Y \sigma) - \nabla_Y(\nabla_X \sigma) - \nabla_{[X,Y]}\sigma$$

for all $X, Y \in \mathcal{X}(B)$ and $\sigma \in \operatorname{Sec}\xi$, or locally

$$R^\beta_{\gamma j i} = \frac{\partial \Gamma^\beta_{j\gamma}}{\partial x^i} - \frac{\partial \Gamma^\beta_{i\gamma}}{\partial x^j} + \Gamma^\beta_{is}\Gamma^s_{j\gamma} - \Gamma^\beta_{js}\Gamma^s_{i\gamma}$$

On the other hand, the curvature of a homogeneous connection $\nabla^{(h)}$ is defined by

$$\overset{(h)}{R}(U,V) = -v[h(U),h(V)],$$

for all $U, V \in \mathcal{X}(E)$, where $h = \mathcal{H} \circ d\pi$, $v = id - h$. Locally

$$\overset{(h)}{R}{}^{\alpha}_{ij} = \frac{\partial \Gamma^{\alpha}_{j}}{\partial x^i} - \frac{\partial \Gamma^{\alpha}_{i}}{\partial x^j} + \Gamma^{\beta}_{j}\frac{\partial \Gamma^{\alpha}_{i}}{\partial y^{\beta}} - \Gamma^{\beta}_{i}\frac{\partial \Gamma^{\alpha}_{j}}{\partial y^{\beta}}. \tag{4}$$

It is known [4] that the two types of curvatures are related as follows:

$$\alpha \circ \overset{(h)}{R}(X^h, Y^h) \circ \sigma = R(X,Y)(\sigma).$$

As far as our construction of a metrical homogeneous connection we can prove the following theorem.

**Theorem 2.** *Let us denote by* $R^{\beta}_{kji}$ *and* $\overset{(m)}{R}{}^{\beta}_{ij}$ *the corresponding curvatures of a linear connection* $\mathcal{H}$ *and the constructed metrical homogeneous connection* $\mathcal{H}^{(m)}$. *Then they are related by*

$$\overset{(m)}{R}{}^{\beta}_{ij} = y^{\delta} R^{\alpha}_{\delta ji}\left(\delta^{\beta}_{\alpha} - \frac{y^{\beta}}{L}\frac{\partial L}{\partial y^{\alpha}}\right).$$

Proof. We calculate all term in (4) using the explicit form (3) of the connection parameters $\overset{(m)}{\Gamma}{}^{\alpha}_{i}$. First we have

$$\begin{aligned}\frac{\partial \overset{(m)}{\Gamma}{}^{\beta}_{i}}{\partial x^j} = \frac{\partial \Gamma^{\beta}_{i\gamma}}{\partial x^j}y^{\gamma} - \frac{y^{\beta}}{L^2}\left(\frac{\partial^2 L}{\partial y^{\alpha}\partial x^j}\Gamma^{\alpha}_{i\gamma}y^{\gamma} + \frac{\partial L}{\partial y^{\alpha}}\frac{\partial \Gamma^{\alpha}_{i\gamma}}{\partial x^j}y^{\gamma} - \frac{\partial^2 L}{\partial x^i \partial x^j}\right)L \\ + \frac{y^{\beta}}{L^2}\left(\frac{\partial L}{\partial x^j}\frac{\partial L}{\partial y^{\alpha}}\Gamma^{\alpha}_{i\gamma}y^{\gamma} - \frac{\partial L}{\partial x^j}\frac{\partial L}{\partial x^i}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \overset{(m)}{\Gamma}{}^{\beta}_{i}}{\partial y^{\gamma}} &= \Gamma^{\beta}_{i\gamma} - \frac{\delta^{\beta}_{\gamma}}{L}\left(\frac{\partial L}{\partial y^{\alpha}}\Gamma^{\alpha}_{i\delta}y^{\delta} - \frac{\partial L}{\partial x^i}\right) \\ &- \frac{y^{\beta}}{L}\left(\frac{\partial^2 L}{\partial y^{\alpha}\partial y^{\gamma}}\Gamma^{\alpha}_{i\delta}y^{\delta} + \frac{\partial L}{\partial y^{\alpha}}\Gamma^{\alpha}_{i\gamma} - \frac{\partial^2 L}{\partial x^i \partial y^{\gamma}}\right) \\ &+ \frac{y^{\beta}}{L^2}\left(\frac{\partial L}{\partial y^{\gamma}}\frac{\partial L}{\partial y^{\alpha}}\Gamma^{\alpha}_{i\delta}y^{\delta} - \frac{\partial L}{\partial x^i}\frac{\partial L}{\partial y^{\gamma}}\right).\end{aligned}$$

Using the relations

$$\frac{\partial^2 L}{\partial y^\alpha \partial y^\gamma} y^\gamma = 0, \quad \frac{\partial^2 L}{\partial x^i \partial y^\gamma} y^\gamma = \frac{\partial L}{\partial x^i}, \quad \frac{\partial L}{\partial y^\gamma} y^\gamma = L,$$

which are consequences of the homogeneity of $L$, we obtain

$$\begin{aligned}
\overset{(m)}{\Gamma}{}^{\gamma}_{j} \frac{\partial \overset{(m)}{\Gamma}{}^{\beta}_{i}}{\partial y^\gamma} &= \Gamma^\beta_{i\gamma}\Gamma^\gamma_{j\delta} y^\delta - \Gamma^\beta_{j\delta}\frac{y^\delta}{L}\left(\frac{\partial L}{\partial y^\alpha}\Gamma^\alpha_{i\delta}y^\delta - \frac{\partial L}{\partial x^i}\right) \\
&\quad - \Gamma^\beta_{i\delta}\frac{y^\delta}{L}\left(\frac{\partial L}{\partial y^\alpha}\Gamma^\alpha_{j\delta}y^\delta - \frac{\partial L}{\partial x^j}\right) - \frac{y^\beta}{L}\Gamma^\gamma_{jm}\Gamma^\alpha_{i\delta}\frac{\partial^2 L}{\partial y^\alpha \partial y^\gamma} y^m y^\delta \\
&\quad - \frac{y^\beta}{L}\frac{\partial L}{\partial y^\alpha}\Gamma^\alpha_{is}\Gamma^s_{j\delta}y^\delta + \frac{y^\beta}{L}\frac{\partial^2 L}{\partial x^i \partial y^\gamma}\Gamma^\gamma_{j\delta}y^\delta \\
&\quad + \frac{y^\beta}{L^2}\frac{\partial L}{\partial y^\gamma}\Gamma^\gamma_{j\delta}y^\delta\left(\frac{\partial L}{\partial y^\alpha}\Gamma^\alpha_{i\delta}y^\delta - \frac{\partial L}{\partial x^i}\right) \\
&\quad + \frac{y^\beta}{L^2}\frac{\partial L}{\partial y^\alpha}\Gamma^\alpha_{i\gamma}y^\gamma\left(\frac{\partial L}{\partial y^\ell}\Gamma^\ell_{j\delta}y^\delta - \frac{\partial L}{\partial x^j}\right) \\
&\quad - \frac{y^\beta}{L^2}\frac{\partial L}{\partial x^i}\left(\frac{\partial L}{\partial y^\alpha}\Gamma^\alpha_{j\delta}y^\delta - \frac{\partial L}{\partial x^j}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\overset{(m)}{\Gamma}{}^{\gamma}_{j} \frac{\partial \overset{(m)}{\Gamma}{}^{\beta}_{i}}{\partial y^\gamma} - \overset{(m)}{\Gamma}{}^{\gamma}_{i} \frac{\partial \overset{(m)}{\Gamma}{}^{\beta}_{j}}{\partial y^\gamma} &= (\Gamma^\beta_{i\gamma}\Gamma^\gamma_{j\delta} - \Gamma^\beta_{j\gamma}\Gamma^\gamma_{i\delta})y^\delta \\
&\quad - \frac{y^\beta}{L}\left(\frac{\partial L}{\partial y^\alpha}\Gamma^\alpha_{i\gamma}\Gamma^\gamma_{j\delta} - \frac{\partial L}{\partial y^\alpha}\Gamma^\alpha_{j\gamma}\Gamma^\gamma_{i\delta}\right)y^\delta \\
&\quad + \frac{y^\beta}{L}\left(\frac{\partial^2 L}{\partial x^i \partial y^\gamma}\Gamma^\gamma_{j\delta} - \frac{\partial^2 L}{\partial x^j \partial y^\gamma}\Gamma^\gamma_{i\delta}\right)y^\delta \\
&\quad - \frac{y^\beta}{L^2}\frac{\partial L}{\partial y^\alpha}\left(\Gamma^\alpha_{j\delta}y^\delta\frac{\partial L}{\partial x^i} - \Gamma^\alpha_{i\delta}y^\delta\frac{\partial L}{\partial x^j}\right).
\end{aligned}$$

Substituting these in (4) we have

$$\overset{(m)}{R}{}^{\beta}_{ij} = \frac{\partial \overset{(m)}{\Gamma}{}^{\beta}_{j}}{\partial x^i} - \frac{\partial \overset{(m)}{\Gamma}{}^{\beta}_{i}}{\partial x^j} + \overset{(m)}{\Gamma}{}^{\gamma}_{j} \frac{\partial \overset{(m)}{\Gamma}{}^{\beta}_{i}}{\partial y^\gamma} - \overset{(m)}{\Gamma}{}^{\gamma}_{i} \frac{\partial \overset{(m)}{\Gamma}{}^{\beta}_{j}}{\partial y^\gamma}$$

$$= \left( \frac{\partial \Gamma^{\beta}_{j\delta}}{\partial x^i} - \frac{\partial \Gamma^{\beta}_{i\delta}}{\partial x^j} + \Gamma^{\beta}_{i\gamma}\Gamma^{\gamma}_{j\delta} - \Gamma^{\beta}_{j\gamma}\Gamma^{\gamma}_{i\delta} \right) y^{\delta}$$

$$- \frac{y^{\beta}}{L}\frac{\partial L}{\partial y^{\alpha}} \left( \frac{\partial \Gamma^{\alpha}_{j\delta}}{\partial x^i} - \frac{\partial \Gamma^{\alpha}_{i\delta}}{\partial x^j} + \Gamma^{\alpha}_{i\gamma}\Gamma^{\gamma}_{j\delta} - \Gamma^{\alpha}_{j\gamma}\Gamma^{\gamma}_{i\delta} \right) y^{\delta}$$

$$= R^{\beta}_{\delta ji} y^{\delta} - \frac{y^{\beta}}{L}\frac{\partial L}{\partial y^{\alpha}} R^{\alpha}_{\delta ji} y^{\delta}. \qquad \square$$

*Remark.* Notice that the operator $H^{\beta}_{\alpha} := \delta^{\beta}_{\alpha} - \frac{y^{\beta}}{L}\frac{\partial L}{\partial y^{\alpha}}$ in Theorem 2 is closely related to the angular metric tensor $h_{\alpha\beta}$ of the Finsler metric. In fact, transvecting by $g_{\beta\gamma}$ we have

$$H^{\beta}_{\alpha} g_{\beta\gamma} = \left( \delta^{\beta}_{\alpha} - \frac{y^{\beta}}{L}\frac{\partial L}{\partial y^{\alpha}} \right) g_{\beta\gamma} = g_{\alpha\gamma} - \frac{\partial L}{\partial y^{\alpha}}\frac{\partial L}{\partial y^{\gamma}} = h_{\alpha\gamma}.$$

Therefore we call the operator $H$ the *angular operator* of the Finsler vector bundle $(\xi, L)$. The angular operator is singular self adjoint operator with rank $m-1$ in the vertical bundle $V\xi$. So we can express the result of Theorem 2 as follows:

$$\overset{(m)}{R}(U,V) = H\left(\ell^v_z(R(d\pi(U), d\pi(V)))\right)$$

where $\ell^v_z \colon \pi^{-1}(\pi(z)) \to V_z E$ is the vertical lift: $\ell^v_z : (y^{\beta}) \mapsto (y^{\beta}(\frac{\partial}{\partial y^{\beta}})_z$.

We consider now the geodesic structures of the connections in the tangent bundle case $\xi = \tau_B$. A curve $\varphi \colon I \to B$ is called a geodesic (or an autoparallel curve) of the connection $\mathcal{H}$ iff its tangent curve $\dot\varphi$ is parallel (i.e. horizontal curve) along $\varphi$ with respect to $\mathcal{H}$.

**Proposition 3.** *Apart from parametrization the geodesics of $\mathcal{H}$ and $\mathcal{H}^{(m)}$ coincide.*

Proof. Consider a geodesic $\varphi \colon I \to B$ of $\mathcal{H}$ with initial vector $v_0 = \dot\varphi(0)$. Define a reparametrization $\gamma \colon I \to I$ as follows: $\gamma(\tau) = L(v_0)\int_0^{\tau} \frac{1}{L(\dot\varphi)}$. Then $\varphi_1 := \varphi \circ \gamma$ holds $\dot\varphi_1 = L(v_0)\frac{1}{L(\dot\varphi)}\dot\varphi = \mu_{L(v_0)}(\eta(\dot\varphi))$. It follows from the Corollary that $\dot\varphi_1$ is a geodesic of $\mathcal{H}^{(m)}$ with initial vector $v_0$. By the local uniqueness of geodesics every geodesic curve of $\mathcal{H}^{(m)}$ can be gained in this way. $\square$

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