Option pricing
Exercises

1. Probability theory

1.1. A blind knife thrower hits the target with probability $1/4$ and he keeps trying till the first hit.

(a) What is the distribution of the number of required trials?

(b) Find the mean and the standard deviation of the required trials.

1.2. Anna and Kate are playing tennis. Anna wins a game with probability 0.4, they play 3 games, and the winner is the player with more games won.

(a) What is the probability that Anna wins the tennis party?

(b) What is the expected value of the number of games won by Anna?

(c) What is the variance of the number of games won by Anna?

1.3. A dice is rolled one hundred times. Find the mean and the variance of the sum of the numbers obtained.

1.4. The following table specifies the joint distribution of (ξ, η):

$\xi \downarrow, \eta \rightarrow$	-1	0	1
-1	p	$3p$	$6p$
1	$5p$	$15p$	$30p$

- Find the value of p.
- Are ξ and η independent? Give the marginal distributions.
- Give the distributions of $\xi + \eta$ and $\xi \cdot \eta$.

1.5. Let the distribution of (ξ, η) be the distribution given in the previous example. Find the probabilities

(a) $P(\eta = i \mid \xi = -1)$, $i = -1, 0, 1$;

(b) $P(\eta < 1 \mid \xi = 1)$;

(c) $P(\xi = 1 \mid \eta \geq 0)$.

1.6. The nominal value of a share is one golden galleon. In a year the value either can doubled, or halved or remain the same – each of the events has the same probability. On the next year the same happens, independently of the events of the previous year. Find the distribution of the value of the share after two years. What is the mean and the variance of the value?

1.7. A fair dice is rolled 100 times. Let ξ denote the number of values 3 obtained from the first 50 rolls, while η denotes the number of even values obtained from the second 50 rolls. Find the variance of $\eta - \xi$.

1.8. In an office the number of letters arriving to the director on a particular day can be modelled by a Poisson random variable with parameter λ. The secretary of the director makes a pre-selection and a letter is forwarded to the director with probability p independently from the other letters. Find the distribution and the mean of the forwarded letters.

1.9. Let ξ be a Poisson random variable with parameter λ. Find the mean of the random variable $\eta = \frac{1}{1+\xi}$.

1.10. On a coin 0 is written on the head side and 1 on the tail side. The coin is tossed twice. Let ξ and η denote the sum and the product of the numbers obtained, respectively. Are ξ and η independent? Find the correlation coefficient of ξ and η.

1.11. In a certain zoo there are two sloths. Every day the first sleeps in half of the time, while the second in one third of the time, independently of the first. Let ξ denote the number of sloths that are awaken during our visit in the zoo. Find the cdf of ξ.

1.12. Check whether the following functions are probability density functions or not.

(a)

$$f(x) = \begin{cases} \frac{\sin(x)}{2}, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$f(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

1.13. The pdf of a random variable ξ equals

$$f(x) = \begin{cases} 0, & \text{if } x \leq 2, \\ \frac{A}{x^3}, & \text{if } x > 2. \end{cases}$$

(a) Find the value of A.
(b) Find the probability that $\xi \in [1, 4]$.
(c) Find the cdf of ξ.

2. Payoff of European options

Consider the following European options written for a fixed stock. What is the payoff as a function of the stock price S_T at maturity?

2.1. We buy 3 European call options with strike price 50$ per share, and also a European put option with the same strike price. Time to expiration is 2 months in both cases. What will be the exact payoff if the stock price is 52$ then?

2.2. We buy 1 European call option with strike price 8$, and also a European put option with strike price 6$. Time to expiration is half a year for both.

2.3. We buy a European call option for 100 shares with exercise price 22$ per share, and we sell another European call option for 100 shares with exercise price 19$. Time to expiration is 1 month.

2.4. We buy a European put option for 100 shares with exercise price 22$ per share, and we sell another European put option for 100 shares with exercise price 19$. Time to expiration is 1 month.

2.5. We buy a European call option with strike price 30$ per share, another European call option with strike price 40$ per share, and we sell 2 European call options with strike price 35$. Time to expiration is 2 months for all. What is the difference, if the sold options have strike price 32$ or 37$?

3. Trading strategies

3.1. How to obtain a long position in a put option with the help of a long call position?

3.2. How to obtain a short position in a call option with the help of a short put position?

3.3. Create a strip trading strategy from the options below. (They all refer to the same stock.)

- European call option with strike price 30$ per share, time to expiration is 6 months.
- European put option with strike price 32$ per share, time to expiration is 6 months.
- European put option with strike price 30$ per share, time to expiration is 6 months.

Let us assume that half a year later the stock price is 31.5$. What is the payoff of this strategy?

3.4. Create a strangle trading strategy from the options below. (They all refer to the same stock.)

- European put option with strike price 10$ per share, time to expiration is 3 months.
- European put option with strike price 10$ per share, time to expiration is 6 months.
- European call option with strike price 13$ per share, time to expiration is 3 months.

Let us assume that half a year later the stock price is 9.50$. What is the payoff of this strategy?

3.5. Create a bull spread strategy from the options below. (They all refer to the same stock.)

- European call option with strike price 8.10$ per share, time to expiration is 1 months.
- European put option with strike price 8.25$ per share, time to expiration is 2 months.
- European put option with strike price 7.80$ per share, time to expiration is 1 months.
- European call option with strike price 8.40$ per share, time to expiration is 1 months.

Let us assume that half a year later the stock price is 10$. What is the payoff of this strategy?

3.6. Create a bear spread strategy from the options below. (They all refer to the same stock.)

- European put option with strike price 22$ per share, time to expiration is 1 months.
- European call option with strike price 21$ per share, time to expiration is 2 months.
- European put option with strike price 20.50$ per share, time to expiration is 1 months.
- European call option with strike price 21.40$ per share, time to expiration is 1 months.

Let us assume that half a year later the stock price is 21$. What is the payoff of this strategy?

3.7. Create a butterfly spread from the options below. (They all refer to the same stock.)

- European call option with strike price 17$ per share, time to expiration is 3 months.
- European call option with strike price 15$ per share, time to expiration is 3 months.
- European put option with strike price 15.50$ per share, time to expiration is 3 months.
- European call option with strike price 16.50$ per share, time to expiration is 3 months.
- European call option with strike price 16$ per share, time to expiration is 3 months.

Let us assume that half a year later the stock price is 15.50$. What is the payoff of this strategy?

4. Forward contracts

4.1. We sign a forward contract today, that is, on 9th October 2018: we would like to buy 20 shares in 6 months. The current stock price is 10$, the risk-free interest rate is 6%. (We apply continuous compounding.) This is a non-dividend-paying stock.

(a) What is the forward price?

(b) What is the value of the position on 9th December 2018 if the stock price then turns out to be 10.40$?

4.2. Consider the previous exercise, and solve it again with the following changes.

(I) The stock pays dividend, namely, 0.50$ on 9th November 2018.

(II) The stock pays dividend, namely, 0.50$ on 9th March 2018.

(III) The stock pays dividend, namely, the average yield per annum on the stock in this 6 months is 1%.

5. Lower bounds for option prices

5.1. Consider a European call option on a non-dividend-paying stock when the stock price is 51$, the strike price is 50$, the time to maturity is 6 months, and the risk-free rate of interest is 12% per annum. Assume that the option price is 3.98$. Is this price arbitrage-free? If not, provide an arbitrage strategy.

5.2. Consider a European call option on a non-dividend-paying stock when the stock price is 20$, the strike price is 18$, the time to maturity is 1 year, and the risk-free rate of interest is 10% per annum. Assume that the option price is 3$. Is this price arbitrage-free? If not, provide an arbitrage strategy.

5.3. Consider a European put option on a non-dividend-paying stock when the stock price is 38$, the strike price is 40$, the time to maturity is 3 months, and the risk-free rate of interest is 10% per annum. Assume that the option price is 1.10$. Is this price arbitrage-free? If not, provide an arbitrage strategy.

5.4. Consider a European put option on a non-dividend-paying stock when the stock price is 37$, the strike price is 40$, the time to maturity is 6 months, and the risk-free rate of interest is 5% per annum. Assume that the option price is 1$. Is this price arbitrage-free? If not, provide an arbitrage strategy.