Discrete mathematics
Exercises with solutions

1 Introduction: sets and functions

2 Mathematical induction

3 Divisors and divisibility

4 Complex numbers

4.1. Solve the following equations among complex numbers.

(a) $x^2+4=0$ (b) $x^2+x+1=0$

Solution.
(a) $x_1=2i$, $x_2=-2i$ (b) $x_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $x_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

4.2. What is the algebraic form of the complex numbers below?

(a) $(-2+3i)(5-2i)$, $i(1+2i)$, $(3+i)(2+3i)$, $(-1+i)(1-2i)(1+2i)$,

(b) $\overline{5-2i}$, $\overline{(3+4i)}(2+i)$,

(c) $(2-i)^3$, i^{100}, i^{2017}, $i^6+3i^5-2i^3+i^2-1$,

(d) $\dfrac{1-2i}{1-3i}$, $\dfrac{5+3i}{i}$, $\dfrac{1-i}{2+i}$, $\dfrac{2-i}{(3-2i)(2+5i)}$.

Solution.
(a) $-4+19i$, $-2+i$, $3+11i$, $-5+5i$
(b) $5+2i$, $10-5i$
(c) $2-11i$, 1, i, $-3+5i$
(d) $\frac{7}{10}+\frac{1}{10}i$, $3-5i$, $\frac{1}{5}-\frac{3}{5}i$, $\frac{2-i}{16+11i}=\frac{21}{377}-\frac{38}{377}i$

4.3. What is the trigonometric form of the complex numbers below?

(a) 2 (b) i (c) $-7i$ (d) $1-i$ (e) $1+i$ (f) $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ (g) $-3-3\sqrt{3}i$ (h) $\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}i$

Solution.

(a) $2(\cos 0+i\cdot\sin 0$
(b) $1\left(\cos\frac{\pi}{2}+i\cdot\sin\frac{\pi}{2}\right)=\cos\frac{\pi}{2}+i\cdot\sin\frac{\pi}{2}$
(c) $7\left(\cos\frac{3\pi}{2}+i\cdot\sin\frac{3\pi}{2}\right)$
(d) $\sqrt{2}\left(\cos\frac{7\pi}{4}+i\cdot\sin\frac{7\pi}{4}\right)$
(e) $\sqrt{2}\left(\cos\frac{\pi}{4}+i\cdot\sin\frac{\pi}{4}\right)$
(f) $1\left(\cos\frac{2\pi}{3}+i\cdot\sin\frac{2\pi}{3}\right)=\cos\frac{2\pi}{3}+i\cdot\sin\frac{2\pi}{3}$
(g) $6\left(\cos\frac{4\pi}{3}+i\cdot\sin\frac{4\pi}{3}\right)$
(h) $1\left(\cos\frac{\pi}{4}+i\cdot\sin\frac{\pi}{4}\right)=\cos\frac{\pi}{4}+i\cdot\sin\frac{\pi}{4}$

4.4. Let $x=3\left(\cos\dfrac{\pi}{9}+i\sin\dfrac{\pi}{9}\right)$ and $y=2\left(\cos\dfrac{\pi}{5}+i\sin\dfrac{\pi}{5}\right)$. Determine the value of the expressions below.

(a) $x \cdot y$, (b) $\dfrac{x}{y}$, (c) x^3, (d) y^5, (e) $\dfrac{1}{x}$, (f) x^2y.

Solution.

(a) $6\big(\cos\frac{14\pi}{45} + i \cdot \sin\frac{14\pi}{45}\big)$

(b) $\frac{3}{2}\Big(\cos\big(-\frac{4\pi}{45}\big) + i \cdot \sin\big(-\frac{4\pi}{45}\big)\Big) = \frac{3}{2}\big(\cos\frac{86\pi}{45} + i \cdot \sin\frac{86\pi}{45}\big)$

(c) $27\big(\cos\frac{\pi}{3} + i \cdot \sin\frac{\pi}{3}\big)$

(d) $32(\cos\pi + i \cdot \sin\pi) = 32(-1 + i \cdot 0) = -32$

(e) $\frac{1}{3}\Big(\cos\big(-\frac{\pi}{9}\big) + i \cdot \sin\big(-\frac{\pi}{9}\big)\Big) = \frac{1}{3}\big(\cos\frac{17\pi}{9} + i \cdot \sin\frac{17\pi}{9}\big)$

(f) $18\big(\cos\frac{19\pi}{45} + i \cdot \sin\frac{19\pi}{45}\big)$

4.5. Compute the second, third and fourth roots of the complex number $z = 81\left(\cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}\right)$. Plot them on the complex plane.

Solution.
Second roots:

$$w_0 = 9\big(\cos\frac{\pi}{10} + i \cdot \sin\frac{\pi}{10}\big) \qquad w_1 = 9\big(\cos\frac{11\pi}{10} + i \cdot \sin\frac{11\pi}{10}\big)$$

Third roots:

$$\begin{aligned} w_0 &= \sqrt[3]{81}\big(\cos\frac{\pi}{15} + i \cdot \sin\frac{\pi}{15}\big) \\ w_1 &= \sqrt[3]{81}\big(\cos\frac{11\pi}{15} + i \cdot \sin\frac{11\pi}{15}\big) \\ w_2 &= \sqrt[3]{81}\big(\cos\frac{21\pi}{15} + i \cdot \sin\frac{21\pi}{15}\big) \end{aligned}$$

Fourth roots:

$$\begin{aligned} w_0 &= 3\big(\cos\frac{\pi}{20} + i \cdot \sin\frac{\pi}{20}\big) & w_1 &= 3\big(\cos\frac{11\pi}{20} + i \cdot \sin\frac{11\pi}{20}\big) \\ w_2 &= 3\big(\cos\frac{21\pi}{20} + i \cdot \sin\frac{21\pi}{20}\big) & w_3 &= 3\big(\cos\frac{31\pi}{20} + i \cdot \sin\frac{31\pi}{20}\big) \end{aligned}$$

4.6. Solve the equations below on the set of complex numbers.

(a) $z^2 - 3iz + 4 = 0$,

(b) $z^3 + z^2 + z = 0$,

(c) $z^5 - z = 0$,

(d) $(1-i)z^2 + (2+4i)z - 3 = 0$.

Solution.

(a) $z_1 = 4i,\ z_2 = -i$

(b) $z_0 = 0,\ z_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

(c) $z_0 = 0,\ z_{1,2} = \pm 1,\ z_{3,4} = \pm i$

(d) $z_{1,2} = \frac{-2-4i\pm\sqrt{4i}}{2-2i}$,
$z_1 = \frac{-2-4i+\sqrt{2}+\sqrt{2}i}{2-2i} = \dots,$
$z_2 = \frac{-2-4i-\sqrt{2}-\sqrt{2}i}{2-2i} = \dots$