# Discrete mathematics
## Exercises

*The exercises marked by * represent higher level exercises.*

## 1. Introduction: sets and functions

**1.1.** Let us consider the sets $A = \{x \in \mathbb{N} \mid x \text{ is even}\}$, $B = \{x \in \mathbb{N} \mid x > 4\}$, and $C = \{x \in \mathbb{N} \mid x < 6\}$. What are the sets below?

$$B \setminus C, \qquad A \setminus (B \cap C), \qquad \overline{B}, \qquad (B \cup C) \setminus A, \qquad B \triangle C$$

**1.2.** Let us consider the sets $A = \mathbb{Z}$, $B = \{x \in \mathbb{Z} \mid x \text{ is even}\}$, $C = \{0, 1, 2, 3, 4\}$, $D = \{x \in \mathbb{N} \mid 3 \le x < 7\}$. What are the sets below?

$$A \setminus B, \qquad B \setminus A, \qquad A \cap B, \qquad C \setminus B, \qquad B \cap \overline{C}, \qquad (A \setminus B) \cup D, \qquad C \triangle D$$

**1.3.** What is the power set of $A = \{a, b, c\}$?

**1.4.** Let $H$ be a universal set, and consider arbitrary subsets $A$, $B$, $C$ of it. With the help of these sets and the set operations, write down the sets below.

(a) Elements of $B$ only;
(b) Elements of exactly 2 sets (out of $A$, $B$ and $C$);
(c) Not elements of all 3 sets;
(d) Elements of at most 1 set;
(e) Elements of at least 1 set;
(f) Elements of at least 2 sets.

**1.5.** Let $A$ be a set of cardinality $m$, and $B$ a set of cardinality $n$. Assume that $m \le n$. What is the cardinality of the sets below at least and at most?

$$A \cup B, \qquad A \cap B, \qquad A \setminus B, \qquad B \setminus A, \qquad A \triangle B, \qquad A \times B$$

**1.6.** Plot the following functions. (The domain is the greatest possible subset of $\mathbb{R}$.) Also provide the formula for the function, if it's not given.

(a) $f(x) = 2x + 3$
(b) $f(x) = -\frac{x}{2}$
(c) $f(x) = 6$
(d) The linear function $f(x)$ which has a graph going through the points $(-3, 1)$ and $(2, -2)$.
(e) $f(x) = 2x^2 + 8x - 10$
(f) $f(t) = t^2 - 6t + 9$
(g) $f(x) = -x^2 - 4$
(h) $f(x) = 2^x$
(i) $f(x) = 0.7^x$
(j) $f(x) = \log_3 x$
(k) $f(x) = \log_{0.5} x$

**1.7.** Plot the following functions. (The domain is the greatest possible subset of $\mathbb{R}$.) Also provide the formula for the function, if it's not given.

(a) $f(x) = 8 - 2x$

(b) $f(x) = \frac{x}{4} + 1$

(c) The linear function $f(x)$ which has a graph going through the points $(-2, 0)$ and $(3, 4)$.

(d) $f(x) = -2x^2 - 7x + 3$

(e) $f(x) = x^2 - 2x + 4$

(f) $f(x) = x^2 - 4x + 4$

(g) $f(x) = 0.99^x$

(h) $f(x) = 3^x$

(i) $f(x) = \log_4 x$

(j) $f(x) = \log_{\frac{1}{3}} x$

**1.8.** Decide whether the functions below are injective, surjective, bijective. Unless otherwise stated, the domain is the greatest possible subset of $\mathbb{R}$ and the image set is $\mathbb{R}$.

(a) $f(x) = x^2$

(b) $g(x) = \sin x$

(c) $h(x) = \sin x \restriction \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Does it change something if the image set is $[-1, 1]$?

(d) $f(x) = x^3$

(e) $f(x) = x^3 - x$

(f) $f(x) = ax + b$

(g) $f(x) = 2^x$

(h) $f(x) = \log_{\frac{1}{3}} x$

# 2. Mathematical induction

**2.1.** Prove the following relations by mathematical induction.

(a) $$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad \forall n \in \mathbb{N};$$

(b) $$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \forall n \in \mathbb{N};$$

(c) $$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \qquad \forall n \in \mathbb{N};$$

(d) $$1 + 3 + 5 + \cdots + (2n-1) = n^2 \qquad \forall n \in \mathbb{N};$$

(e) $$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3} \qquad \forall n \in \mathbb{N};$$

(f) $$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3} \qquad \forall n \in \mathbb{N};$$

(g) $$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} \qquad \forall n \in \mathbb{N};$$

(h) $1 \cdot 4 + 2 \cdot 7 + \cdots + n(3n+1) = n(n+1)^2 \qquad \forall n \in \mathbb{N};$

(i) $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\cdots\left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+2}{2n+2} \qquad \forall n \in \mathbb{N};$

(j) $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1 \qquad \forall n \in \mathbb{N};$

(k) $6|(n^3 - n) \qquad \forall n \in \mathbb{N};$

(l) $6|(n^3 + 5n) \qquad \forall n \in \mathbb{N};$

(m) $5|(2^{4n+1} + 3) \qquad \forall n \in \mathbb{N};$

(n) $3|(n^3 + 5n + 6) \qquad \forall n \in \mathbb{N};$

(o) $9|(10^n + 3 \cdot 4^{n+2} + 5) \qquad \forall n \in \mathbb{N};$

(p) $4|(7^n + 10n - 5) \qquad \forall n \in \mathbb{N};$

(q) $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15}$ is an integer for all $n \in \mathbb{N}$;

(r) $(n+1)! > 2^{n+3}$ if $n \geq 5$, $n \in \mathbb{N}$;

(s)* $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq 2(\sqrt{n+1} - 1) \qquad \forall n \in \mathbb{N};$

(t)* $\binom{2n}{n} < 4^{n-1}$, if $n \geq 5$, $n \in \mathbb{N}$;

(u) $\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n} \qquad \forall n \in \mathbb{N};$

(v)* $n^3 < 2^{n+1}$ if $n > 8$, $n \in \mathbb{N}$;

(w) $\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} \qquad \forall n \in \mathbb{N}.$

**2.2.** Let us assume that $n \geq 4$ elderly women all know a different gossip. Furthermore, if any two of them talk on the phone, they share all the gossips they know. Prove by induction that it's possible for all of them to get to know every gossip via $2n - 4$ phone calls.

# 3. Divisors and divisibility

**3.1.** Prove the divisibility relations below.

(a) $9|(10^{19} + 53)$

(b) $36|(10^{17} - 64)$

(c) $6|(10^7 - 88)$

(d) $12|(10^{16} + 44)$

**3.2.** Which digits can we use as $a$ and $b$ for the divisibility relations to hold?

(a) $33|\overline{52ab71}$

(b) $36|\overline{762a4b}$

(c) $45|\overline{61a24b}$

(d) $72|\overline{44a21b}$

**3.3.** Prove the divisibility relations below.

(a) $200|(199^3 - 199)$

(b) $7|(11^9 - 4^9)$

(c) $200|(101^3 + 99^3)$

(d)* $99|(11^{22} - 22^{11})$

**3.4.** Prove that the product of 4 consecutive numbers is always divisible by 24.

**3.5.** Prove that if you write down a number of 3 digits two times after each other, then the obtained number is divisible by 13.

**3.6.** Calculate the greatest common divisor of the following pair of numbers with Euclidean algorithm.

(a) 672 and 360,

(b) 455 and −312,

(c) 1225 and 216,

(d) 680 and −845,

(e) 783 and 1160,

(f) 3751 and 1240.

**3.7.** Can the numbers $n - 12$, $n + 3$ and $n + 5$ be primes at the same time?

**3.8.** Let $p > 3$ be a prime. Show that $24|(p^2 - 1)$.

**3.9.** Calculate the number of zero digits at the end of 100!.

**3.10.** Calculate the greatest common divisor and least common multiple of the following pairs of numbers.

(a) 450 and 420,

(b) 539 and 364,

(c) 1260 and 14850,

(d) 663 and 308,

(e) 495 and 300,

(f) 990 and 420.

**3.11.** How many positive divisors do the following numbers have?

(a) 252,

(b) 600,

(c) 528.

**3.12.** Give an example for a natural number having exactly 6 positive divisors. What is the smallest natural number having exactly 6 positive divisors?

**3.13.** How many positive divisors of 7560 is coprime to 15?

**3.14.** Solve the linear Diophantine equations below if possible.

(a) $14x - 18y = 6$,

(b) $15x + 28y = 12$,

(c) $12x - 15y = 26$,

(d) $21x - 15y = 12$,

(e) $495x + 300y = 15$,

(f) $18x + 28y = 10$.

**3.15.** Prove that the following equations are not solvable among the integers.

(a) $n^{k+1} = (n + 1)^k$,

(b) $a^6 + 25a = 7425$,

(c) $k(k^4 + 1) = 3267$.

**3.16.** Why is it impossible to have 2017 as the sum of two primes?

**3.17.*** Is it possible, that the sum of 2007 consecutive integers is a prime?

**3.18.** Arthur has 1420 Forints and he would like to spend it all on chocolate. There are two types of chocolate in the store: 1 bar of milk chocolate costs 70 Forints, while 1 bar of dark chocolate is 80 Forints. What possibilities does Arthur have?

**3.19.*** Peter bought a bouquet of 20 flowers for 1430 Forints. There are yellow, pink and purple flowers in it, which cost 50, 70 and 80 Forints, respectively. How many flowers are there in the bouquet from the different colors if we know that there are no more than 10 pieces from any of them?

**3.20.** Solve the following congruences if possible.

(a) $3x \equiv 5 \pmod 7$

(b) $12x \equiv 8 \pmod{16}$

(c) $9x \equiv 15 \pmod{12}$

(d) $5x \equiv 4 \pmod{11}$

(e) $5x \equiv 24 \pmod{13}$

(f) $14x \equiv 8 \pmod{21}$

(g) $11x \equiv 12 \pmod{18}$

(h) $30x \equiv 48 \pmod{58}$

**3.21.** Compute the remainder of

(a) $39^{28}$, when divided by 29;

(b) $17^{40}$, when divided by 25;

(c) $23^{81}$, when divided by 50;

(d) $17^{18}$, when divided by 40;

(e)* $54^{55^{56}}$, when divided by 13;

(f)* $38^{39^{40}}$, when divided by 11.

**3.22.** What are the last two digits of $19^{81}$?

# 4. Complex numbers

**4.1.** Solve the following equations among complex numbers.

(a) $x^2 + 4 = 0$

(b) $x^2 + x + 1 = 0$

**4.2.** What is the algebraic form of the complex numbers below?

(a) $(-2+3i)(5-2i)$, $\quad i(1+2i)$, $\quad (3+i)(2+3i)$, $\quad (-1+i)(1-2i)(1+2i)$,

(b) $\overline{5-2i}$, $\quad \overline{(3+4i)}(2+i)$,

(c) $(2-i)^3$, $\quad i^{100}$, $\quad i^{2017}$, $\quad i^6 + 3i^5 - 2i^3 + i^2 - 1$,

(d) $\dfrac{1-2i}{1-3i}$, $\quad \dfrac{5+3i}{i}$, $\quad \dfrac{1-i}{2+i}$, $\quad \dfrac{2-i}{(3-2i)(2+5i)}$.

**4.3.** What is the trigonometric form of the complex numbers below?

(a) $2$

(b) $i$

(c) $-7i$

(d) $1-i$

(e) $1+i$

(f) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

(g) $-3-3\sqrt{3}i$

(h) $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

**4.4.** Let $x = 3\left(\cos\dfrac{\pi}{9} + i\sin\dfrac{\pi}{9}\right)$ and $y = 2\left(\cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}\right)$. Determine the value of the expressions below.

(a) $x \cdot y$, (b) $\dfrac{x}{y}$, (c) $x^3$, (d) $y^5$, (e) $\dfrac{1}{x}$, (f) $x^2y$.

**4.5.** Compute the second, third and fourth roots of the complex number $z = 81\left(\cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}\right)$. Plot them on the complex plane.

**4.6.** Solve the equations below on the set of complex numbers.

(a) $z^2 - 3iz + 4 = 0$,

(b) $z^3 + z^2 + z = 0$,

(c) $z^5 - z = 0$,

(d) $(1-i)z^2 + (2+4i)z - 3 = 0$.

# 5. Polynomials

**5.1.** Factorize the polynomials below over $\mathbb{R}$.

(a) $2x^2 - 2x - 12$

(b) $-x^3 - 3x^2 + 4x$

(c) $4x^2 - 4x - 3$

(d) $x^3 + 3x^2 + 3x + 1$

(e) $x^3 - 6x^2 + 12x - 8$

(f) $x^3 - 3x^2 - x + 3$

(g) $x^4 - 16$

(h) $x^4 - 2x^2 + 1$

**5.2.** Give a minimal degree polynomial with real coefficients such that

(a) $-3$ is a root of it with multiplicity 2 and 1 is a simple root;

(b) 5 and $i$ are its roots;

(c) $i$ is a root of it with multiplicity 2.

**5.3.** Do the divisions below.

(a) $(2x^3 - x^2 - 5x - 2) \div (x - 2)$

(b) $(2x^4 - 5x^3 - 7x^2 + 14x - 6) \div (x - 3)$

(c) $(-3x^3 + 48x) \div (x - 4)$

(d) $(6x^3 + 9x^2 - 5x - 4) \div (2x + 1)$

(e) $(6x^4 - 7x^3 + 5x^2 - 2x) \div (2x^2 - x + 1)$

**5.4.** Do Euclidean division on the polynomials below.

(a) $2x^4 - 3x^3 + 4x^2 - 5x + 6$, $x^2 - 3x + 1$

(b) $2x^5 - 5x^3 - 8x$, $x + 3$

(c) $x^5 - 3x^4 + 1$, $x^2 + x + 1$

(d) $x^4 + x^3 - 3x^2 - 4x - 1$, $x^3 + x^2 - x - 1$

(e) $x^4 - 10x^2 + 1$, $x^4 - 4\sqrt{2}x^3 + 6x^2 + 4\sqrt{2}x + 1$

**5.5.** Use Horner's method to evaluate the polynomials below at the given values.

(a) $p(x) = 2x^5 - 3x^4 + x^3 - 5x^2 + 2x - 4$, $p(2) =?$, $p(-1) =?$

(b) $p(x) = 2x^6 - 3x^4 + x^2 + 2x + 1$, $p(-2) =?$

(c) $p(x) = x^5 - 3x^4 + 4x^2 + 3x + 2$, $p(-2) =?$, $p(3) =?$

(d) $p(x) = 6x^3 + 9x^2 - 5x - 4$, $p\left(-\frac{1}{2}\right) =?$

# 6. Combinatorics

**6.1.** There are 10 participants at a running competition. How many different orders can they finish in?

**6.2.** There are 10 participants at a running competition. How many possibilities are there for the podium (that is, for the first 3 places)?

**6.3.** You are eating at Emile's restaurant and the waiter informs you that you have (a) two choices for appetizers: soup or juice; (b) three for the main course: a meat, fish, or vegetable dish; and (c) two for dessert: ice cream or cake. How many possible choices do you have for your complete meal?

**6.4.** How many actual four digit numbers (they can not start with zero) can be formed from the digits 0, 1, 2, 3, 4, 5, 6?

**6.5.** We have 12 books on the shelf. How many ways can the books be arranged on the shelf if 3 particular books have to be next to each other

(a) if the order of the three books does not count?
(b) if the order of the three books does count?

**6.6.** In how many ways can one arrange 4 math books, 3 chemistry books, 2 physics books, and 1 biology book on a bookshelf so that all the math books are together, all the chemistry books are together, and all the physics books are together?

**6.7.** In how many ways can 7 people be arranged at a round table?

**6.8.** In how many ways can 5 men and 5 women be arranged at a round table if neither two men, nor two women can sit next to each other?

**6.9.** In how many ways can one fill a toto coupon (14 matches, three possible results: 1, 2 or X)?

**6.10.** Find the number of possible choices of four cards of four different colours from a deck of ordinary cards (4 colours, 13 cards per colour). What is the result if we require that the four cards should be of different values?

**6.11.** Find the number of possible paths from the origin to the point $(5, 3)$ if we can walk only on points with integer coordinates and we can step only upwards and right.

**6.12.** In Circus Maximus the tamer has to lead 5 lions and 4 tigers to the ring, but tigers can not follow each other because they fight. In how many ways can he do it if the animals can be distinguished?

**6.13.** A deck of ordinary cards is shuffled and 10 cards are dealt.

(a) In how many cases will we have aces among the 10 cards?
(b) Exactly one ace?
(c) At most one ace?
(d) Exactly two aces?
(e) At least two aces?

**6.14.** In how many ways can we choose four dancing pairs from 12 girls and 15 boys?

**6.15.** Prove that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

**6.16.** Are there more

- 9-digit numbers with all different digits,
- or 10-digit numbers with all different digits?

**6.17.** Find the number of possible arrangements of 8 castles on the chess board in a way that they do not hit each other? What is the result if we can distinguish between the castles?

**6.18.** In a box there are 5 red, 7 blue and 3 green balls. We take them out one by one, and write down the colors in order. How many possibilities are there for the sequence of colors?

**6.19.** There is a group of 10 going for a journey, and at the hotel they booked a room with 2 beds, a room with 3 and a room with 5 beds. How many possibilities are there for distributing the group among the rooms?

**6.20.** There are 15 (different) books and 12 children. In how many ways can we distribute the books among the children?

**6.21.** There are 15 apples and 12 children. In how many ways can we distribute the apples among the children?

**6.22.** Three postmen has to deliver six letters. Find the number of possible distributions of the letters.

**6.23.** Find the number of possible arrangements of 4 boys and 6 girls such that there are no boys standing next to each other.

**6.24.*** Find the number of real 5-digit numbers having 3 odd and 2 even digits.

**6.25.** In the canteen we can by four types of snacks. In how many ways can we buy 12 of them?

**6.26.** In how many ways can we choose a group of 4 from 5 boys and 5 girls, such that there are at least two girls among them?

# 7. Vector spaces

**7.1.** Show that the vectors $(1, 2)$ and $(-3, 2)$ are linearly independent in $\mathbb{R}^2$.

**7.2.** Are the vectors below linearly dependent or independent in $\mathbb{R}^3$?

(a) $v_1 = (1, 0, 1)$, $v_2 = (1, 1, 1)$;
(b) $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (1, 1, 1)$;
(c) $v_1 = (1, 2, 2)$, $v_2 = (-1, -1, 2)$, $v_3 = (2, 3, 0)$;
(d) $v_1 = (1, 1, 2)$, $v_2 = (2, 3, -1)$, $v_3 = (-1, 2, -17)$;
(e) $v_1 = (1, 1, 2)$, $v_2 = (-2, -2, -4)$;
(f) $v_1 = (-1, 2, 1)$, $v_2 = (2, -3, 1)$, $v_3 = (1, 1, 2)$;
(g) $v_1 = (-1, 2, 1)$, $v_2 = (2, -3, 1)$, $v_3 = (1, 1, 2)$, $v_4 = (5, 2, 0)$.

**7.3.** What are the coordinates in $\mathbb{R}^2$ of the vector $(1,2)$ with respect to the bases below?

(a) $(1,0)$, $(0,1)$;

(b) $(-1,2)$, $(1,1)$;

(c) $(3,1)$, $(1,2)$;

(d) $(1,-1)$, $(1,3)$.

**7.4.** What are the coordinates in $\mathbb{R}^3$ of the vector $(1,-1,2)$ with respect to the bases below?

(a) $(1,0,0)$, $(0,1,0)$, $(0,0,1)$;

(b) $(1,0,0)$, $(0,1,0)$, $(1,1,1)$;

(c) $(-1,2,1)$, $(2,-3,1)$, $(1,1,2)$;

(d) $(1,2,2)$, $(-1,-1,3)$, $(2,3,0)$.

# 8. Matrices, determinants

**8.1.** Consider the matrices $A$, $B$, $C$ and $D$:

$$A=\begin{pmatrix}4&-1&0\\1&5&-3\end{pmatrix}\quad B=\begin{pmatrix}0&3&-2\\1&1&4\end{pmatrix}\quad C=\begin{pmatrix}2&-1&1\\-1&3&10\\4&0&4\end{pmatrix}\quad D=\begin{pmatrix}1\\-2\\7\end{pmatrix}$$

Do the following operations if possible.

$$A+B,\ 3A-2B,\ A^T+B^T,\ AD,\ CB,\ CD,\ AC+BC,\ (A+B)C$$

**8.2.** Let

$$v=\begin{pmatrix}4\\-3\end{pmatrix},\quad w=\begin{pmatrix}0\\-1\end{pmatrix},\quad A=\begin{pmatrix}-1&3\\2&-4\end{pmatrix},$$
$$B=\begin{pmatrix}4&2\\-2&-3\end{pmatrix},\quad C=\begin{pmatrix}2&-3\\-3&-5\end{pmatrix},\quad E=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Calculate the matrices below.

$$Av,\quad Aw\quad A(u+v),\quad w^TA^T,\quad A+B,\quad (-2C)^T,\quad 2A-3B,\quad (A+B)^T,\quad AB,\quad BA,$$
$$B^TA^T,\quad AE,\quad EA,\quad AC,\quad A(B+C),\quad C(A-2B),\quad A^TA,\quad AA^T.$$

**8.3.** Calculate the determinants below with the rule of Sarrus.

(a) $\begin{pmatrix}1&3\\1&-2\end{pmatrix}$

(b) $\begin{pmatrix}5&-2\\2&1\end{pmatrix}$

(c) $\begin{pmatrix}2&1\\4&2\end{pmatrix}$

(d) $\begin{pmatrix}1&1&1\\1&2&3\\1&4&9\end{pmatrix}$

(e) $\begin{pmatrix}2&0&0\\-5&1&2\\3&8&-7\end{pmatrix}$

(f) $\begin{pmatrix}\cos\alpha&\sin\alpha&0\\-\sin\alpha&\cos\alpha&0\\0&0&1\end{pmatrix}$

**8.4.** Calculate the determinants below with Laplace expansion or Gaussian elimination.

(a) $\begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & -1 & 2 \\ 0 & 0 & 5 \\ 0 & 1 & -5 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 & -1 \\ -6 & -6 & 5 \\ 4 & -4 & 3 \end{pmatrix}$ (e) $\begin{pmatrix} -3 & 0 & -1 \\ -1 & -6 & 5 \\ 2 & 0 & -3 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & -1 & 1 \\ 4 & -2 & 5 \\ -2 & 1 & -5 \end{pmatrix}$

(g) $\begin{pmatrix} 5 & 1 & 2 & 7 \\ 3 & 0 & 0 & 2 \\ 1 & 3 & 4 & 5 \\ 2 & 0 & 0 & 3 \end{pmatrix}$ (h) $\begin{pmatrix} -2 & 2 & 0 & 1 \\ 4 & -2 & -1 & 1 \\ -4 & -2 & 4 & -8 \\ 8 & -6 & 1 & 0 \end{pmatrix}$ (i) $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$

**8.5.** Determine the inverse of the matrices below if it exists.

(a) $\begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 2 & -12 \\ -1 & 0 & 2 \\ 2 & 2 & -10 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & -1 & 2 \\ 1 & -3 & -4 \\ 2 & 2 & 5 \end{pmatrix}$ (e) $\begin{pmatrix} -2 & 1 & 4 \\ 1 & 0 & 3 \\ -1 & 2 & -3 \end{pmatrix}$ (f) $\begin{pmatrix} -1 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (h) $\begin{pmatrix} -1 & 3 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{pmatrix}$ (i) $\begin{pmatrix} 1 & -3 & 2 \\ -6 & 4 & -1 \\ 1 & 3 & -6 \end{pmatrix}$

# 9. Systems of linear equations

**9.1.** Solve the following systems of linear equations if possible.

(a)
$$\begin{aligned} x_1 - x_2 + x_3 &= 6 \\ 2x_1 + x_2 - x_3 &= 5 \\ -3x_1 + 2x_2 + 5x_3 &= 3 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 + x_2 - x_3 &= 5 \\ x_1 - x_2 + x_3 &= 6 \\ x_1 + 2x_2 - 2x_3 &= -1 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 4x_2 &= 5 \\ 2x_1 + 8x_2 &= 10 \end{aligned}$$

(d)
$$\begin{aligned} x_1 - x_2 + 3x_3 + 5x_4 &= 2 \\ 2x_1 + 2x_2 \phantom{+ 3x_3} + x_4 &= 0 \\ -x_1 - 3x_2 + 3x_3 + 4x_4 &= 2 \end{aligned}$$

(e)
$$\begin{aligned} -3x_1 + x_2 + 2x_3 &= -2 \\ 4x_1 - 6x_2 - x_3 &= 17 \\ x_1 - 5x_2 + x_3 &= 10 \end{aligned}$$

(f)
$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 2z &= 5 \\ 3x + 5y + 6z &= 14 \\ 2x + 4y + 5z &= 11 \\ x + 2y + 3z &= 6 \end{aligned}$$

**9.2.** Solve the following systems of linear equations if possible.

(a)
$$\begin{aligned} -2x_1 - x_2 + 4x_3 &= 3 \\ 2x_1 + 3x_2 - x_3 &= 1 \\ -4x_1 - 10x_2 - 5x_3 &= -12 \end{aligned}$$

(c)
$$\begin{aligned} -2x_1 + 2x_2 + x_3 &= 0 \\ 6x_1 - 3x_2 - 4x_3 &= -8 \\ -4x_1 + x_2 + x_3 &= 4 \end{aligned}$$

$$\text{(e)}\quad \begin{aligned} 2x_1 - 4x_2 + 2x_3 &= -2 \\ -4x_1 + 6x_2 - x_3 &= 5 \\ x_1 - 2x_3 &= 0 \end{aligned}$$

$$\text{(d)}\quad \begin{aligned} 3x_1 + 2x_2 - 4x_3 - x_4 &= 4 \\ -x_1 - 2x_2 + 3x_3 + 5x_4 &= -5 \\ 2x_1 + 4x_2 - 2x_3 - 5x_4 &= 8 \end{aligned}$$

$$\text{(b)}\quad \begin{aligned} -4x_1 - 4x_2 + 2x_3 &= -2 \\ -2x_1 - 7x_2 + 3x_3 &= 6 \\ 2x_1 + 12x_2 - 5x_3 &= -13 \end{aligned}$$

$$\text{(f)}\quad \begin{aligned} -x_1 + 3x_2 - 2x_3 &= 3 \\ 2x_1 - 4x_2 + 9x_3 &= 3 \\ -x_1 + 7x_2 + 7x_3 &= 20 \\ x_1 - 5x_2 - 2x_3 &= -11 \end{aligned}$$

**9.3.** What is the rank of the matrices below?

$$\begin{pmatrix} 3 & 1 & -1 \\ 0 & 1 & -4 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & -1 \\ -6 & -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 0 \\ -2 & 2 & -6 \\ 0 & 4 & -4 \\ 3 & -3 & 9 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 7 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 1 & -1 \\ -9 & -8 & -1 & 6 \\ -6 & 0 & -8 & -8 \\ 9 & 8 & 7 & 5 \end{pmatrix}, \quad \begin{pmatrix} -2 & 4 \\ 1 & 3 \\ 0 & 2 \\ -5 & 2 \end{pmatrix}$$