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ON EXACTNESS, DEFINABILITY AND VAGUENESS IN PARTIAL APPROXIMATION SPACES

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Abstract

In this paper, lower/upper, boundary, and negative regions of set approximations, the fundamental concepts of classical rough set theory, have been considered as primitive ones. Assuming that they are independent of each other, a generalized framework for their investigations is outlined. Its main building blocks are base sets and definable sets. Lower/upper approximations, boundaries and negative sets are all considered as definable sets and their mutual interactions are studied. Lastly exact/rough sets are discussed. In generalized framework, four groups of formulae are defined for representing different variants of rough sets. They emphasize distinct features of roughness, and so it may be of highly importance which one is used in practical applications. Some possible choices appeared in authors' publications are mentioned.

Introduction

In Pawlak's rough set theory (PAWLAK 1982, 1991, PAWLAK, SKOWRON 2007a) and its standard generalizations (BANERJEE, CHAKRABORTY 2004, BONIKOWSKI et al. 1998, PAWLAK, SKOWRON 2007b, YAO 2003, YAO, YAO 2012), lower/upper, boundary and negative regions/operators of set approximations are not independent. Usually, two of them are sufficient to define the others.

In this paper, these dependences are relaxed and lower/upper, boundary and negative operators are viewed as *approximation primitives*. As an initial

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step, it is assumed that they are independent of each other, and then their mutual interactions will be investigated.

At first, a general basic framework for our study is outlined (CSAJBÓK 2013, CSAJBÓK, MIHÁLYDEÁK 2012a, 2012b). In the approximation process, the main building blocks are base sets (equivalence classes in the standard case) and definable sets (unions of equivalence classes) obtained by some set operations on base sets. In a general setting, the base sets do not necessarily form a covering. Definable sets constitute all the available knowledge about the objects of interest. Consequently, lower/upper approximations, boundaries and negative sets should all be considered as definable sets.

Vagueness has a central role in the motivation basis of rough set theory. In Pawlak's information-based proposal, it was expressed by the boundary regions of sets represented by the difference of upper and lower approximations. Exactness and roughness of sets are defined via these differences as well. However, in general situations, the above notions and their relations become trickier. For instance, we can definitely say neither that a definable set is exact nor that an exact set is definable. Another important example is that in standard (i.e., equivalence-based) rough set theory boundaries can be defined in three equivalent ways. In generalized theories, however, these definitions lead to different notions of boundaries.

Reformulating of classical rough set theory

Let U be a nonempty finite set of objects.

The well-known Pawlak's rough set theory can be *reformulated* as follows.

- Let ε be an equivalence relation on U. The partition of U generated by ε is called the *base system*, denoted by $\mathfrak{B}$. Its members, the equivalence classes, are called *base sets*.
- Any union of base sets are referred to as *definable* sets. Their collection is denoted by $\mathfrak{D}_{\mathfrak{B}}$. By definition, $\emptyset$ is a definable set.
- Let S be a subset of U ($S \subseteq U$). The *lower approximation* of S, denoted by $\mathsf{l}(S)$, is the union of all base sets that are the subsets of S. The *upper approximation* of S, denoted by $\mathsf{u}(S)$, is the union of all base sets that have nonempty intersection with S. $\langle \mathsf{l}, \mathsf{u} \rangle$ is called the *approximation pair*.
- The *boundary* of S is $\mathsf{bnd}(S) = \mathsf{u}(S) \setminus \mathsf{l}(S)$.
- The *negative region* of S is $\mathsf{n}(S) = (u(S))^c$, where $(\cdot)^c$ denotes the set complementation operator.

According to the above notations, three highly important notions in connection with set approximation can be defined:

- S is *exact* if $\mathsf{l}(S) = \mathsf{u}(S)$;

- S is *rough* if it is not exact;
- S is *definable* if $S \in \mathfrak{D}_{\mathfrak{B}}$.

Relying on the classical Pawlak's set approximation theory, four basic components of an initial set-theoretical general approximation framework can be identified.

1. *Domain*: a set of sets whose members are approximated.
2. *Base system*: a beforehand defined set of some distinguished sets of the domain which serves as the basis of approximations.
3. *Definable sets*: a set of sets whose members are
   - derived from the base system (the base sets are always definable), and
   - designated for possible lower and upper approximations, boundaries and negative regions of the members of the domain.
4. *Approximation pair*: determines lower and upper approximations of the members of the domain by using definable sets.

Generalization of classical rough set theory: the first step

Pawlak's theory is based on a partition of the universe U. By the classical Pawlak's terminology:

- the domain is 2^U (the power set of U);
- the base systems is $U / \varepsilon \subset 2^U$ (the partition of U generated by the equivalence relation ε); its members are the base sets which are called ε-elementary sets;
- $\mathfrak{D}_{\mathfrak{B}}$ is the set of definable sets, where $U / \varepsilon \subseteq \mathfrak{D}_{\mathfrak{B}} (\subseteq 2^U)$, $\emptyset \in \mathfrak{D}_{\mathfrak{B}}$ and any union of ε-elementary sets is also definable.
- $\mathsf{l}(S) = \underline{\varepsilon}(S) = \cup\{B \in \mathfrak{B} \mid B \subseteq S\}$, $\mathsf{u}(S) = \overline{\varepsilon}(S) = \cup\{B \in \mathfrak{B} \mid B \cap S \neq \emptyset\}$.

Accordingly, at the first phase, the notions of base system, definable sets and approximation pair are generalized.

Generalizations of the base system The base system can be generalized by giving up the pairwise disjoint property of U/ε and/or the covering of the universe. Table 1 summarizes the possible opportunities. The most general case gives up the mutually disjoint property *and* the covering of the universe.

Table 1

Relaxing partition

Mutual disjoint property	Covering of the universe	Theory / Framework
✓	✓	Pawlakian rough set theory
∅	✓	Covering-based rough set rheory
✓	∅	One-layered partial approximation framework
∅	∅	**Partial approximation framework**

Formally, the generalized base system $\mathfrak{B} \subseteq 2^U$ is a nonempty family of nonempty subsets of U which does not necessarily cover the universe. The structure which is obtained in such a way is called the *partial approximation framework*.

If $\cup\, \mathfrak{B} = U$, the base system is called *total*, otherwise *partial*. Nevertheless, the base sets belonging to partial base system may be mutually disjoint. In this case, $\mathfrak{B}$ is called *partial partition* or *one-layered* (CIUCCI et al. 2014a).

Generalization of definable sets Let the set of definable sets $\mathfrak{D}_{\mathfrak{B}} \subseteq 2^U$ be an extension of $\mathfrak{B}$ in such a way that $\mathfrak{B} \subseteq \mathfrak{D}_{\mathfrak{B}}$ and $\emptyset \in \mathfrak{D}_{\mathfrak{B}}$. The members of $\mathfrak{D}_{\mathfrak{B}}$ are called *definable sets*.

Many types of $\mathfrak{D}_{\mathfrak{B}}$ can be defined (CSAJBÓK, MIHÁLYDEÁK 2012b). Several examples are the following. $\mathfrak{D}_{\mathfrak{B}}$ is

- *strictly union type* if $D_1, D_2 \in \mathfrak{D}_{\mathfrak{B}}$, then $D_1 \cup D_2 \in \mathfrak{D}_{\mathfrak{B}}$ and $\mathfrak{D}_{\mathfrak{B}}$ does not contain any other set;
- *intersection type* if $D_1, D_2 \in \mathfrak{D}_{\mathfrak{B}}$, then $D_1 \cap D_2 \in \mathfrak{D}_{\mathfrak{B}}$;.
- *Boolean type* if $\mathfrak{D}_{\mathfrak{B}}$ is a Boole algebra generated by $\mathfrak{B}$.

In a base system $\mathfrak{B}$, our background knowledge about the objects of interest appears. $\mathfrak{D}_{\mathfrak{B}}$, the set of definable sets, constitutes how the beforehand given background knowledge can be used in an approximation process. From the logical point of view, strictly union type and intersection type of $\mathfrak{D}_{\mathfrak{B}}$ mean that the disjunction and conjunction of concepts represented by base sets are enabled, respectively. Intersection type works well only together with strictly union type. They together mean that during an approximation process, the conjunction and disjunction of the background properties of the objects of interest are enabled. In the case of Boolean type, partial negation is also enabled beyond the conjunction and disjunction.

Generalization of the approximation pair An approximation pair has to meet the following minimal requirements:

(1) $\mathsf{l}(2^U), \mathsf{u}(2^U) \subseteq \mathfrak{D}_{\mathfrak{B}}$ – it means that lower/upper approximations must be *definable*;

(2) the functions l and u must be *monotone*, i.e., for all $S_1, S_2 \in 2^U$ if $S_1 \subseteq S_2$, then $\mathsf{l}(S_1) \subseteq \mathsf{l}(S_2)$ and $\mathsf{u}(S_1 \subseteq \mathsf{u}(S_2)$;

(3) $\mathsf{u}(\emptyset) = \emptyset$ (*normality* of u);

(4) if $S \subseteq U$, then $\mathsf{l}(S) \subseteq \mathsf{u}(S)$ – it is called the *weak approximation property*.

Requirements (3) and (4) immediately imply that $\mathsf{l}\,(\emptyset) = \emptyset$ (*normality* of l).

In general, it is reasonable to assume that the base sets are exactly approximated from lower side. In certain cases, it may also be required of definable sets. Accordingly, a weak approximation pair $\langle \mathsf{l}, \mathsf{u} \rangle$ is

(5) *granular* if $\mathsf{l}(B) = B$ $(B \in \mathfrak{B})$ (granularity of $\mathfrak{B}$, or l is granular);

(6) *standard* if $\mathsf{l}(D) = D$ $(D \in \mathfrak{D}_{\mathfrak{B}})$ (granularity of $\mathfrak{D}_{\mathfrak{B}}$, or l is standard).

It is an important question how lower and upper approximations relate to the approximated set itself. Conditions (1) – (6) may permit that $\mathsf{l}(S) \not\subseteq S$ and/or $S \not\subseteq \mathsf{u}(S)$, or even $\mathsf{l}(S) \cap S = \emptyset$ and/or $S \cap \mathsf{u}(S) = \emptyset$. These observations motivate the following definitions.

Let $S \subseteq U$. A weak approximation pair $\langle \mathsf{l}, \mathsf{u} \rangle$ is

(7) *lower semi-strong* if $\mathsf{l}(S) \subseteq S$, i.e., l is contractive;

(8) *upper semi-strong* if $S \subseteq \mathsf{u}(S)$, i.e., is extensive;

(9) *strong* if it is lower semi-strong and upper semi-strong at the same time, i.e., each subset $S \subseteq U$ is bounded by $\mathsf{l}(S)$ and $\mathsf{u}(S)$: $\mathsf{l}(S) \subseteq S \subseteq \mathsf{u}(S)$;

(10) Pawlakian if

- $\mathsf{l}(S) = \cup\{B \mid B \in \mathfrak{B}, B \subseteq S\}$,
- $\mathsf{u}(S) = \cup\{B \mid B \in \mathfrak{B}, B \cap S \neq \emptyset\}$,

provided that $\mathfrak{D}_{\mathfrak{B}}$ is strictly union type.

The Pawlakian approximation pair $\langle \mathsf{l}, \mathsf{u} \rangle$ is a direct generalization of the pair $\langle \underline{\varepsilon}, \overline{\varepsilon} \rangle$. Clearly, it is a strong one.

Further generalization: boundaries

Definable sets constitute all the available primary and derived knowledge about the objects of interest. Consequently, like as lower and upper approximations are definable sets, boundaries must be definable as well.

Let $S \subseteq U$. In classical rough set theory, the following definitions of boundaries are equivalent, but they lead to different notions of boundaries in the general case.

(1) $\mathsf{b}_1(S) = \mathsf{u}(S) \setminus \mathsf{l}(S)$

(2) $\mathsf{b}_2(S) = \cup(C^{\mathsf{u}}(S) \setminus C^{\mathsf{l}}(S))$, where

- $C^{\mathsf{l}}(S) = \{B \mid B \in \mathfrak{B}, B \subseteq \mathsf{l}(S)\}$
- $C^{\mathsf{u}}(S) = \{B \mid B \in \mathfrak{B}, B \subseteq \mathsf{u}(S)\}$

(3) $\mathsf{b}_3(S) = \cup\{B \mid B \in \mathfrak{B}, B \cap S \neq \emptyset, B \not\subseteq S\}$

At first, it should be noted that none of $\mathsf{b}_1(S)$, $\mathsf{b}_2(S)$ and $\mathsf{b}_3(S)$ will be definable necessarily, hence these formulae cannot be generalized directly. Nevertheless, under different conditions one or more of them become definable.

- If the approximation pair $\langle \mathsf{l}, \mathsf{u} \rangle$ is Pawlakian and $\mathfrak{B}$ is one-layered, b_1 is definable.
- If $\mathfrak{D}_{\mathfrak{B}}$ is strictly union type, b_2 and b_3 are also definable. In addition, if $\langle \mathsf{l}, \mathsf{u} \rangle$ is Pawlakian, then $\mathsf{b}_1(S) \subseteq \mathsf{b}_2(S)$, where $\mathsf{b}_1(S) = \mathsf{b}_2(S) \Leftrightarrow \mathsf{b}_2(S) \cap \mathsf{l}(S) = \emptyset$ $(S \subseteq U)$ also holds (CSAJBÓK, MIHÁLYDEÁK 2013).

Another observation that both b_1 and b_2 are „independent" of the set S itself, they solely depend on the lower and upper approximations of S. It may happen that $S \cap \mathsf{b}_1(S) = \emptyset$ or $S \cap \mathsf{b}_2(S) = \emptyset$ even if $\mathsf{b}_1(S), \mathsf{b}_2(S) \neq \emptyset$.

On the contrary, b_3 only depends on the set S (of course, with respect to the given base system), it is independent of the lower and upper approximations of S. Furthermore, if $\mathsf{b}_3(S) \neq \emptyset$, $S \cap \mathsf{b}_3(S) \neq \emptyset$. This type of boundary can be viewed as a real representation of the informal notion of boundaries.

Further generalization: negative regions

In classical rough set theory, the negative region is $\mathsf{n}(S) = (u(S))^c$ $(S \subseteq U)$. Of course, it must be definable as well. However, even in covering-based rough set theory, $(u(S))^c$ is not definable necessarily. A possible solution may be that the lower approximation of $(u(S))^c$ is taken, i.e., $\mathsf{n}(S) = \mathsf{l}((u(S))^c)$. In this way, definable negative regions are obtained.

In Pawlak's rough set theory, the lower approximation $\mathsf{l}(S)$ of S is also called the *positive region*, and $\mathsf{l}(S) \cap \mathsf{n}(S) = \emptyset$ trivially holds owing to their constructions. In the general case, however, the weak approximation property $\mathsf{l}(S) \subseteq \mathsf{u}(S)$ immediately implies that $\mathsf{l}(S) \cap \mathsf{l}((u(S))^c) = \mathsf{l}(S) \cap \mathsf{n}(S) = \emptyset$ also holds. It is in accordance with the informal meaning of positive and negative properties.

It is a question to be investigated whether negative regions of sets can be treated as an approximation primitive, i.e., independently of lower/upper approximations and boundaries. If the answer is yes, the next question is that the relation $\mathsf{l}(S) \cap \mathsf{n}(S) = \emptyset$ as a minimum requirement for positive and negative regions is required or not.

Exact sets

As before, at first, the case of classical rough set theory is investigated. Accordingly, $S \subseteq U$ is *exact* if $\mathsf{l}(S) = \mathsf{u}(S)$, and so $\mathsf{l}(S) = S = \mathsf{u}(S)$ by the construction of l and u in classical rough set theory. This definition only works well when the approximation pair is strong, i.e., $\mathsf{l}(S) \subseteq S \subseteq \mathsf{u}(S)$. For instance, if the base system is total and the approximation pair $\langle \mathsf{l}, \mathsf{u} \rangle$ is Pawlakian, $\langle \mathsf{l}, \mathsf{u} \rangle$ is strong. However, when the base system is partial, the approximation pair generally is not strong even if it is a Pawlakian one. Indeed, if $\langle \mathsf{l}, \mathsf{u} \rangle$ is Pawlakian, it is surely lower semi-strong, i.e., $\mathsf{l}(S) \subseteq S$ $(S \subseteq U)$, but $S \subseteq \mathsf{u}(S)$ does not hold for all $S \subseteq U$.

Moving on, let $\mathfrak{B}$ be partial and $\langle \mathsf{l}, \mathsf{u} \rangle$ be Pawlakian. It may happen that $\mathsf{l}(S) = \mathsf{u}(S) \subsetneqq S$ for a set $S \subseteq U$. On the contrary, it may occur for a definable set S that $\mathsf{l}(S) \subsetneqq \mathsf{u}(S)$ holds. These observations mean that neither „an exact set is definable" nor „a definable set is exact" holds necessarily according to the classical definition of exactness.

In general case, exact sets can be defined, e.g., by lower and upper approximations as follows: $S = \mathsf{l}(S)$ or $S = \mathsf{u}(S)$. If both $S = \mathsf{l}(S)$ and $S = \mathsf{u}(S)$ hold, S can be viewed as absolutely exact set (CIUCCI et al. 2014b).

It is also a question to be investigated whether the notion of exact set could be an approximation primitive.

Rough sets

In rough set theory, a set $S \subseteq U$ is *rough* if it is not exact, i.e., $\mathsf{l}(S) \neq \mathsf{u}(S)$ according to the classical Pawlakian definition. However, there may be a number of sets with the same lower and upper approximations. Therefore, the pair $\langle \mathsf{l}(S)\ \mathsf{u}(S) \rangle$ represents a class of sets which is called the *rough set*. Its further definitions and detailed investigations can be found, e.g., in BANERJEE, CHAKRABORTY (2004) and the references therein.

In generalized framework, the following definitions of rough sets may arise:
1. $\langle \mathsf{l}(S), \mathsf{u}(S) \rangle$ (it is the classical Pawlakian approach);
2. $\langle \mathsf{l}(S), \mathsf{n}(S) \rangle$;
3. $\langle \mathsf{l}(S), \mathsf{b}_i(S) \rangle$ ($i = 1,2,3$);
4. $\langle \mathsf{u}(S), \mathsf{b}_i(S) \rangle$ ($i = 1,2,3$).

These formulae generate the same partition of the power set of U in classical rough set theory. This is because the operators n, b_i ($i = 1,2$) can be derived from from the operators l and u. The operator b_3 is independent of l and u, but the boundaries b_1, b_2, b_3, are equivalent in classical rough set theory. Therefore, in this case, any one of the previous formulae can represent rough sets. It should be noted that in these representations, exact sets appear as special rough sets.

However, these formulae may lead to different partitions of the power set of U in generalized framework. Therefore, formulae 1–4 may represent distinct variants of rough sets emphasizing their different possible features. Consequently, it may be of highly importance in practical applications of generalized set approximation which formula represents rough sets.

If it is laid emphasis on the positive/negative features of concepts, the second formula must be applied. The situation in which the relation $\mathsf{l}(S) \cap \mathsf{n}(S) = \emptyset$ holds is well known and studied in detail as orthopair (CIUCCI 2011, CIUCCI 2014). A possible application was discussed in (CSAJBÓK, MIHÁLYDEÁK 2011). However, in case of conflicting information, there may be something both positive and negative which situation has to be handled in some way as well. This is which happens, for instance, in paraconsistent logics (PRIEST et al. 1989, 2013, PRIEST 2002), Atanassov's IFS (ATANASSOV 1986, 1999, 2012, CATTANEO, CIUCCI 2006), generalized orthopairs (CIUCCI et al. 2014a) and relative sets (MOUSAVI, JABEDAR-MARALANI 2001, 2002).

A number of studies deal with the relationship between rough set theory and fuzzy set theory. There are many opportunities to establish relations between them (DUBOIS, PRADE 1987, DUBOIS, PRADE 1990, DUBOIS, PRADE 1992, CHAKRABORTY 2011, YAO, ZHANG 2000). In this case the third version of the definition of rough sets is the relevant one.

Fuzziness in partial approximation framework was investigated in (CSAJBÓK, MIHÁLYDEÁK 2013). The investigation was started from Pawlakian approximation pair, and created three partial membership functions: optimistic, average, and pessimistic ones. These fuzzy functions provide *special type of fuzziness* on the universe: all of them are *partial functions* and derived from the observed data relatively to available knowledge about the objects of the universe. In (MIHÁLYDEÁK 2013), relying on the above partial fuzzy functions, three different semantic notions of logical consequence relations were introduced in partial first-order logic.

Lastly, it should be noted that if the notion of exact sets is viewed as an approximation primitive, the representation of rough sets cannot be defined independently from exact sets.

Conclusions and future work

In this paper, starting from the classical Pawlakian rough set theory, general formulations of approximation spaces have been outlined. Three underlying approximation components (domain, base systems, and definable sets) and five approximation primitives (lower/upper, boundary and negative operators) have been identified. Possible basic structures of these components and mutual interactions of approximation primitives have been investigated. Some questions to be answered concerning approximation primitives have been posed.

Further investigations of algebraic structures of underlying approximation components are required. Next, it should be important to take a survey of algebraic structures of rough sets in different approximation spaces.

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